École Normale Supérieure de Lyon **Année 2021 - 2022**

Département de Mathématiques **M1 - Analyse avancée**

TD 4: Weak topologies

Exercise 1.

1. Let E be a l.c.t.v.s whose topology is generated by a separating family of seminorms $(p_\alpha)_{\alpha\in I}$. Prove that a sequence $(x_n)_n$ of elements in E converges to some $x \in E$ if and only if for all $\alpha \in I$, the sequence $(p_\alpha(x - x_n))_n$ converges to 0.
2. Let E be a Banach space. By using the previous question, give a characterization of weakly converging sequences in terms of continuous linear forms.

Exercise 2. Let X be a normed vector space.

1. Let $(u_n)_n$ be a weakly convergent sequence in X. Justify that (u_n) is bounded and that the weak limit u of $(u_n)_n$ satisfies $\|u\| \leq \liminf_{n\to+\infty} \|u_n\|$.
2. Suppose that the sequence $(\varphi_n)_n$ in X^* is converging strongly to some $\varphi \in X^*$. Show that for any sequence $(u_n)_n$ in X that converges weakly to $u \in X$, then the sequence $(\varphi_n(u_n))_n$ converges to $\varphi(u)$.
3. Assume that X is a Hilbert space. Let $(u_n)_n$ be a sequence in X that converges weakly to $u \in X$ and such that $(\|u_n\|)_n$ converges to $\|u\|$. Prove that $(u_n)_n$ converges strongly to u.

Exercise 3. The purpose of this exercise is to present three obstructions to strong convergence in $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{T}^d)$. In the following, $\varphi \in C_c^\infty(\mathbb{R}^d)$ denotes a compactly supported smooth function being not identically equal to zero.

1. (Loss of mass) Let ν be a vector of norm 1. Prove that the sequence $(\varphi(\cdot - n\nu))_n$ converges weakly to zero in $L^2(\mathbb{R}^d)$, but not strongly.
2. (Concentration) Prove that the sequence $(n^{d/2}\varphi(n\,\cdot))_n$ converges weakly to zero in $L^2(\mathbb{R}^d)$, but not strongly.
3. (Oscillations) We now consider $w \in L^2(\mathbb{T}^d)$ a non-constant function. Prove that the sequence $(w(n\,\cdot))_n$ converges weakly but not strongly to $\frac{1}{2\pi}\int_0^{2\pi} w$ in $L^2(\mathbb{T}^d)$.

Exercise 4. Let E be a Banach space.

1. Show that if E is finite-dimensional, then the weak topology $\sigma(E, E^*)$ and the strong topology coincide.
2. We assume that E is infinite-dimensional.
   (a) Show that every weak open subset of E contains a straight line.
   (b) Deduce that $B = \{x \in E : \|x\| < 1\}$ has an empty interieur for the weak topology.
   (c) Let $S = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. What is the weak closure of S ?

Exercise 5. Let E be an infinite-dimensional Banach space. Prove that the weak topology on E is not metrizable.
Hint: Recall that any open weak set contains a line.

Exercise 6.

1. (Mazur's lemma) Let E be a Banach space and $(u_n)_n$ be a sequence in E weakly converging to $u_\infty \in E$. Show that u_∞ is a strong limit of finite convex combinations of the u_n.
2. (Banach-Sacks' property) Show that if E is in addition a Hilbert space, we can extract a subsequence converging to u_∞ strongly in the sens of Cesàro.

Exercise 7 (Schur's property for $\ell^1(\mathbb{N})$).

1. Recall why weak and strong topologies always differ in an infinite dimensional norm vector space.

The aim is to prove that a sequence of $\ell^1(\mathbb{N})$ converges weakly if and only if it converges strongly. Take $(u^n)_n$ a sequence in $\ell^1(\mathbb{N})$ weakly converging to 0.

2. Show that for all k, $\lim_{n\to\infty} u_k^n \to 0$.
3. Show that if $u_n \not\to 0$ in $\ell^1(\mathbb{N})$, one can additionally assume that $\|u^n\|_{\ell^1} = 1$.
4. Define via a recursive argument two increasing sequences of $\mathbb{N}$, $(a_k)_k$ and $(n_k)_k$, such that

$$\forall k \geq 0, \qquad \sum_{j=a_k}^{a_{k+1}-1} |u_j^{n_k}| \geq \frac{3}{4}.$$

5. Show that there exists $v \in \ell^\infty(\mathbb{N})$ such that $(v, u^{n_k})_{\ell^2} \geq \frac{1}{2}$ for all k. Conclude.