École Normale Supérieure de Lyon **Année 2021 - 2022**

Département de Mathématiques **M1 - EDP**

TD 1: First elliptic equations

Exercise 1. Let $\Omega = (0,1)$. Establish the following Poincaré inequality

$$\forall f \in H_0^1(\Omega), \quad \|f\|_{L^2(\Omega)} \le \frac{1}{\pi}\|f'\|_{L^2(\Omega)},$$

and prove that the constant $1/\pi$ is optimal.
Hint: Use Fourier series.

Exercise 2. Let $\Omega = (0,1)$. The purpose of this exercice is to prove with a variational method that given a function $f \in L^2(\Omega)$, there exists a unique function $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying

$$-u'' + \sinh(u) = f \quad \text{in } L^2(\Omega). \tag{1}$$

1. Preliminaries: Let H be a real Hilbert space and $J : H \to \mathbb{R}$ be a continuous convex functional. We assume that J is coercive, that is, $J(x) \to +\infty$ when $\|x\| \to +\infty$. Prove then that there exists $x_\star$ in H such that $J(x_\star) = \inf_{x\in H} J(x)$.

2. In this question, we prove that there exists a unique $u \in H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega), \quad \int_0^1 (u'(x)v'(x) + \sinh(u(x))v(x) - f(x)v(x))\,\mathrm{d}x = 0. \tag{2}$$

   To that end, we introduce the functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined for all $v \in H_0^1(\Omega)$ by

$$J(v) = \int_0^1 \left(\frac{1}{2}|v'(x)|^2 + \cosh(v(x)) - f(x)v(x)\right)\mathrm{d}x.$$

   a) Check that the functional J is well-defined, strictly convex and coercive.
   b) Prove that the functional J is differentiable on $H_0^1(\Omega)$ and give the expression of its derivative.
   c) Deduce from the preliminary question that the variational problem (2) admits a unique solution $u \in H_0^1(\Omega)$.

3. Prove that the unique function $u \in H_0^1(\Omega)$ satisfying (2) belongs to $H^2(\Omega)$ and is also the unique function that satisfies (1).

4. When the function f is continuous on $[0,1]$, check that $u \in C^2(\bar{\Omega})$ is a strong solution of (1), in the sense that

$$\forall x \in [0,1], \quad -u''(x) + \sinh(u(x)) = f(x).$$

Exercise 3. Let $\Omega = (0,1)$. We aim at proving that there exists a unique $u \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfying

$$\begin{cases} -u'' + u = \cos(u), \\ u(0) = u(1) = 0. \end{cases}$$

1. Given $v \in L^2(\Omega)$, check that the following problem

$$\begin{cases} -u'' + u = \cos(v), \\ u(0) = u(1) = 0, \end{cases}$$

admits a unique solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$.
Hint: Use Riesz' representation theorem in $H_0^1(\Omega)$.

2. Conclude by using the Banach-Picard fixed point theorem on the space $L^2(\Omega)$.

Exercise 4. Let ρ be a compactly supported C^∞ function on $\mathbb{R}^3$. We are looking for a function $u \in C^2(\mathbb{R}^3)$ satisfying

$$-\Delta u = \rho, \tag{3}$$

under the following decreasing conditions at infinity

$$x \mapsto |x|u(x) \text{ is bounded}, \quad x \mapsto |x|^2 \nabla u(x) \text{ is bounded}. \tag{4}$$

1. Check that the function $x \mapsto 1/|x|$ is of class C^2 on $\mathbb{R}^3 \setminus \{0\}$ and compute its Laplacian.

2. Let Ω be a smooth open subset of $\mathbb{R}^3$. We denote by $n(x)$ the unit normal vector exiting at $x \in \partial\Omega$ and $\mathrm{d}\sigma$ the measure surface on $\partial\Omega$. We consider two functions u, v of class C^2 on $\overline{\Omega}$. By using Stokes' formula, prove Green's formula for the Laplacian:

$$\int_\Omega (v\Delta u - u\Delta v)\,\mathrm{d}x = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) \mathrm{d}\sigma(x).$$

3. For $0 < \alpha < \beta$, we define the following sphere and annulus

$$S_\alpha = \{x \in \mathbb{R}^3 : |x| = \alpha\} \quad \text{and} \quad A_{\alpha,\beta} = \{x \in \mathbb{R}^3 : \alpha \le |x| \le \beta\}.$$

Let $0 < \varepsilon < R$. We consider $u \in C^2(\mathbb{R}^3)$ satisfying (4). For all $x \in \mathbb{R}^3$, prove the following identity

$$\frac{1}{\varepsilon^2} \int_{S_\varepsilon} u(x+y)\,\mathrm{d}\sigma(y) = \int_{A_{\varepsilon,R}} \frac{(-\Delta u)(x+y)}{|y|}\,\mathrm{d}y + \mathcal{O}\left(\frac{1}{R}\right) + \mathcal{O}(\varepsilon).$$

4. Prove that the unique solution of (3) satisfying (4) is given for all $x \in \mathbb{R}^3$ by

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\,\mathrm{d}y.$$

5. Let $p \in [1, 3)$. Check that there exists a constant C_p independent of ρ, such that

$$\|\nabla u\|_{L^\infty(\mathbb{R}^3)} \le C_p \|\rho\|_{L^p(\mathbb{R}^3)}^{p/3} \|\rho\|_{L^\infty(\mathbb{R}^3)}^{1-p/3}.$$

Hint: Consider the domains $\{|x-y| \le r\}$ and $\{|x-y| > r\}$, and optimize with respect to r.

6. Prove the following formula:

$$\|\nabla u\|_{L^2(\mathbb{R}^3)}^2 = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y.$$