TD 4: Elliptic regularity and maximum principles

Exercise 1 (Control of the L^∞ norm). Let Ω be an open bounded subset of $\mathbb{R}^d$ of class C^2. Let $A \in C^1(\overline{\Omega}, S_d(\mathbb{R}))$ satisfying the following ellipticity condition

$$\exists \alpha > 0, \forall (x,\xi) \in \Omega \times \mathbb{R}^d, \quad A(x)\xi \cdot \xi \geq \alpha |\xi|^2. \tag{1}$$

Let $f \in L^2(\Omega)$ and $u \in H^1_0(\Omega)$ be the weak solution of the following Dirichlet problem

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

1. In this question, we assume that $d \leq 3$. Show that there exists a constant $C \geq 0$ depending only on Ω and d such that

$$\|u\|_{L^\infty(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{2}$$

2. We assume that $\Omega = B(0,R)$ where $R \in (0,1)$.
   (a) Compute Δv when $v(x) = \psi(|x|)$ is a radial function.
   (b) By considering the function $u(x) = \ln|\ln|x||$ and the case $A(x) = \mathrm{I}_d$, discuss the validity of the estimate (2) when $d \geq 4$.

Remark: One can prove (this is a bit technical) that when $d \geq 4$ and $f \in L^p(\Omega)$, where $p > d/2$, there exists a positive constant $C > 0$ only depending on d, Ω and p such that the following estimate, somehow analogous to (2), holds

$$\|u\|_{L^\infty(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)}).$$

Exercise 2 (Hölder regularity). The purpose of this exercise is to show a gain of derivatives in Hölder spaces for the solution u to the Poisson equation $-\Delta u = \rho$, where $\rho \in C^0(\mathbb{R}^3)$ is a function with compact support. Let G be the Green function of the Laplacian in dimension 3. Let us recall that the function

$$u(x) = (G * \rho)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|} \, \mathrm{d}y,$$

is solution of the Poisson equation in $\mathbb{R}^3$. We assume that $\rho \in C^\alpha(\mathbb{R}^3)$ for a given $\alpha \in (0,1)$, and we set

$$[\rho]_{\dot{C}^\alpha(\mathbb{R}^3)} = \sup_{x \neq z \in \mathbb{R}^3} \frac{|\rho(x) - \rho(y)|}{|x-y|^\alpha} < +\infty.$$

Let K be a compact of $\mathbb{R}^3$. We want to prove that $u, \nabla u \in C^\alpha(K)$ and that there exists a positive constant $c > 0$ only depending on K, d, α and on the support of ρ such that

$$[u]_{\dot{C}^\alpha(K)} + [\nabla u]_{\dot{C}^\alpha(K)} \leq c[\rho]_{\dot{C}^\alpha(\mathbb{R}^3)}. \tag{3}$$

1. Show that $u \in C^\alpha(K)$ and that the estimate (3) holds for u.
2. By introducing a cut-off function ω_ε of the form $\omega_\varepsilon(x) = \theta(\varepsilon^{-1}|x|)$ and considering the approximation $u_\varepsilon = (G\omega_\varepsilon) * \rho$, prove that $\nabla u \in C^\alpha(K)$ and that the estimate (3) holds for the function ∇u.

Remark: By using similar techniques, one can prove that for all $\delta \in (0, \alpha)$, we have $\nabla^2 u \in C^\delta(K)$ and also that there exists a positive constant $c' > 0$ depending only on K, d, α, δ and the support of the function ρ such that

$$[\nabla^2 u]_{\dot{C}^\delta(K)} \leq c'[\rho]_{\dot{C}^\alpha(\mathbb{R}^3)}.$$

Exercise 3 (Weak maximum principle). Let Ω be a bounded open subset of $\mathbb{R}^d$ with smooth boundary and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $\Delta u \leq 0$ on Ω. Proof by hand that

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

Hint: Assume first that $\Delta u < 0$.

Exercise 4 (Weak maximum principle for weak solutions). Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. We consider the following operator $L = -\operatorname{div}(A(x)\nabla\cdot)$, where $A \in L^\infty(\Omega, M_d(\mathbb{R}))$ satisfies the following ellipticity assumption

$$\exists\alpha > 0, \forall(x, \xi) \in \Omega \times \mathbb{R}^d, \quad A(x)\xi \cdot \xi \geq \alpha|\xi|^2.$$

We want to prove that if $u \in H_0^1(\Omega)$ is a weak solution of the equation $Lu \leq 0$, then $u \leq 0$ a.e. in the set Ω.

1. Prove that there exists a non-negative function $G \in C^1(\mathbb{R})$ with bounded derivative such that $G' > 0$ on $(0, +\infty)$ and $G' = 0$ on $(-\infty, 0]$.
2. Check that we have
$$\int_\Omega |\nabla u(x)|^2 (G' \circ u)(x)\, \mathrm{d}x \leq 0.$$
3. Conclude.

Exercise 5 (No solution). Let $L > 0$. We aim at proving that when $L \gg 1$ is large enough, there is no smooth solution u satisfying $-u'' = e^u$ in $(0, L)$ and $u(0) = u(L) = 0$. We assume by contradiction that such a solution $u \in C^0[0, L] \cap C^2(0, L)$ exists.

1. Given $\varepsilon > 0$, we consider the function $w = u + \varepsilon$. Give the equation satisfied by this new function w.
2. We consider the family of functions $(v_\lambda)_{\lambda\geq 0}$ defined on $[0, L]$ by $v_\lambda(x) = \lambda\sin(\pi x/L)$. Give the equations satisfied by these functions.
3. Prove that when $L \gg 1$ is large enough, the function w is a sub-solution of the equation established in the above question. Check moreover that when $0 < \lambda \ll 1$ is sufficiently small, then $v_\lambda < w$ on $[0, L]$.
4. Let us now start increasing λ until the graphs of v_λ and w touch at some point
$$\lambda_0 = \sup\big\{\lambda \geq 0 : \forall x \in [0, L],\ v_\lambda(x) < w(x)\big\}.$$
By considering the function $p = v_{\lambda_0} - w$ and using the weak maximum principle, obtain a contradiction.