TD 9: Convolution of distributions

Exercise 1 (Examples of convolutions). Compute the following convolutions:

1. $\delta_a * \delta_b$ in $\mathbb{R}^d$,
2. $T * \delta_a$, with $T \in \mathcal{D}'(\mathbb{R}^d)$,
3. $(x^p \delta_0^{(q)}) * (x^m \delta_0^{(n)})$,
4. $\delta_0^{(k)} * (x^m H)$,
5. $\mathbb{1}_{[a,b]} * \mathbb{1}_{[c,d]}$,
6. $\mathbb{1}_{[0,1]} * (xH)$.

Exercise 2 (Associativity and convolution). Show that the convolution product is not associative without assumptions on the supports by considering the distributions 1, δ_0' and H in $\mathcal{D}'(\mathbb{R})$, where H is the Heaviside function.

Exercise 3. We will study the behavior of the convolution of distributions with respect to the convolution product.

1. Let $T \in \mathcal{D}'(\mathbb{R}^d)$ be compactly supported, $V \in \mathcal{D}'(\mathbb{R}^d)$ and $(V_n)_n$ be a sequence of distributions in $\mathcal{D}'(\mathbb{R}^d)$. Prove that if $V_n \to V$ in $\mathcal{D}'(\mathbb{R}^d)$, then $V_n * T \to V * T$ in $\mathcal{D}'(\mathbb{R}^d)$.
2. Show that there exist two sequences of distributions T_n and V_n tending to 0 in $\mathcal{D}'(\mathbb{R})$ and such that $T_n * V_n \to \delta_0$.

Exercise 4 (Regularization by polynomials). For $n \in \mathbb{N}^*$, we define the polynomial P_n on $\mathbb{R}^d$ by

$$P_n(x) = \frac{n^d}{\pi^{d/2}} \left(1 - \frac{|x|^2}{n}\right)^{n^3}.$$

1. What is the limit in $\mathcal{D}'(\mathbb{R}^d)$ of the sequence $(P_n)_n$?
2. Deduce that any compactly supported distribution is the limit in $\mathcal{D}'(\mathbb{R}^d)$ of a sequence of polynomials.

Exercise 5 (Convolution and translations). Let $F : \mathcal{D}(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ be a continuous linear map. We say that F commutes with translations when $\tau_x \circ F = F \circ \tau_x$ for all $x \in \mathbb{R}^d$.

1. Check that if there exists $T \in \mathcal{D}'(\mathbb{R}^d)$ such that, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $F(\varphi) = T * \varphi$, then F commutes with translations.
2. Show that for all $T \in \mathcal{D}'(\mathbb{R}^d)$, and all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we have $\langle T, \varphi \rangle = T * \check{\varphi}(0)$, where $\check{\varphi}(x) = \varphi(-x)$.
3. Prove that if F commutes with translations, then there exists $T \in \mathcal{D}'(\mathbb{R}^d)$ such that, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $F(\varphi) = T * \varphi$.

Exercise 6 (The extension of the convolution).

1. Let $\varphi \in C^\infty(\mathbb{R}^d)$ and $T \in \mathcal{D}'(\mathbb{R}^d)$ such that $\operatorname{supp}(T) \cap \operatorname{supp}(\varphi)$ is compact. Show that $\langle T, \varphi \rangle$ can be defined in a meaningful way.

2. Let $T, S \in \mathcal{D}'(\mathbb{R}^d)$ satisfying the following property: for every compact K in $\mathbb{R}^d$,

$$D_K = \big\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x \in \operatorname{supp} T,\ y \in \operatorname{supp} S,\ x+y \in K\big\}$$

is compact. Show that in this case, $T * S$ and $S * T$ are well-defined and are equal.

3. Compute the distribution $(x^p H) * (x^q H)$ for all $p, q \in \mathbb{N}$, where H is the Heaviside function.

Exercise 7 (Linear differential equations). Define $\mathcal{D}'_+(\mathbb{R}) = \{T \in \mathcal{D}'(\mathbb{R}) : \operatorname{supp} T \subset \mathbb{R}_+\}$.

1. By using Exercice 6, show that the convolution of two elements of $\mathcal{D}'_+(\mathbb{R})$ is well-defined and gives an element of $\mathcal{D}'_+(\mathbb{R})$. *In the following, we admit that $\mathcal{D}'(\mathbb{R}_+)$ is a commutative algebra for the convolution.* What is the identity element for the convolution in $\mathcal{D}'_+(\mathbb{R})$?
2. Show that for all $a \in \mathbb{R}$ and $T, S \in \mathcal{D}'_+(\mathbb{R})$, we have $(e^{ax}T) * (e^{ax}S) = e^{ax}(T * S)$.
3. For any $T \in \mathcal{D}'_+(\mathbb{R})$, let T^{-1} denote the inverse of T in $\mathcal{D}'_+(\mathbb{R})$ for the convolution whenever it exists. Check that T^{-1} is unique when it exists.
4. Compute H^{-1} and $(\delta_0' - \lambda\delta_0)^{-1}$ for all $\lambda \in \mathbb{R}$ whenever they exist.
5. Let P be a polynomial that splits in $\mathbb{R}$, compute $[P(D)\delta_0]^{-1}$.
6. Solve the following system in $\mathcal{D}'_+(\mathbb{R}) \times \mathcal{D}'_+(\mathbb{R})$

$$\begin{cases} \delta_0'' * X + \delta_0' * Y = \delta_0, \\ \delta_0' * X + \delta_0'' * Y = 0. \end{cases}$$