École Normale Supérieure de Lyon **Année 2021 - 2022**

Département de Mathématiques **M1 - Analyse avancée**

TD 3: Hahn-Banach theorem and locally convex topological vector spaces

Exercise 1 (Towards duality). Let E be a normed vector space.

1. Let G be a vector subspace of E and $g : G \to \mathbb{R}$ be a continuous linear form. Show that there exists a continuous linear form f over E that extends g, and such that
$$\|f\|_{E^*} = \|g\|_{G^*}.$$
When E is an Hilbert space, prove that this extension is unique.
2. Assume that $E = \ell^1(\mathbb{N})$. Give the example of a continuous linear form of norm 1, defined on a strict vector subspace of E, which admits an infinite number of linear continuous extensions of norm 1 over E.
3. Assume that E is a Banach space. Let B be a subset of E such that
$$\forall f \in E^*, \quad \sup_{x \in B} f(x) < +\infty.$$
Prove that B is bounded.

Exercise 2 (Hahn-Banach theorems for complex spaces). Let E be a vector space over $\mathbb{C}$. Let M be a vector subspace of E and let $f : M \to \mathbb{C}$ be a $\mathbb{C}$-linear form. Suppose that there is a semi-norm $p : E \to [0, \infty)$ such that
$$\forall x \in M, \quad |f(x)| \leq p(x).$$
Prove that there there exists a linear form $F : E \to \mathbb{C}$ extending f, and such that $|F| \leq p$.

Exercise 3 (Hahn-Banach Theorem without the axiom of choice.). Let E be a real separable Banach space and p be a norm on E. Let M be a linear subspace of E and $\varphi : M \to \mathbb{R}$ be a linear functional which is dominated by p. Prove that φ can be extended to a linear functional $E \to \mathbb{R}$ which remains dominated by p.

Exercise 4 (Separation of convex sets in Hilbert spaces). Let H be an Hilbert space.

1. Let $C \subset H$ be a convex, closed and non-empty set. Prove that any $v \notin C$ can be strictly separated by C by a closed hyperplane, *i.e.* there exists $u_0 \in H$ such that
$$\forall u \in C, \quad \langle u_0, u\rangle < \langle u_0, v\rangle.$$
2. Let $C_1, C_2 \subset H$ be convex, closed and non-empty disjoint sets, C_1 being moreover compact. Prove that C_1 and C_2 can be strictly separated by a closed hyperplane, *i.e.* there exists $u_0 \in H$ such that
$$\sup_{u \in C_1} \langle u_0, u\rangle < \inf_{u \in C_2} \langle u_0, u\rangle.$$

Exercise 5 (Convex sets that cannot be separated). Let H be the Hilbert space $L^2([-1,1])$. For every $\alpha \in \mathbb{R}$, let $C_\alpha \subset H$ be the subset of continuous functions $u : [-1,1] \to \mathbb{R}$ such that $u(0) = \alpha$. Prove that C_α is a convex dense subset of H. Deduce that, if $\alpha \neq \beta$, then C_α and C_β are convex disjoint subsets that cannot be separated by a continuous linear form.

Exercise 6 (Banach limit).

1. Let $s : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ be the shift operator, defined by $s(x)_i = x_{i+1}$ for all $i \in \mathbb{N}$ and $x \in \ell^\infty(\mathbb{N})$. Prove the existence of a continuous linear function $\Lambda \in (\ell^\infty(\mathbb{N}))'$ satisfying $\Lambda \circ s = \Lambda$ and
$$\forall u \in \ell^\infty(\mathbb{N}), \quad \liminf_{n\to+\infty} u_n \leq \Lambda(u) \leq \limsup_{n\to+\infty} u_n.$$
Such a linear form Λ is called Banach limit.
Hint: Consider the vector space of bounded sequences that converge in the sense of Cesàro.
2. Deduce that there exists a function $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}_+$ which satisfies
   - (i) $\mu(\mathbb{N}) = 1$,
   - (ii) μ is finitely additive: $\forall A, B \subset \mathbb{N}$ with $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$,
   - (iii) μ is left-invariant: $\forall k \in \mathbb{N}$ and $A \subset \mathbb{N}$, $\mu(k + A) = \mu(A)$.

Exercise 7 (L^p spaces with $0 < p < 1$). Let $p \in (0,1)$ and L^p be the set of real-valued measurable functions u defined over $[0,1]$, modulo almost everywhere vanishing functions, for which the following quantity is finite:
$$\|u\|_p = \left(\int_0^1 |u|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

1. Show that L^p is a vector space and that $d(u,v) = \|u - v\|_p^p$ is a distance. Prove that (L^p, d) is complete.
2. Let $f \in L^p$ and $n \geq 1$ be a positive integer. Prove that there exist some points $0 = x_0 < x_1 < \ldots < x_n = 1$ such that for all $i = 0, \ldots, n-1$,
$$\int_{x_i}^{x_{i+1}} |f|^p \, \mathrm{d}x = \frac{1}{n} \int_0^1 |f|^p \, \mathrm{d}x.$$
3. Prove that the only convex open domain in L^p containing $u \equiv 0$ is L^p itself. Deduce that the space L^p is not locally convex.
Hint: Introduce the functions $g_i^n = n f \mathbb{1}_{[x_i, x_{i+1}]}$.
4. Show that the (topological) dual space of L^p reduces to $\{0\}$.