
TD 6: Weak-$*$ topology

Exercise 1. (Warm-up exercise) Let E and F be two Banach spaces, and $T : E \to F$ be a linear map. Show that T is strongly continuous (*i.e.* continuous from $(E, \|\cdot\|_E)$ to $(F, \|\cdot\|_F)$) if and only if T is weakly continuous (*i.e.* continuous from $(E, \sigma(E, E^*))$ to $(F, \sigma(F, F^*))$).

Exercise 2 (Weak-$*$ topology and metrics). Let E be a separable real normed vector space. Let $(u_n)_n$ be a dense sequence in $B_E(0,1)$. By considering the following metric d on the unit ball of E^*,

$$d(f,g) = \sum_{n=0}^{+\infty} \frac{1}{2^n} |(f-g)(u_n)|, \quad f, g \in B_{E^*}(0,1),$$

prove that the weak-$*$ topology on $B_{E^*}(0,1)$ is metrizable.

Exercise 3 (Weak-$*$ closed hyperplanes).

1. In $\ell^\infty(\mathbb{N})$ we consider
$$C = \left\{ u \in \ell^\infty(\mathbb{N}) : \liminf_n u_n \geq 0 \right\}.$$
Show that C is strongly closed but not weakly-$*$ closed.

Let us now consider E a normed vector space.

2. Let $\varphi : E^* \to \mathbb{R}$ a linear form continuous for the $\sigma(E^*, E)$ topology. Show that:
$$\exists u \in E, \forall \ell \in E^*, \quad \varphi(\ell) = \ell(u).$$
3. Show that an hyperplane $H \subset E^*$ which is closed for the weak-$*$ topology is the kernel of $\mathrm{ev}_u : \varphi \mapsto \varphi(u)$ for some $u \in E$.

Exercise 4 (Eberlein-Šmulian's theorem). The aim of the exercice is to prove the following result:

> Let A a subset of a Banach space E. If A is relatively compact for the weak topology, then A is sequentially relatively compact (still for the weak topology of E).

1. Recall why the result is direct if E^* is separable.
2. Let $(a_n)_n$ be a sequence in A. We denote $F := \overline{\mathrm{vect}\{a_n : n \in \mathbb{N}\}}$. Show that there exists a sequence of linear continuous form $(\phi_n)_n$ such that for any $u \in F$,
$$\|u\| = \sup_n |\phi_n(u)|.$$
Show that $(F, \sigma(F, F^*))$ is metrisable on any weak compact of F.
3. Conclude.
4. Show that the result is wrong for the weak-$*$ topology.
Hint: Work in the space $\ell^\infty(\mathbb{N})^*$.

Remark: the converse implication is also true.

Exercise 5 (Dunford-Pettis' Theorem). The objective of the exercise is to prove Dunford-Pettis' theorem:

> Let $\Omega \subset \mathbb{R}^d$ be a bounded set and $(f_n)_n$ be a bounded sequence in $L^1(\Omega)$. Then, the set $\{f_n\}$ is sequentially compact for the weak topology $\sigma(L^1, L^\infty)$ if and only if the sequence $(f_n)_n$ is equi-integrable.

1. Recall the definition of equi-integrability.

First we prove the reciprocal: let $(f_n)_n$ be a bounded and equi-integrable sequence in L^1.

2. Show that we can reduce to the case where the f_n are non-negative.
3. Let $f_n^k = \mathbf{1}_{f_n \leq k} f_n$. Show that $\sup_n \|f_n - f_n^k\|_{L^1} \to 0$.
4. Show that there exists an extraction (n') such that for all $k \in \mathbb{N}$, $f_{n'}^k \rightharpoonup f^k$ in L^1.
5. Prove that $(f^k)_k$ is an increasing sequence and deduce that there exists some $f \in L^1$ such that $f^k \to f$ in L^1.
6. Conclude that $f_{n'} \rightharpoonup f$ in L^1.

Now we want to prove the direct implication. Let $(f_n)_n$ be a bounded sequence in $L^1(\Omega)$ satisfying $f_n \rightharpoonup f \in L^1(\Omega)$. We consider $\mathcal{X}$ the set of indicator functions and, for a fixed $\varepsilon > 0$, we also consider the sets X_n defined for all $n \geq 0$ by:

$$X_n := \left\{ \mathbf{1}_A \in \mathcal{X} : \forall k \geq n, \ \left| \int_A (f_k - f) \,\mathrm{d}x \right| \leq \varepsilon \right\}.$$

7. Show that $\mathcal{X}$ and X_n are closed in $L^1(\Omega)$.
8. Using a Baire's argument, show that $(f_n)_n$ is equi-integrable.
9. Conclude.

Exercise 6 (Egorov's theorem).

1. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and $(g_n)_n$ be a sequence of measurable functions such that $(g_n)_n$ converge a.e. to some measurable function g. Show that for all $\varepsilon > 0$, there exists a measurable set $E_\varepsilon \subset \Omega$ such that $\mu(E_\varepsilon^c) < \varepsilon$ and $(g_n)_n$ converges uniformly in E_ε.
2. Let $(f_n)_n$ be a sequence in $L^1(\Omega)$ with $f_n \rightharpoonup f \in L^1(\Omega)$, and $(g_n)_n$ be a bounded sequence in $L^\infty(\Omega)$ satisfying $g_n \to g$ a.e. Show that $f_n g_n \rightharpoonup fg$ in $L^1(\Omega)$.
   Hint: Use Dunford-Pettis' theorem.

Exercise 7 (L^1 is not a dual space). Show that the closed unit ball of $L^1([0,1])$ does not admit extremal points. Deduce that $L^1([0,1])$ is not the dual space of a normed vector space.
Hint: Use Krein-Milman's theorem.