TD 3: Hahn-Banach theorems

Exercise 1 (Hahn-Banach Theorem without the axiom of choice).

1. Let (E, d_E) and (F, d_F) be metric spaces, (F, d_F) being complete, $D \subset E$ be a dense subset and $f : (D, d_E) \to (f, d_F)$ be a uniformly continuous function. Then, there exists a unique continuous function $F : (E, d_E) \to (F, d_F)$ such that $F_{|D} = f$. Moreover, prove that the function F is uniformly continuous.
2. Let E be a real separable Banach space and p be a continuous seminorm on E. Let M be a linear subspace of E and $\varphi : M \to \mathbb{R}$ be a linear functional which is dominated by p. Without using the axiom of choice, prove that φ can be extended to a linear functional $E \to \mathbb{R}$ which remains dominated by p.

Exercise 2 (Separation in Hilbert spaces without the Hahn-Banach theorem). In this exercise, the use of the axiom of choice is prohibited. Let H be an Hilbert space.

1. Let $C \subset H$ be a convex, closed and non-empty set. Prove that any $v \notin C$ can be strictly separated by C by a closed hyperplane, *i.e.* there exists $v_0 \in H$ such that
$$\forall u \in C, \quad \langle v_0, u\rangle < \langle v_0, v\rangle.$$
2. Let $C_1, C_2 \subset H$ be convex, closed and non-empty disjoint sets, C_1 being moreover compact. Prove that C_1 and C_2 can be strictly separated by a closed hyperplane, *i.e.* there exists $u_0 \in H$ such that
$$\sup_{u\in C_1} \langle u_0, u\rangle < \inf_{u\in C_2} \langle u_0, u\rangle.$$

Exercise 3 (First uses of the Hahn-Banach theorem). Let E be a normed vector space.

1. Let G be a vector subspace of E and $g : G \to \mathbb{R}$ be a continuous linear form. Recall why there exists a continuous linear form f over E that extends g, and such that
$$\|f\|_{E^*} = \|g\|_{G^*}.$$
When E is an Hilbert space, prove that this extension is unique.
2. Assume that $E = \ell^1(\mathbb{N})$. Give the example of a continuous linear form of norm 1, defined on a strict vector subspace of E, which admits an infinite number of linear continuous extensions of norm 1 over E.
3. Assume that E is a Banach space.
   (a) Prove that for all $x \in E$,
   $$\|x\| = \max_{f\in E^*:\|f\|_{E^*}\leq 1} |f(x)|.$$
   (b) Let B be a subset of E such that
   $$\forall f \in E^*, \quad \sup_{x\in B} f(x) < +\infty.$$
   Prove that B is bounded.

Exercise 4 (Convex sets that cannot be separated). Let H be the Hilbert space $L^2([-1,1])$. For every $\alpha \in \mathbb{R}$, let $C_\alpha \subset H$ be the subset of continuous functions $u : [-1,1] \to \mathbb{R}$ such that $u(0) = \alpha$. Prove that C_α is a convex dense subset of H. Deduce that, if $\alpha \neq \beta$, then C_α and C_β are convex disjoint subsets that cannot be separated by a continuous linear form.

Exercise 5 (Banach limit).

1. Let $s : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ be the shift operator, defined by $s(x)_i = x_{i+1}$ for all $i \in \mathbb{N}$ and $x \in \ell^\infty(\mathbb{N})$. Prove the existence of a continuous linear function $\Lambda \in (\ell^\infty(\mathbb{N}))'$ satisfying $\Lambda \circ s = \Lambda$ and
$$\forall u \in \ell^\infty(\mathbb{N}), \quad \liminf_{n\to+\infty} u_n \leq \Lambda(u) \leq \limsup_{n\to+\infty} u_n.$$
Such a linear form Λ is called Banach limit.
Hint: Consider the vector space of bounded sequences that converge in the sense of Cesàro.
2. Deduce that there exists a function $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}_+$ which satisfies
   - (i) $\mu(\mathbb{N}) = 1$,
   - (ii) μ is finitely additive: $\forall A, B \subset \mathbb{N}$ with $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$,
   - (iii) μ is left-invariant: $\forall k \in \mathbb{N}$ and $A \subset \mathbb{N}$, $\mu(k + A) = \mu(A)$.

Exercise 6 (Finite-dimensional case).

1. Let $C \subset \mathbb{R}^d$ be a convex set such that $C \neq \mathbb{R}^d$, and $x_0 \notin C$. Prove that there exists an affine hyperplane that separates C and $\{x_0\}$.
2. Does this result hold in an infinite dimensional space ?

Exercise 7 (Convex hull). Let E be a locally convex topological vector space (abbreviated l.c.t.v.s. in the following). One says that H is a closed half-space if there exists a $\varphi \in E^*$ and $a \in \mathbb{R}$ such that $H = \{u \in E \mid \varphi(u) \leq a\}$.

1. If C is a convex subset of E, show that its closure $\overline{C}$ is also convex.
2. Let A be a closed convex subset of E. Show that A is the intersection of the closed half-spaces containing A.
3. Deduce that $\overline{co(A)}$ is the intersection of the closed half-spaces containing A for any subset A of E, where $co(A)$ denotes the convex hull of the set A, that is, the smallest convex set that contains A.

Exercise 8 (Density criterion).

1. Let E be a real normed vector space and $F \subset E$ be a vector subspace such that $\overline{F} \neq E$. Prove that there exists $\varphi \in E' \setminus \{0\}$ such that $\varphi(u) = 0$ for all $u \in F$.
2. *Application:* Let $(a_n)_n$ be a sequence in $]1, +\infty[$ that diverges to $+\infty$. Prove that the set
$$W = \operatorname{span}\left\{x \in [0,1] \mapsto \frac{1}{x - a_n} : n \geq 0\right\},$$
is dense in the space $\mathcal{C}^0([0,1])$ equipped with the norm $\|\cdot\|_\infty$.
Hint: While considering a continuous linear form that vanishes on W, introduce a generating function.