TD 7: Reflexivity

Exercise 1. Let $(E, \|\cdot\|)$ be a reflexive space and B_E be its unit ball. Show that for all $f \in E^*$, there exists $x_f \in B_E$, such that $\|f\|_{E^*} = |f(x_f)|$, i.e. the supremum in the definition of the norm operator is in fact a maximum.

Exercise 2. The aim of this exercise is to prove by two different methods that the space $(C^0([0,1]), \|\cdot\|_\infty)$ of continuous real-valued functions on $[0,1]$ is not reflexive.

1. Method by compactness.
   (a) Define $\varphi \in C([0,1])^*$ by
   $$\varphi(f) = \int_0^{\frac{1}{2}} f(t)\,\mathrm{d}t - \int_{\frac{1}{2}}^1 f(t)\,\mathrm{d}t, \quad f \in C^0([0,1]),$$
   and show that $\|\varphi\| = 1$.
   (b) Prove that $|\varphi(f)| < 1$ for all $f \in C^0([0,1])$ such that $\|f\|_\infty \leq 1$.
   (c) Conclude that the space $C^0([0,1])$ is not reflexive.
2. Method by separability.
   (a) Prove that if E is a Banach space and its dual E^* is separable, then E is separable.
   (b) Show that $C([0,1])$ is separable.
   (c) Prove that $C([0,1])^*$ is not separable.
   Hint: Consider the functions $\delta_t : C([0,1]) \to \mathbb{R}$ defined by $\delta_t(f) = f(t)$ for any $t \in [0,1]$.
   (d) Conclude that $C([0,1])$ is not isomorphic to $C([0,1])^{**}$ as Banach spaces.
   Remark: This is stronger than not being reflexive.

Exercise 3.

1. Let E be a reflexive, separable Banach space. Let $(u_n)_n$ be a bounded sequence in E. Show that one can extract a subsequence $(u_{n'})_{n'}$ which converges weakly in E.
   Remark: the condition "separable" is not necessary thanks to exercise 5.
2. Does this result hold when E is not reflexive ?

Exercise 4. Let E be a normed vector space. Show that any weakly compact set of E is bounded for the norm.

Exercise 5 (Eberlein-Šmulian's theorem). The aim of the exercise is to prove the following result:

> Let A be a subset of a normed vector space E. If A is weakly compact, then A is weakly sequentially compact.

1. Assume that E^* is separable. Recall the key argument that gives the result.

Let $(a_n)_n$ be a sequence in A. We set $F := \overline{\mathrm{vect}\{a_n : n \in \mathbb{N}\}}$ and set $\tilde{A} := A \cap F$.

2. Show that $\tilde{A}$ is weakly compact in F.
3. Show that the unit ball of F^* admits a countable subset $\{\phi_k : k \in \mathbb{N}\}$ such that

$$\forall x \in F, \quad \|x\| = \sup_k |\phi_k(x)|.$$

In the following, we denote by σ the weak topology on $\tilde{A}$ and by τ the topology generated by the semi-norms $|\phi_k|$, $k \in \mathbb{N}$.

4. Show that that $(\tilde{A}, \tau)$ is Hausdorff and that the identity map $\mathrm{Id}_{\sigma,\tau} : (\tilde{A}, \sigma) \to (\tilde{A}, \tau)$ is continuous.
5. Deduce that $(\tilde{A}, \tau)$ is compact and that $\mathrm{Id}_{\sigma,\tau}$ is an homeomorphism.
   Hint: show that the image of a closed set by $Id_{\sigma,\tau}$ is closed.
6. Show that $(\tilde{A}, \sigma(F, F^*))$ is metrizable.
7. Show that one can extract a subsequence $(a_{n_k})_k$ converging weakly in F (to some limit a), and that $(a_{n_k})_k$ converges also weakly to a in E.
8. Show that the result is wrong for the weak-$*$ topology.
   Hint: consider the dual of $\ell^\infty(\mathbb{N})$.