
TD 4: HEAT EQUATION

EXERCISE 1 (Heat kernel). Let $d \geq 1$ and $E_d \in \mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_x^d)$ be the tempered distribution defined by

$$E_d(t, x) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}} \mathbb{1}_{]0, +\infty[}(t).$$

Prove that E_d is a fundamental solution of the heat operator, that is, satisfies

$$\left(\partial_t - \frac{1}{2}\Delta\right)E_d = \delta_{(t,x)=(0,0)} \quad \text{in } \mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_x^d).$$

Check that E_d is unique under the condition $\text{Supp } E_d \subset \mathbb{R}_+ \times \mathbb{R}^d$.

EXERCISE 2 (Heat equation on \mathbb{R}^d). Let $u_0 \in L^2(\mathbb{R}^d)$. We consider the homogeneous heat equation posed on the whole space \mathbb{R}^d :

$$(1) \quad \begin{cases} \partial_t u - \frac{1}{2}\Delta u = 0 & \text{on } (0, +\infty) \times \mathbb{R}^d, \\ u(0, \cdot) = u_0 & \text{on } \mathbb{R}^d. \end{cases}$$

- (Regularity) Compute explicitly the solution of the equation (1). What is its regularity ?
- (Energy estimate) Show that for all $t \geq 0$,

$$\|u(t, \cdot)\|_{L^2(\mathbb{R}^d)}^2 + \int_0^t \|\nabla u(s, \cdot)\|_{L^2(\mathbb{R}^d)}^2 ds = \|u_0\|_{L^2(\mathbb{R}^d)}^2.$$

- (Maximum principle) Show that if $u_0 \in L^\infty(\mathbb{R}^d)$, then $u(t, \cdot) \in L^\infty(\mathbb{R}^d)$ for all $t \geq 0$ and

$$\sup_{t \geq 0} \|u(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}.$$

- (Infinite speed of propagation) Prove that if $u_0 \geq 0$ is a function being not identically equal to zero and non-negative, then $u > 0$ in $\mathbb{R}_+ \times \mathbb{R}^d$.

EXERCISE 3 (Spectral theory). Let Ω be a bounded open subset of \mathbb{R}^d .

- Prove that there exists a continuous operator $T \in \mathcal{L}(L^2(\Omega), H_0^1(\Omega))$ satisfying $\langle f, v \rangle_{L^2(\Omega)} = \langle Tf, v \rangle_{H_0^1(\Omega)}$ for all $f \in L^2(\Omega)$ and $v \in H_0^1(\Omega)$.
- Let $\iota : H_0^1(\Omega) \rightarrow L^2(\Omega)$ be the canonical injection. Check that the operator $T \circ \iota : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ is non-negative, selfadjoint, one to one and compact.
- Deduce that the spectrum of the Laplacian operator $-\Delta$ with Dirichlet boundary condition is a sequence $(\lambda_n)_{n \geq 0}$ of positive real numbers which is increasing and diverges to $+\infty$, and also that there exists a Hilbert basis $(e_n)_{n \geq 0}$ of $H_0^1(\Omega)$ composed of eigenfunctions of $-\Delta$ and such that $-\Delta e_n = \lambda_n e_n$ for all $n \geq 0$.
- Compute explicitly those eigenvalues and those eigenfunctions when $d = 1$ and $\Omega = (0, 1)$.

EXERCISE 4 (Heat equation on bounded domains). Let Ω be a bounded open subset of \mathbb{R}^d with regular boundary, $T > 0$ be a final time, $u_0 \in L^2(\Omega)$ be an initial datum and $f \in L^2((0, T), L^2(\Omega))$ be a source term. We aim at proving that there exists a unique solution $u \in L^2((0, T), H_0^1(\Omega)) \cap C^0([0, T], L^2(\Omega))$ to the following heat equation with Dirichlet boundary conditions

$$(2) \quad \begin{cases} \partial_t u - \Delta u = f & \text{a.e. in } (0, T) \times \Omega, \\ u = 0 & \text{a.e. on } (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0 & \text{a.e. in } \Omega. \end{cases}$$

We will also check that this solution satisfies the following energy estimate for all $0 \leq t \leq T$,

$$(3) \quad \|u(t, \cdot)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla u(s, \cdot)\|_{L^2(\Omega)}^2 ds \leq C \left(\|u_0\|_{L^2(\Omega)}^2 + \int_0^t \|f(s, \cdot)\|_{L^2(\Omega)}^2 ds \right),$$

where $C > 0$ is a positive constant only depending on Ω . In the following, we consider $(e_n)_{n \geq 0}$ a Hilbert basis of $L^2(\Omega)$ composed of eigenfunctions of the operator $-\Delta$. Moreover, we set λ_n the eigenvalue associated with the eigenfunction e_n .

1. We first prove that there exists a unique $u \in L^2((0, T), H_0^1(\Omega)) \cap C^0([0, T], L^2(\Omega))$ satisfying

$$\begin{cases} \frac{d}{dt} \langle u(t, \cdot), v \rangle_{L^2(\Omega)} + \int_{\Omega} \nabla u(t, \cdot) \cdot \nabla v = \langle f(t, \cdot), v \rangle_{L^2(\Omega)} & \forall v \in H_0^1(\Omega), \forall t \in (0, T), \\ u(0, \cdot) = u_0. \end{cases}$$

- a) Define properly this variational formulation.
 - b) Give the formal expansion in the Hilbert basis $(e_n)_{n \geq 0}$ of such a solution.
 - c) Prove that this expansion converges in $L^2((0, T), H_0^1(\Omega))$ and also in $C^0([0, T], L^2(\Omega))$.
 - d) Conclude.
2. We now want to prove that this weak solution u is a strong solution, that is, is solution of the problem (2).
 - a) Check that the boundary condition and the initial value condition hold.
 - b) * Prove that $\partial_t u - \Delta u = f$ a.e. in $(0, T) \times \Omega$.

3. When $f = 0$, check that

$$\forall t \geq 0, \quad \|u(t, \cdot) - \langle u_0, e_0 \rangle_{L^2(\Omega)} e_0\|_{L^2(\Omega)} \leq e^{-\lambda_1 t} \|u_0\|_{L^2(\Omega)}.$$

EXERCISE 5 (Maximum principle). Let Ω be a bounded open subset of \mathbb{R}^d with smooth boundary, $T > 0$ be a final time, $u_0 \in H_0^1(\Omega)$ be an initial datum and $f \in L^2((0, T), L^2(\Omega))$ be a source term. We consider $u \in L^2((0, T), H_0^1(\Omega)) \cap C^0([0, T], L^2(\Omega))$ the unique solution of the problem (2). Prove that when $f \geq 0$ a.e. in $(0, T) \times \Omega$ and $u_0 \geq 0$ a.e. in Ω , then $u \geq 0$ a.e. on $(0, T) \times \Omega$.

Hint: Admit that $\partial_t u \in L^2((0, T), L^2(\Omega))$ and $u \in L^2((0, T), H^2(\Omega)) \cap C^0([0, T], H_0^1(\Omega))$.

*Application *:* Assume now that $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $f \in L^\infty([0, +\infty) \times \Omega)$. Show that

$$\sup_{t \geq 0} \|u(t, \cdot)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} + \frac{\text{diam}(\Omega)^2}{2d} \sup_{t \geq 0} \|f(t, \cdot)\|_{L^\infty(\Omega)}.$$