TD 11: Reviews

Exercise 1. We consider the vector space $E = C^\infty([0,1],\mathbb{R})$ equipped with the following metric

$$d(f,g) = \sum_{k\geq 0} \frac{1}{2^k} \min\left(1, \|f^{(k)} - g^{(k)}\|_\infty\right).$$

1. Check that E is a Fréchet space.
2. Prove that any closed and bounded subset of E is compact.
3. Can the topology of E be defined by a norm ?

Exercise 2. For all $n \geq 0$, we set e^n the sequence which every term is zero, except the n^{th} which is 1. Recall that $c_0(\mathbb{N})$ denotes the subspace of $l^\infty(\mathbb{N})$ of sequences that converge to zero. Let

$$S = \left\{\varphi \in c_0(\mathbb{N})^* : \sum_{n=0}^{+\infty} \varphi(e^n) = 0\right\}.$$

1. Justify that S is well-defined and show that S is strongly closed in $c_0(\mathbb{N})^*$.
2. Show that S is weakly closed in $c_0(\mathbb{N})^*$, i.e. closed for the $\sigma(c_0(\mathbb{N})^*, c_0(\mathbb{N})^{**})$-topology.
3. Show that S is not weakly-$*$ closed in $c_0(\mathbb{N})^*$, i.e. not closed for the $\sigma(c_0(\mathbb{N})^*, c_0(\mathbb{N}))$-topology.

Exercise 3 (Banach limit).

1. Let $s : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ be the shift operator, defined by $s(x)_i = x_{i+1}$ for all $i \in \mathbb{N}$ and $x \in \ell^\infty(\mathbb{N})$. Prove the existence of a continuous linear function $\Lambda \in (\ell^\infty(\mathbb{N}))'$ satisfying $\Lambda \circ s = \Lambda$ and
$$\forall u \in \ell^\infty(\mathbb{N}), \quad \liminf_{n\to+\infty} u_n \leq \Lambda(u) \leq \limsup_{n\to+\infty} u_n.$$
Such a linear form Λ is called Banach limit.
Hint: Consider the vector space of bounded sequences that converge in the sense of Cesàro.
2. Deduce that there exists a function $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}_+$ which satisfies
   - (i) $\mu(\mathbb{N}) = 1$,
   - (ii) μ is finitely additive: $\forall A, B \subset \mathbb{N}$ with $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$,
   - (iii) μ is left-invariant: $\forall k \in \mathbb{N}$ and $A \subset \mathbb{N}$, $\mu(k + A) = \mu(A)$.

Exercise 4. Let H be a real Hilbert space and $J : H \to \mathbb{R}$ be a continuous convex functional. We assume that J is coercive, that is, $J(x) \to +\infty$ when $\|x\| \to +\infty$. Prove then that there exists $x_\star$ in H such that $J(x_\star) = \inf_{x\in H} J(x)$.

Exercise 5. Let $T : L^2[0,1] \to L^2[0,1]$ be the operator defined by

$$(Tf)(x) = \int_0^1 e^{-|x-y|} f(y)\,\mathrm{d}y.$$

1. Prove that T is well-defined, selfadjoint, compact and that $\|T\| \leq 1$.
2. Let $g = Tf$, where $f \in C^0[0,1]$. Check that g is in $C^2[0,1]$ and satisfies

$$g'' - g = -2f, \quad g(0) = g'(0), \quad g(1) = -g'(1).$$

3. Reciprocally, let $g \in C^2[0,1]$ satisfying $g(0) = g'(0)$ and $g(1) = g'(1)$. We set $f = (g - g'')/2$. Check that $g = Tf$.
4. Prove that $\operatorname{Im} T$ is dense in $L^2[0,1]$. Is 0 an eigenvalue of T ?
5. Let $f \in C^0[0,1]$ and $g = Tf$. Check that

$$2\langle Tf, f\rangle_{L^2} = |g(0)|^2 + |g(1)|^2 + \int_0^1 |g(x)|^2 \, \mathrm{d}x + \int_0^1 |g'(x)|^2 \, \mathrm{d}x.$$

Deduce that $2\langle Tf, f\rangle_{L^2} \geq \|Tf\|_{L^2}^2$.

6. Prove that $\sigma(T) \subset [0,1]$.
7. For all $\lambda \in (0,1]$, we set $a_\lambda = \sqrt{(2-\lambda)/\lambda}$. Check that

$$\lambda \in \sigma(T) \cap (0,1] \Longleftrightarrow (1 - a_\lambda^2) \sin a_\lambda + 2a_\lambda \cos a_\lambda = 0.$$

8. Deduce that $\sigma(T) = \{0\} \cup \{\lambda_n : n \geq 0\}$, with

$$\frac{2}{1 + (\pi/2 + n\pi)^2} < \lambda_n < \frac{2}{1 + (n\pi)^2}.$$

Exercise 6. Prove that there is no distribution $T \in \mathscr{D}'(\mathbb{R})$ such that

$$T(\varphi) = \int_{\mathbb{R}} \exp\left(\frac{1}{x^2}\right) \varphi(x) \, \mathrm{d}x, \quad \varphi \in C_0^\infty(\mathbb{R} \setminus \{0\}).$$

Hint: Construct a sequence $(\varphi_n)_n$ *converging to zero in* $C_0^\infty(\mathbb{R})$ *such that each* φ_n *is supported in* $\{1/n \leq |x| \leq 2/n\}$ *and* $(T(\varphi_n))_n$ *converges to* $+\infty$.