TD 10: Pseudo-differential operators II

Exercise 1. Let $K : \mathbb{R}^{2d} \to \mathbb{C}$ be a continuous function. Assume that there exists $A > 0$ such that

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x,y)|\,\mathrm{d}y \le A, \quad \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x,y)|\,\mathrm{d}x \le A.$$

For all $u \in C_0^\infty(\mathbb{R}^d)$, we set

$$(Pu)(x) = \int_{\mathbb{R}^d} K(x,y)u(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d.$$

1. Check that Pu is well-defined and belongs to $L^\infty(\mathbb{R}^d)$.
2. We will prove Schur's lemma, stating that P can be uniquely extended to a bounded operator in $L^2(\mathbb{R}^d)$ satisfying $\|P\|_{\mathcal{L}(L^2)} \le A$.
   a) By using Cauchy-Schwarz' inequality, check that for all $u \in C_0^0(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

   $$|(Pu)(x)|^2 \le A \int_{\mathbb{R}^d} |K(x,y)||u(y)|^2\,\mathrm{d}y.$$

   b) Conclude.

Exercise 2. The purpose of this exercise is to prove Calderón-Vaillancourt's theorem: any pseudo-differential operator $\mathrm{Op}(a)$, with $a \in S^0$, is bounded in $L^2(\mathbb{R}^d)$.

1. We first assume that $a \in S^{-(d+1)}$.
   a) Check that $\mathrm{Op}(a)$ can be written

   $$\mathrm{Op}(a)u(x) = \int_{\mathbb{R}^d} K(x,y)u(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d.$$

   where K is a kernel to be precised.
   b) Prove that the function $(x,y) \in \mathbb{R}^{2d} \mapsto (1+|x-y|^{d+1})K(x,y)$ is bounded.
   c) Prove the theorem by using Exercise 1.
2. Prove with an induction that for all $k \in \{0,\dots,d\}$, the theorem is true when $a \in S^{k-(d+1)}$. *Hint: Consider the operator* $\mathrm{Op}(a)^*\,\mathrm{Op}(a)$.
3. The previous question implies in particular that the theorem holds when $a \in S^{-1}$. We now assume that $a \in S^0$.
   a) Prove that if $M > 0$ is large enough, there exist symbols $c \in S^0$ and $r \in S^{-1}$ such that

   $$\mathrm{Op}(c)^*\,\mathrm{Op}(c) = M\,\mathrm{Id} - \mathrm{Op}(a)^*\,\mathrm{Op}(a) + \mathrm{Op}(r).$$

   b) Conclude.

Exercise 3. Let $m \in \mathbb{R} \cup \{-\infty\}$ and $a \in S^m$.

1. Recall the expression of the kernel K of the operator $\operatorname{Op}(a)$.
2. Prove that when $m = -\infty$, K belongs to $C^\infty(\mathbb{R}^{2d})$.
3. Let $x, y \in \mathbb{R}^d$ such that $x \neq y$. We consider $\varphi, \psi \in C_0^\infty(\mathbb{R}^d)$ satisfying

   a) $\varphi = 1$ is a neighborhood of x,

   b) $\psi = 1$ is a neighborhood of y,

   c) $\operatorname{supp} \varphi \cap \operatorname{supp} \psi = \emptyset$.

   Show that $M_\varphi \operatorname{Op}(a) M_\psi$ belongs to $\operatorname{Op}(S^{-\infty})$, where M_φ and M_ψ denote the multiplication by φ and ψ respectively.
4. Compute the kernel of the operator $M_\varphi \operatorname{Op}(a) M_\psi$ as a function of K.
5. Prove that K is C^∞ in a neighborhood of (x, y).

Exercise 4.

1. Let $a \in C^\infty(\mathbb{R}^{2d})$ and $\chi \in C^\infty(\mathbb{R}^d)$ satisfying

   $$\chi(\xi) \neq 0 \Longleftrightarrow 1/2 < |\xi| < 2.$$

   For all $\lambda \geq 1$, we set $a_\lambda(x, \xi) = \chi(\xi) a(x, \lambda\xi)$. Prove that the following conditions are equivalent:

   a) $a \in S^m$,

   b) $\forall (\alpha, \beta) \in \mathbb{N}^{2d}, \exists C_{\alpha,\beta} > 0, \forall \lambda \geq 1,\ \|\partial_\xi^\alpha \partial_x^\beta a_\lambda\|_{L^\infty} \leq C\lambda^m$.
2. Let $f \in C^k(\mathbb{R}^d)$ satisfying that f and $\partial^\alpha f$ are bounded for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| = k$.

   a) Prove that there exists a positive constant $c > 0$ independent on f such that for all $\beta \in \mathbb{N}^d$ satisfying $0 \leq |\beta| \leq k$,

   $$\|\partial^\beta f\|_{L^\infty} \leq c\bigg(\|f\|_{L^\infty} + \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^\infty}\bigg).$$

   b) Prove that for all $\beta \in \mathbb{N}^d$ satisfying $0 \leq |\beta| \leq k$,

   $$\|\partial^\beta f\|_{L^\infty} \leq c\|f\|_{L^\infty}^{1-|\beta|/k} \bigg(\sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^\infty}\bigg)^{|\beta|/k}.$$

   Hint: Consider the function $g : x \in \mathbb{R}^d \mapsto f(\lambda x)$ for a well-chosen $\lambda > 0$.
3. Let $a \in S^m$. Assume that there exists $\mu > 0$ and $c > 0$ such that

   $$\forall (x, \xi) \in \mathbb{R}^{2d}, \quad |a(x, \xi)| \leq c\langle\xi\rangle^\mu.$$

   Prove that $a \in S^{\mu+\varepsilon}$ for all $\varepsilon > 0$.
4. Let A be a nilpotent pseudo-differential operator, *i.e.* satisfying $A^k = 0$ for some $k \geq 1$.

   a) Prove that $A \in \operatorname{Op}(S^{-\infty})$.

   b) Give a non-trivial example when $k = 2$.