**École Normale Supérieure de Lyon** **Année 2021 - 2022**
**Département de Mathématiques** **M1 - Analyse avancée**

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# TD 4: Geometric Hahn-Banach theorem and Fréchet spaces

**Exercise** 1 (Finite-dimensional case). Let $C \subset \mathbb{R}^d$ be a convex set such that $C \neq \mathbb{R}^d$, and $x_0 \notin C$. Prove that there exists an affine hyperplane that separates $C$ and $\{x_0\}$.

**Exercise** 2 (Convex hull). Let $E$ be a locally convex topological vector space (abbreviated l.c.t.v.s. in the following). One says that $H$ is a closed half-space if there exists a $\varphi \in E^*$ and $a \in \mathbb{R}$ such that $H = \{u \in E \mid \varphi(u) \leq a\}$.

1. If $C$ is a convex subset of $E$, show that its closure $\overline{C}$ is also convex.
2. Let $A$ be a closed convex subset of $E$. Show that $A$ is the intersection of the closed half-spaces containing $A$.
3. Deduce that $\overline{co(A)}$ is the intersection of the closed half-spaces containing $A$ for any subset $A$ of $E$, where $co(A)$ denotes the convex hull of the set $A$, that is, the smallest convex set that contains $A$.

**Exercise** 3 (Density criterion).

1. Let $E$ be a real normed vector space and $F \subset E$ be a vector subset such that $\overline{F} \neq E$. Prove that there exists $\varphi \in E' \setminus \{0\}$ such that $\varphi(u) = 0$ for all $u \in F$.
2. *Application:* Let $(a_n)_n$ be a sequence in $]1, +\infty[$ that diverges to $+\infty$. Prove that the set

$$W = \text{vect}\left\{x \in [0,1] \mapsto \frac{1}{x - a_n} : n \geq 0\right\},$$

is dense in the space $\mathcal{C}^0([0,1])$ equipped with the norm $\|\cdot\|_\infty$.
*Hint: While considering a continuous linear form that vanishes on $W$, introduce a generating function.*

**Exercise** 4 (Extreme points). Let $K$ be a subset of a vector space $E$. A point $a \in K$ is called an *extremal point* of $K$ if, whenever $a = \theta b + (1-\theta)c$ with $\theta \in (0,1)$ and $b, c \in K$, one has $b = c$. A subset[1] $S$ of $K$ is called an *extremal subset* of $K$ if, for all $a$ in $S$ such that $a = \theta b + (1-\theta)c$ with $\theta \in (0,1)$ and $b, c \in K$, one has $b \in S$ and $c \in S$.

1. In a Hilbert space, what are the extremal points of the unit closed ball ? What about the open ball ?
2. Let $c_0$ denote the space of real sequences $(a_n)_{n\in\mathbb{N}}$ converging to zero. We endow $c_0$ with the norm $\|\cdot\|_\infty$. Show that the closed unit ball of $c_0$ does not admit extremal points.
3. Let $I \subset \mathbb{R}$ be an interval. Show that the closed unit ball of $L^1(I)$ does not admit extremal points.

**Exercise** 5 (Krein-Milman theorem). The aim of this exercise is to prove the following statement.

**Theorem 1** (Krein-Milman)**.** *Let $E$ be a l.c.t.v.s. and $K$ be a non-empty convex compact subset of $E$. Then $K$ coincides with the closed convex envelop of its extremal points.*

[1] This notion is only used in Exercice 5

1. The first step is to show the existence of an extremal point in $K$. Let $\mathcal{P}$ be the set of non-empty closed extremal subsets of $K$, endowed with the order "$A \prec B$ if and only if $B \subset A$". Show that $\mathcal{P}$ admits a maximal element which is reduced to a point.
   *Hint: If a maximal element $S$ is composed of more than one point, choose a continuous linear form separating points of $S$ and consider the set of points reaching the maximum of this form on $S$.*
2. Define $\tilde{K} = \overline{co}(ext(K))$ the closed convex hull of the extremal points of $K$, and show that $\tilde{K}$ and $K$ coincide.
3. *Application:* An $n \times n$ matrix with real entries is bi-stochastic if its entries are non-negative, and the sum of the entries of either rows or columns equals 1. One denotes $SM_n(\mathbb{R})$ the set of bistochastic matrices. Show that every matrix in $SM_n(\mathbb{R})$ is actually a convex combination of permutation matrices.

**Exercise** 6. Let $X$ and $Y$ be l.c.t.v.s. We consider $(p_\alpha)_{\alpha\in A}$ (resp. $(q_\beta)_{\beta\in B}$) a countable family of continuous semi-norms which is separating and generates the topology of $X$ (resp. of $Y$). Let $T : X \to Y$ be a linear map. Prove that $T$ is continuous if and only if for all $\beta \in B$, there exists a finite set $I \subset A$ and a positive constant $c > 0$ such that for all $u \in X$,

$$q_\beta(Tu) \leq c \sum_{\alpha\in I} p_\alpha(u).$$

**Exercise** 7 (Space of continuous functions). Let $U$ be an open subset of $\mathbb{R}^d$ and $(K_n)_n$ be an exhaustive sequence of compacts of $U$.

1. Prove that $C^0(U)$ is a Fréchet space for the distance
$$d(f, g) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \min(1, p_n(f - g)),$$
   defined by the semi norms $p_n(f) = \sup_{x\in K_n} |f(x)|$.
2. A subset $B \subset C^0(U)$ is said to be bounded if for any neighborhood $V$ of 0, there exists $\lambda > 0$ such that $\lambda B \subset V$. Prove that if $B$ is a subset of equibounded functions of $C^0(U)$, that is $\sup_{f\in B} \|f\|_\infty < \infty$, then $B$ is bounded.
3. Let us consider $(f_n)_n$ a sequence of continuous function on $U$ such that $f_n : U \to [0, n]$ with $f_n = 0$ on $K_n$ and $f_n = n$ on $U \setminus K_{n+1}$. Show that $\cup_n\{f_n\}$ is a bounded subset of $C^0(U)$.
4. Prove that the space $C^0(\mathbb{R})$ is not locally bounded, that is, the origin does not have a bounded neighborhood.

**Exercise** 8 (Space of $C^\infty$ functions). We consider the $E = C^\infty([0, 1], \mathbb{R})$ equipped with the following metric

$$d(f, g) = \sum_{k\geq 0} \frac{1}{2^k} \min\left(1, \|f^{(k)} - g^{(k)}\|_\infty\right).$$

1. Check that $E$ is a Fréchet space.
2. Prove that any closed and bounded (*cf* the previous exercise) subset of $E$ is compact.
3. Can the topology of $E$ be defined by a norm ?