École Normale Supérieure de Lyon **Année 2021 - 2022**

Département de Mathématiques **M1 - Analyse avancée**

TD 12: Sobolev spaces and PDEs

Exercise 1 (Agmon's and Brezis-Gallouët's type inequalities).

1. Prove that there exists a positive constant $c > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^3)$,

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq c\,\|u\|_{H^1(\mathbb{R}^3)}^{1/2}\|u\|_{H^2(\mathbb{R}^3)}^{1/2}.$$

Hint: Setting $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ *and considering* $R > 0$, *use the following decomposition*

$$\|\widehat{u}\|_{L^1(\mathbb{R}^3)} = \int_{|\xi|\leq R} \langle\xi\rangle|\widehat{u}(\xi)|\frac{\mathrm{d}\xi}{\langle\xi\rangle} + \int_{|\xi|>R} \langle\xi\rangle^2|\widehat{u}(\xi)|\frac{\mathrm{d}\xi}{\langle\xi\rangle^2}.$$

2. Show similarly that there exists a positive constant $c > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^2)$,

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq c\Big(1 + \|u\|_{H^1(\mathbb{R}^2)}\sqrt{\log(1+\|u\|_{H^2(\mathbb{R}^2)})}\Big).$$

Exercise 2. Let $U = (0,1)$.

1. Prove that the following continuous embeddings hold

$$W^{1,1}(U) \hookrightarrow C^0(\bar{U}) \quad \text{and} \quad W^{1,p}(U) \hookrightarrow C^{0,1-1/p}(\bar{U}) \quad \text{when } p \in (1,\infty],$$

with the convention $1/\infty = 0$.

2. Prove that for all $1 \leq p < \infty$, the space $W_0^{1,p}(U)$ is given by

$$W_0^{1,p}(U) = \big\{u \in W^{1,p}(U) : u(0) = u(1) = 0\big\}.$$

Exercise 3 (Poincaré's inequality). Let $p \in [1,+\infty)$ and let U be an open subset of $\mathbb{R}^d$.

1. Assume that U is bounded in one direction, meaning that U is contained in the region between two parallel hyperplanes. Prove Poincaré's inequality: there exists $c > 0$ such that for every $f \in W_0^{1,p}(U)$,

$$\|f\|_{L^p(U)} \leq c\|\nabla f\|_{L^p(U)}.$$

As a consequence, $\|\nabla \cdot\|_{L^p(U)}$ defines a norm on $W_0^{1,p}(U)$ which is equivalent to $\|\cdot\|_{W^{1,p}(U)}$.
Hint: Consider first the case $U \subset \mathbb{R}^{d-1}\times[-M,M]$.

2. Assume that U is bounded. Prove Poincaré-Wirtinger's inequality: there exists a constant $c > 0$ such that for any $f \in W^{1,p}(U)$ satisfying $\int_U f = 0$,

$$\|f\|_{L^p(U)} \leq c\|\nabla f\|_{L^p(U)}.$$

Exercise 4 (Duality). Let U be an open subset of $\mathbb{R}^d$ and let $p \in (1, +\infty)$.

1. Prove that for all $F \in W_0^{1,p}(U)'$, there exist $f_0, f_1, \dots, f_d \in L^q(U)$ (with $\frac{1}{p} + \frac{1}{q} = 1$) such that for all $g \in W_0^{1,p}(U)$,
$$\langle F, g \rangle_{W_0^{1,p}(U)', W_0^{1,p}(U)} = \int_U f_0 g \, \mathrm{d}x + \sum_{i=1}^d \int_U f_i \partial_i g \, \mathrm{d}x.$$
2. Prove that we also have
$$\|F\|_{W_0^{1,p}(U)'} \leq \left(\sum_{i=0}^d \|f_i\|_{L^q(U)}^q \right)^{\frac{1}{q}}.$$
3. Assuming that U is bounded, prove that we may take $f_0 = 0$.

Exercise 5 (A minimization problem). Let $U \subset \mathbb{R}^3$ be open, bounded with smooth boundary. The purpose is to prove that the following elliptic problem has a non-trivial weak solution
$$\begin{cases} -\Delta u = u^3 & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

1. Prove that there exists a solution to the following minimization problem
$$\inf \left\{ \|\nabla v\|_{L^2(U)} : v \in H_0^1(U), \ \|v\|_{L^4(U)} = 1 \right\}. \tag{1}$$
Hint: Since $d = 3$ here, the continuous embedding $H_0^1(U) \hookrightarrow L^q(U)$ holds for all $1 \leq q \leq 6$, and is moreover compact when $1 \leq q < 6$. Moreover, $\|\nabla \cdot \|_{L^2(U)}$ defines a norm on $H_0^1(U)$ which is equivalent to $\| \cdot \|_{W^1(U)}$ as a consequence of Poincaré's inequality, which is proven in Exercise 3.
2. Check that if the function $v \in H_0^1(U)$ solves (1), there exists a positive constant $\lambda > 0$ such that $-\Delta v = \lambda v^3$ in $\mathcal{D}'(U)$.
3. Conclude.