TD 5: Weak topologies (II)

Exercise 1. Let E and F be two Banach spaces, and $T : E \to F$ be a linear map. Show that T is strongly continuous (*i.e.* continuous from $(E, \|\cdot\|_E)$ to $(F, \|\cdot\|_F)$) if and only if T is weakly continuous (*i.e.* continuous from $(E, \sigma(E, E^*))$ to $(F, \sigma(F, F^*))$).

Exercise 2. Let E be a separable real normed vector space. Let $(u_n)_n$ be a dense sequence in $B_E(0, 1)$. By considering the following metric d on the unit ball of E^*,

$$d(f, g) = \sum_{n=0}^{+\infty} \frac{1}{2^n} |(f - g)(u_n)|, \quad f, g \in B_{E^*}(0, 1),$$

prove that the weak-$*$ topology on $B_{E^*}(0, 1)$ is metrizable.

Exercise 3 (Goldstine lemma). Let X be a Banach space. For any $x \in X$, let us define the evaluation $\mathrm{ev}_x : \varphi \in X^* \mapsto \varphi(x) \in \mathbb{R}$. We can therefore consider the following application

$$J : \begin{cases} X & \to & X^{**} \\ x & \mapsto & \mathrm{ev}_x \end{cases}$$

For any normed vector space E, we denote by B_E its closed unit ball.

1. Check that J is an isometry and that $J(X)$ is strongly closed in X^{**}.
2. Let E be a normed vector space. Determine all the linear forms on E^* which are continuous for the weak-$*$ topology $\sigma(E^*, E)$.
3. By using the Hahn-Banach theorem, prove that $J(B_X)$ is dense in $B_{X^{**}}$ for the weak-$*$ topology $\sigma(X^{**}, X^*)$.

Exercise 4.

1. In $\ell^\infty(\mathbb{N})$ we consider
$$C = \{x \in \ell^\infty(\mathbb{N}) : \liminf_n x_n \geq 0\}.$$
Show that C is strongly closed but not weakly-$*$ closed.
2. Let E be a normed vector space. Show that an hyperplane $H \subset E^*$ which is closed for the weak-$*$ topology $\sigma(E^*, E)$ is the kernel of $\mathrm{ev}_x : \varphi \mapsto \varphi(x)$ for some $x \in E$.

Exercise 5. Let $(E, \|\cdot\|)$ be a reflexive space and B_E be its unit ball. Show that for all $\varphi \in E^*$, there exists $x_\varphi \in B_E$, such that $\|\varphi\|_{E^*} = |\varphi(x_\varphi)|$, i.e. the supremum in the definition of the norm operator is in fact a maximum.

Exercise 6. The aim of this exercise is to prove by two different methods that the space $(C^0([0,1]), \|\cdot\|_\infty)$ of continuous real-valued functions on $[0,1]$ is not reflexive.

1. Method by compactness.
   (a) Define $\varphi \in C^0([0,1])^*$ by
   $$\varphi(f) = \int_0^{1/2} f(t)\,\mathrm{d}t - \int_{1/2}^1 f(t)\,\mathrm{d}t, \quad f \in C^0([0,1]),$$
   and show that $\|\varphi\| = 1$.
   (b) Prove that $|\varphi(f)| < 1$ for all $f \in C^0([0,1])$ such that $\|f\|_\infty \leq 1$.
   (c) Conclude that the space $C^0([0,1])$ is not reflexive.
2. Method by separability.
   (a) Prove that if E is a Banach space and its dual E^* is separable, then E is separable.
   (b) Show that $C([0,1])$ is separable.
   (c) Prove that $C([0,1])^*$ is not separable.
   Hint: Consider the functions $\delta_t : C([0,1]) \to \mathbb{R}$ defined by $\delta_t(f) = f(t)$ for any $t \in [0,1]$.
   (d) Conclude that $C([0,1])$ is not isomorphic to $C([0,1])^{**}$ as Banach spaces.
   Remark: This is stronger than not being reflexive.

Exercise 7.

1. Let E be a reflexive, separable Banach space. Let $(u_n)_n$ be a bounded sequence in E. Show that one can extract a subsequence $(u_{n'})_{n'}$ which converges weakly in E.
2. Does this result hold when E is not reflexive ?

Exercise 8. Let E be a reflexive Banach space and $I : E \to \mathbb{R}$ be a continuous, convex and coercive functional, in the sense that there exist $\alpha > 0$ and $M \geq 0$ such that for all $x \in E$,

$$I(x) \geq \alpha \|x\|_E - M.$$

We also consider $A \subset E$ a non-empty, closed and convex set. Prove that the functional I admits a minimum on A.

Exercise 9. Let B denote the closed unit ball of $L^1([0,1])$. Recall that a function $f \in B$ is called an extreme point if, whenever $f = \theta f_1 + (1-\theta) f_2$ with $\theta \in (0,1)$ and $f_1, f_2 \in B$, one has $f_1 = f_2$. Prove that B does not admit extremal points. Deduce that there is no isometry between $L^1([0,1])$ and the topological dual of a normed vector space.
Hint: We admit Krein-Milman's theorem, stating that any non-empty convex compact subset of any l.c.t.v.s coincides with the closed convex envelop of its extremal points.