TD 7: Compact operators

Exercise 1. Let H be a Hilbert space.

1. Prove that $\mathcal{K}(H)$ is closed in $\mathcal{L}(H)$.
2. Let $T \in \mathcal{K}(H)$ and $S \in \mathcal{L}(H)$. Prove that the operators TS and ST are also compact.

Exercise 2. Let H be a Hilbert space and $T : H \to H$ be linear and continuous.

1. Prove that the following assertions are equivalent
   - (i) T is compact.
   - (ii) For any sequence $(x_n)_n$ that weakly converges in H, the sequence $(Tx_n)_n$ strongly converges in H.
   - (iii) T is the limit in $\mathcal{L}(H)$ of finite rank operators.
2. We now assume that H is infinite-dimensional and that T is compact. Is the operator T right or left invertible ?
   Application: Study the compactness of the shift operator $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ defined for all $x = (x_n)_n \in l^2(\mathbb{N})$ by $(Tx)_0 = 0$ and $(Tx)_n = x_{n-1}$ for all $n \geq 1$.
3. When T is compact, prove that T^* is also a compact operator.

Exercise 3. Let $(\lambda_n)_n$ a sequence of complex numbers, and $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the operator defined by

$$T((x_n)_n) = (\lambda_n x_n)_n, \quad (x_n)_n \in l^2(\mathbb{N}).$$

1. Check that the operator T is well-defined and bounded on $l^2(\mathbb{N})$ if and only if the sequence $(\lambda_n)_n$ is bounded.
2. Prove that the operator T is compact if and only if the sequence $(\lambda_n)_n$ converges to 0.

Exercise 4. Let $T : L^2(0,1) \to L^2(0,1)$ be the Volterra operator defined by

$$(Tf)(x) = \int_0^x f(y)\,\mathrm{d}y, \quad f \in L^2(0,1),\ x \in [0,1].$$

1. Check that $Tf \in C^0[0,1]$ for all $f \in L^2(0,1)$.
2. Prove that the operator T is compact.
3. Compute the adjoint of the operator T.
4. Deduce that the operator TT^* is the following
$$(TT^*f)(x) = \int_0^1 \min(x,y) f(y)\,\mathrm{d}y, \quad f \in L^2(0,1),\ x \in [0,1].$$
5. Justify that the operator TT^* is compact, selfadjoint and non-negative.
6. Prove that the set of eigenvalues of the operator T^*T is given by
$$\left\{ \frac{1}{(n\pi + \pi/2)^2} : n \geq 0 \right\}.$$
7. Deduce the value of the norm of the operator T.