TD 2: Weak formulation of elliptic equations

Exercise 1 (Ellipticity). For each of the following linear differential operator L, give the symbol, the principal symbol of L, and discuss the ellipticity and uniform ellipticity.

1. $Lu(x) = -\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u, \quad x \in \Omega \subset \mathbb{R}^d,$
2. $Lf(x,v) = v \cdot \nabla_x f + F(x) \cdot \nabla_v f, \quad x, v \in \mathbb{R}^d,\ F\colon \mathbb{R}^d \to \mathbb{R}^d,$
3. $Lu(t,x) = \partial_t u - \Delta u, \quad t > 0,\ x \in \mathbb{R}^d,$
4. $Lu(t,x) = \partial_t u - i\Delta u, \quad t > 0,\ x \in \mathbb{R}^d.$

Exercise 2 (Faber-Krahn inequality). Let Ω be an open bounded subset of $\mathbb{R}^d$ with $d \geq 3$ and $V \in L^\infty(\Omega)$ such that $V \geq 0$. We consider the problem

$$\begin{cases} -\Delta u = Vu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

1. Give the definition of a weak solution to (1).
2. Can you apply the Lax-Milgram theorem here?
3. Let $r > \frac{d}{2}$. Show that there is a constant $c_d > 0$ depending on d only such that, if (1) has a non-trivial weak solution, then
$$|\Omega|^{\frac{2}{d}-\frac{1}{r}} \|V\|_{L^r(\Omega)} \geq c_d.$$
Hint: Use the following Sobolev inequality
$$\|u\|_{L^{2^*}(\Omega)} \leq M_d \|\nabla u\|_{L^2(\Omega)}, \quad \frac{1}{2^*} = \frac{1}{2} - \frac{1}{d},$$
which holds for all $u \in H_0^1(\Omega)$, where M_d depends on d only.
4. What do you obtain in the particular case $V = \lambda = \text{cst}$?

Exercise 3 (Dirichlet problem). Let Ω be an open bounded subset of $\mathbb{R}^d$, $f \in L^2(\Omega)$ and $F \in L^2(\Omega)^d$. Show that the following elliptic problem with Dirichlet boundary condition

$$\begin{cases} -\Delta u = f - \operatorname{div} F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique weak solution $u \in H_0^1(\Omega)$.

Exercise 4 (Neumann problem). Let Ω be an open bounded subset of $\mathbb{R}^d$ with smooth boundary, the exterior unit normal being denoted by n, and $f \in L^2(\Omega)$. Show that, for all $\mu > 0$, the elliptic problem with Neumann boundary condition

$$\begin{cases} -\Delta u + \mu u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

has a unique weak solution $u \in H^1(\Omega)$. In the case $\mu = 0$, give a necessary condition on $\int_\Omega f$ to the existence of a weak solution to (2).

Exercise 5 (Fourier condition). Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with smooth boundary, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ and $\lambda > 0$. We consider the following elliptic problem with Fourier boundary condition

$$\left\{ \begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \lambda u + \frac{\partial u}{\partial n} &= g \quad \text{on } \partial\Omega. \end{aligned} \right. \tag{3}$$

1. Give the variational formulation of the problem (3).
2. Prove that there exists a positive constant $C_\Omega > 0$ only depending on Ω such that for all $u \in H^1(\Omega)$,
$$\|u\|^2_{L^2(\Omega)} \leq C_\Omega\big(\|\nabla u\|^2_{L^2(\Omega)} + \lambda\|\gamma_0 u\|^2_{L^2(\partial\Omega)}\big),$$
where γ_0 denotes the trace operator $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$.
3. Prove that (3) has a unique weak solution.
4. * Is this weak solution a strong solution ?

Exercise 6 (The method of continuity).

1. Solve the equation $u - \Delta u = f$ on $\mathbb{T}^d$ and show that it defines a map $L^2(\mathbb{T}^d) \to H^2(\mathbb{T}^d)$.
2. Let X, Y be some Banach spaces. Let $(T_t)_{t\in[0,1]}$ be a *continuous* path of linear operators from X to Y satisfying
$$\exists C \geq 0, \forall u \in X, \forall t \in [0,1], \quad \|u\|_X \leq C\|T_t u\|_Y. \tag{4}$$
Prove that T_0 is surjective if and only if T_1 is surjective as well.
3. Let $(a_{i,j})_{1\leq i,j\leq d}$ be a family of maps of class C^1 on $\mathbb{T}^d$. We assume that the following ellipticity condition holds
$$\exists \alpha > 0, \forall x \in \mathbb{T}^d, \forall \xi \in \mathbb{R}^d, \quad a_{i,j}(x)\xi_i\xi_j \geq \alpha|\xi|^2.$$
We define the path $(T_t)_{t\in[0,1]}$ of operators $H^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ by the formula
$$T_t u = u - \partial_i(a^{(t)}_{ij}(x)\partial_j u), \quad a^{(t)}_{ij} = t a_{ij} + (1-t)\delta_{ij}.$$
   (a) Show that $t \mapsto T_t$ is continuous.
   (b) Check that (4) is satisfied.
   (c) Conclude.

Exercise 7 (Resolution by minimization). Let $\Omega \subset \mathbb{R}^3$ be open, bounded with smooth boundary. The purpose is to prove that the following elliptic problem has a non-trivial weak solution

$$\left\{ \begin{aligned} -\Delta u &= u^3 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \right.$$

1. Prove that there exists a solution to the following minimization problem
$$\inf\big\{\|\nabla v\|_{L^2(\Omega)} : v \in H^1_0(\Omega),\ \|v\|_{L^4(\Omega)} = 1\big\}. \tag{5}$$
Recall: Since $d = 3$ here, the continuous embedding $H^1_0(\Omega) \hookrightarrow L^q(\Omega)$ holds for all $1 \leq q \leq 6$, and is moreover compact when $1 \leq q < 6$.
2. Prove that if the function $v \in H^1_0(\Omega)$ solves (5), there exists a positive constant $\lambda > 0$ such that $-\Delta v = \lambda v^3$ weakly in Ω.
3. Conclude.