TD 8: Distributions

Exercise 1. Let $f : \mathbb{R} \to \mathbb{R}_+$ be a non-negative C^∞ function. Assume that the function f'' is bounded.

1. Prove that the following pointwize estimation holds

$$\forall x \in \mathbb{R}, \quad |f'(x)|^2 \leq 2f(x)\|f''\|_{L^\infty(\mathbb{R})}.$$

2. Can we have an estimate of the following form

$$\forall x \in \mathbb{R}, \quad |f'(x)| \leq c_f f(x),$$

where the positive constant $c_f > 0$ depends on the function f ?

Exercise 2.

1. Let H be the Heaviside function. Show that $H' = \delta_0$ in $\mathcal{D}'(\mathbb{R})$.
2. Give an example of distribution of order n for all $n \in \mathbb{N}$.
3. Let $\Omega \subset \mathbb{R}^d$ be an open set and $T \in \mathcal{D}'(\Omega)$. We consider $f \in C^\infty(\Omega)$ which vanishes on the support of T. Do we have $fT = 0$ in $\mathcal{D}'(\Omega)$?

Exercise 3. Let $\Omega \subset \mathbb{R}^d$ be an open set. Prove that we have an injection of $L^1_{loc}(\Omega)$ in $\mathcal{D}'(\Omega)$.

Exercise 4 (An example of distribution). Show that the formula

$$\langle T, \varphi \rangle = \sum_{n \geq 0} \varphi^{(n)}(n), \quad \varphi \in \mathcal{D}(\mathbb{R}),$$

defines a distribution $T \in \mathcal{D}'(\mathbb{R})$. What about its order ?

Exercise 5 (Convergence of distributions). Do the following series

$$\sum_{n \geq 0} \delta_n^{(n)} \quad \text{and} \quad \sum_{n \geq 0} \delta_0^{(n)},$$

converge in $\mathcal{D}'(\mathbb{R})$?

Exercise 6 (Principal value of $1/x$). We define $\mathrm{p.v.}(1/x)$ as follows

$$\langle \mathrm{p.v.}(1/x), \varphi \rangle = \lim_{\varepsilon \to 0} \left(\int_{|x| > \varepsilon} \frac{\varphi(x)}{x} \, \mathrm{d}x \right), \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

1. Show that the above limit exists and defines a distribution. Compute its order.
2. Show that $\mathrm{p.v.}(1/x)$ is the derivative of $\log|x|$ in the sense of distributions.
3. Compute $x \, \mathrm{p.v.}(1/x)$.

4. Let $T \in \mathcal{D}'(\mathbb{R})$ which satisfies $xT = 1$. Show that there exists a constant $c \in \mathbb{R}$ such that $T = \mathrm{p.v.}(1/x) + c\,\delta_0$.
5. Show that $|x|^{\alpha-2}x \to \mathrm{p.v.}(1/x)$ in $\mathcal{D}'(\mathbb{R})$ as $\alpha \to 0^+$.

Exercise 7. Solve the equation $T' = 0$ in $\mathcal{D}'(\mathbb{R})$.

Exercise 8 (Jump formula). Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 on $\mathbb{R}^*$. We say that f has a jump at 0 if the limits $f(0^\pm) = \lim_{x\to 0^\pm} f(x)$ exist, and we denote by $[[f(0)]] = f(0^+) - f(0^-)$ the height of the jump. We denote by $\{f'\}$ the derivative of the regular part of f, *i.e.*

$$\{f'\}(x) = \begin{cases} f'(x) & \text{if } f \text{ is differentiable at } x \\ 0 & \text{otherwise} \end{cases}$$

1. Show that in the sense of distributions:

$$f' = \{f'\} + [[f(0)]]\delta_0.$$

2. Let $(x_n)_{n\in\mathbb{Z}}$ be an increasing sequence such that $\lim_{n\to-\infty} x_n = -\infty$ and $\lim_{n\to+\infty} x_n = +\infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise C^1 function presenting jumps at every x_n. Show that in the sense of distributions,

$$f' = \{f'\} + \sum_{n\in\mathbb{Z}} [[f(x_n)]]\delta_{x_n}.$$

Exercise 9 (Punctual support). Let $T \in \mathcal{D}'(\mathbb{R}^d)$ such that $\operatorname{supp} T = \{0\}$. We consider $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that $\psi = 1$ in a neighborhood of $\overline{B(0,1)}$ and $\operatorname{supp}\psi \subset B(0,2)$. We set $\psi_r(x) = \psi(x/r)$ for all $r > 0$ and $x \in \mathbb{R}^n$.

1. Recall why T has a finite order, which will be denoted $m \geq 0$ in the following.
2. Show that for all $r > 0$, $\psi_r T = T$.
3. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ satisfying that for all $p \in \mathbb{N}^n$ with $|p| \leq m$, $\partial^p\varphi(0) = 0$. Check that $\langle T, \varphi\rangle = 0$.
4. Prove that there exist some real numbers $a_p \in \mathbb{R}$ such that $T = \sum_{|p|\leq m} a_p \delta_0^{(p)}$.

Exercise 10 (Support and order). Let T be the linear map defined for all $\varphi \in \mathcal{D}(\mathbb{R})$ by

$$\langle T, \varphi\rangle = \lim_{n\to+\infty} \left(\sum_{j=1}^{n} \varphi\left(\frac{1}{j}\right) - n\varphi(0) - (\log n)\varphi'(0) \right).$$

1. Check that $\langle T, \varphi\rangle$ is well defined for all $\varphi \in \mathcal{D}(\mathbb{R})$, and that T is a distribution of order less than or equal to 2.
2. What is the support S of T ?
3. What is the order of T ?
   Hint: Use test functions of the form

$$\varphi_k(x) = \psi(x) \int_0^x \int_0^y \varphi(kt)\,\mathrm{d}t\mathrm{d}y,$$

where $\varphi \in \mathcal{D}(0,1)$ *has integral* 1 *and* $\psi \in \mathcal{D}(-1,2)$ *satisfies* $0 \leq \psi \leq 1$ *and* $\psi = 1$ *on* $[0,1]$.