TD 7: Travelling waves

Exercise 1. We aim at proving that there are traveling waves solutions for the Fisher-KPP equation

$$\partial_t u - \partial_{xx} u = u(1-u), \quad t > 0,\ x \in \mathbb{R}, \tag{1}$$

i.e. solutions of the form $u(t,x) = \phi(x - ct)$ for some function $\phi : \mathbb{R} \to [0,1]$ and $c \in \mathbb{R}$. Precisely, we are interested in traveling wavefronts, i.e. satisfying $\lim_{+\infty} \phi = 0$ and $\lim_{-\infty} \phi = 1$.

1. Check that a traveling wave is solution of the equation (1) if and only if the wave profile ϕ satisfies the following ordinary equation,

$$\phi''(z) + c\phi'(z) + \phi(z)(1 - \phi(z)) = 0, \quad z \in \mathbb{R},$$

where $z = x - ct$ denotes the co-moving frame.

2. Write this equation as a system of two first order ordinary equations.
3. Study the stationary points of this system.
4. Explain why such a traveling wave does not exist when $0 < c < 2$.
5. Admitting that such a travelling wave exists when $c \geq 2$, prove that the wave profile ϕ has the following asymptotics

$$\phi(z,c) = \frac{1}{1+e^{z/c}} + \frac{1}{c^2}\frac{e^{z/c}}{(1+e^{z/c})^2}\ln\left(\frac{4e^{z/c}}{(1+e^{z/c})^2}\right) + \mathcal{O}(c^{-4}).$$

Hint: Set $\varepsilon = 1/c^2$ *and* $\xi = z/c$, *and consider the expansion of* ϕ *in powers of* ε, *that is,* $\phi(\xi,\varepsilon) = \phi_0(\xi) + \varepsilon\phi_1(\xi) + \varepsilon^2\phi_2(\xi) + \cdots$

Exercise 2. We still consider the Fisher-KPP equation (1). The purpose is now to deal with the appearance of propagation speeds in the reality. Assume that the initial condition of the equation (1) is given by

$$u(0,x) = e^{-a|x|}, \quad x \in \mathbb{R},$$

where $a > 0$ is a positive constant.

1. By considering supersolutions of the form

$$\overline{u}(t,x) = e^{\pm s_a(x \pm c_a t)}, \quad t > 0,\ x \geq 0,$$

where $c_a > 0$ and $s_a > 0$ are positive constants depending on a, establish an estimate of the form

$$\forall t \geq 0, \forall x \in \mathbb{R}, \quad |u(t,x)| \leq e^{-s_a(|x| - c_a t)}.$$

2. Deduce that

$$\forall c > a + \frac{1}{a}, \quad \lim_{t\to+\infty} \sup_{|x|\geq ct} |u(t,x)| = 0, \quad \text{when } 0 < a < 1,$$

$$\forall c > 2, \qquad \lim_{t\to+\infty} \sup_{|x|\geq ct} |u(t,x)| = 0, \quad \text{when } a \geq 1.$$

3. Draw a picture, admitting that

$$\forall 0 < c < a + \frac{1}{a}, \quad \lim_{t\to+\infty} \sup_{|x|\leq ct} |1-u(t,x)| = 0, \quad \text{when } 0 < a < 1,$$

$$\forall 0 < c < 2, \qquad \lim_{t\to+\infty} \sup_{|x|\leq ct} |1-u(t,x)| = 0, \quad \text{when } a \geq 1.$$

Remark: Those limits can be obtained by constructing adapted subsolutions.

4. Comment.

Exercise 3. Rabies may infect all warm-blooded animals, also birds, and also humans, and affects the central nervous system. Vaccines are available (but expensive); but no further cure is known. The spread seems to occur in waves, e.g. one coming from the Polish-Russian border; the spread velocity is approx. 30-60 km/year.

Let us consider two groups of foxes:

. Susceptible foxes (S), with no diffusion (as they are territorial),

. Infective foxes (I), with diffusion (loss of sense of territory), constant death rate.

The infection rate is assumed to be proportional to their densities, no reproduction or further spread:

$$\begin{cases} \partial_t S = -rIS, & t>0,\ x\in\mathbb{R}, \\ \partial_t I = rIS - aI + \nu\partial_{xx}^2 I, & t>0,\ x\in\mathbb{R}. \end{cases}$$

The non-dimensionalised version of the above system is the following:

$$\begin{cases} \partial_t S = -IS, & t>0,\ x\in\mathbb{R}, \\ \partial_t I = IS - mI + \partial_{xx}^2 I, & t>0,\ x\in\mathbb{R}, \end{cases}$$

with $m = a/(rS_0)$, S_0 being the initial (maximum) susceptible density. We look for a travelling wave solution of this system of the form

$$S(t,x) = S(x-ct) = S(z) \quad \text{and} \quad I(t,x) = I(x-ct) = I(z),$$

where $z = x - ct$, the wave fronts S and I satisfying $0 \leq S \leq 1$ and $0 \leq I \leq 1$.

1. Write the system of ODEs satisfied by the functions S and I.
2. Justify the following boundary conditions: $S(+\infty) = 1$, $I(+\infty) = 0$, $S'(-\infty) = 0$, $I(-\infty) = 0$.
3. Check that

$$S(-\infty) - m\ln S(-\infty) = 1.$$

Deduce the fraction of susceptibles which survive the "rabies wave" (draw a picture).

4. Draw the phase plane associated with the system satisfied by S and I.
5. Explain why $c = 2\sqrt{1-m}$ is the minimal wave speed.
6. Draw the shapes of the wave fronts S and I.