TD 2: Locally convex topological vector spaces and Fréchet spaces

In the following, "locally convex topological vector space" will be abbreviated as l.c.t.v.s.

Exercise 1. Let E be a locally convex topological vector space whose topology is induced by a (separating) countable family of semi-norms $(p_n)_{n\in\mathbb{N}}$. We define

$$d(x,y)=\sum_{n=0}^{+\infty}\frac{1}{2^n}\frac{p_n(x-y)}{1+p_n(x-y)},\quad x,y\in E.$$

Let us prove that the topology induced by d and the topology induced by the family of seminorms $(p_n)_{n\in\mathbb{N}}$ coincide.

1. Show that $g:[0,\infty)\to\mathbb{R}$ defined by $g(t)=\frac{t}{1+t}$ is an increasing sub-additive function and give its image. Deduce that d is a translation invariant distance on E.
2. Give a basis of neighbourhoods of 0_E for the topology induced by the family of semi-norms, and show that every neighbourhood of 0_E contains an open ball for the distance d.
3. Show that every open ball for the distance d centered on 0_E contains a neighbourhood of 0_E for the topology induced by the family of semi-norms.
4. Conclude.

More generally, let us consider a continuous bounded function $g:[0,+\infty)\to\mathbb{R}_+$ and

$$d_g(x,y)=\sum_{n=0}^{+\infty}\frac{1}{2^n}g(p_n(x-y)),\quad x,y\in E$$

5. Under what condition on g does d_g defines a distance on E whose topology coincide with the one induced by the family of seminorms $(p_n)_{n\in\mathbb{N}}$?

Exercise 2. Let X and Y be l.c.t.v.s. We consider $(p_\alpha)_{\alpha\in A}$ (resp. $(q_\beta)_{\beta\in B}$) a countable family of continuous semi-norms which is separating and generates the topology of X (resp. of Y). Let $T:X\to Y$ be a linear map. Prove that T is continuous if and only if for all $\beta\in B$, there exists a finite set $I\subset A$ and a positive constant $c>0$ such that for all $u\in X$,

$$q_\beta(Tu)\le c\sum_{\alpha\in I}p_\alpha(u).$$

Exercise 3 (Space of continuous functions). Let U be an open subset of $\mathbb{R}^d$ and $(K_n)_n$ be an exhaustive sequence of compacts of U.

1. Prove that $C^0(U)$ is a Fréchet space for the distance

$$d(f,g)=\sum_{n=0}^{+\infty}\frac{1}{2^n}\min(1,p_n(f-g)),$$

defined by the semi norms $p_n(f)=\sup_{x\in K_n}|f(x)|$.

2. Recall that a subset $B \subset C^0(U)$ is said to be bounded if for any neighborhood V of 0, there exists $\lambda > 0$ such that $\lambda B \subset V$. Prove that if B is a subset of equibounded functions of $C^0(U)$, that is $\sup_{f\in B} \|f\|_\infty < \infty$, then B is bounded.
3. Let us consider $(f_n)_n$ a sequence of continuous function on U such that $f_n : U \to [0, n]$ with $f_n = 0$ on K_n and $f_n = n$ on $U \setminus K_{n+1}$. Show that $\cup_n\{f_n\}$ is a bounded subset of $C^0(U)$.
4. Prove that the space $C^0(\mathbb{R})$ is not locally bounded, that is, the origin does not have a bounded neighborhood.

Exercise 4 (Space of C^∞ functions). We consider the vector space $E = C^\infty([0, 1], \mathbb{R})$ equipped with the following metric

$$d(f, g) = \sum_{k\geq 0} \frac{1}{2^k} \min\left(1, \|f^{(k)} - g^{(k)}\|_\infty\right).$$

1. Check that E is a Fréchet space.
2. Prove that any closed and bounded (*cf* the previous exercise) subset of E is compact.
3. Can the topology of E be defined by a norm ?

Exercise 5 (L^p spaces with $0 < p < 1$). Let $p \in (0, 1)$ and L^p be the set of real-valued measurable functions u defined over $[0, 1]$, modulo almost everywhere vanishing functions, for which the following quantity is finite:

$$\|u\|_p = \left(\int_0^1 |u(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}.$$

1. Show that L^p is a vector space and that $d(u, v) = \|u - v\|_p^p$ is a distance. Prove that (L^p, d) is complete.
2. Let $f \in L^p$ and $n \geq 1$ be a positive integer. Prove that there exist some points $0 = x_0 < x_1 < \ldots < x_n = 1$ such that for all $i = 0, \ldots, n-1$,
$$\int_{x_i}^{x_{i+1}} |f|^p \,\mathrm{d}x = \frac{1}{n}\int_0^1 |f|^p \,\mathrm{d}x.$$
3. Prove that the only convex open domain in L^p containing $u \equiv 0$ is L^p itself. Deduce that the space L^p is not locally convex.
Hint: Introduce the functions $g_i^n = nf\mathbb{1}_{[x_i,x_{i+1}]}$.
4. *Bonus*: Show that the (topological) dual space of L^p reduces to $\{0\}$.