TD 7: Maximum principles and stability of steady states

Exercise 1. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $T > 0$ be a final time and $Q_T = (0, T] \times \Omega$. We consider the following differential operator

$$L = -\sum_{i,j=1}^{d} a^{i,j}(t,x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} b^i(t,x)\partial_{x_i} + c(t,x), \quad (t,x) \in Q_T,$$

the coefficients $a^{i,j}, b^i$ and c being bounded on $\overline{Q_T}$, with moreover $a^{i,j} = a^{j,i}$. We assume that the operator $\partial_t + L$ is uniformly parabolic, that is,

$$\exists \theta > 0, \forall (t,x) \in Q_T, \forall \xi \in \mathbb{R}^d, \quad \sum_{i,j=1}^{d} a^{i,j}(t,x)\xi_i\xi_j \geq \theta|\xi|^2.$$

State as many maximum principles as you can for the parabolic operator $\partial_t + L$.

Exercise 2. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $T > 0$ be a positive time and $Q_T = (0, T] \times \Omega$. We also consider $f \in C^1(\mathbb{R})$ a smooth function. Let $u, v \in C^2(Q_T) \cap C^0(\overline{Q_T})$ be two functions satisfying

$$\begin{cases} \partial_t v - \Delta v - f(v) \leq \partial_t u - \Delta u - f(u) & \text{in } Q_T, \\ v \leq u & \text{on } \Gamma_T. \end{cases}$$

Prove that $v \leq u$ on Q_T.

Application: Consider $u \in C^2(Q_T) \cap C^0(\overline{Q_T})$ a solution of the equation

$$\begin{cases} \partial_t u - \Delta u = u(1-u)(u-a) & \text{in } Q_T, \\ u = 0 & \text{on } (0,T] \times \partial\Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega, \end{cases}$$

where $0 < a < 1$ is a positive constant and u_0 is a smooth initial datum satisfying $0 \leq u_0 \leq 1$ in Ω. Prove that the function u is bounded as follows

$$\forall (t,x) \in Q_T, \quad 0 \leq u(t,x) \leq 1.$$

Can you be more precise when assuming $0 \leq u_0 < a$ in Ω ?

Exercise 3. Let $L > 0$. Prove that there exists a critical length $L_c > 0$ such that the equation

$$\begin{cases} q'' + q(1-q) = 0 & x \in (0, L), \\ q(0) = q(L) = 0, \end{cases} \tag{1}$$

has a non-trivial non-negative solution if and only if $L > L_c$.
Hint: The function $H(q_1, q_2) = q_1^2/2 + q_2^2/2 - q_1^3/3$ is a Lyapunov function for this equation.

Exercise 4. Let $0 < L < \pi$ be a length, $u_0 \in L^2(0,L)$ be an initial datum such that $0 \le u_0 \le 1$ a.e. and u be the solution of the Fisher-KPP equation

$$\text{(2)} \qquad \begin{cases} \partial_t u - \partial_{xx} u = u(1-u), & t > 0,\ x \in (0,L), \\ u(t,0) = u(t,L) = 0, & t > 0, \\ u(0,x) = u_0(x), & x \in (0,L), \end{cases}$$

1. Prove the following estimate

$$\forall t \ge 0, \quad \|u(t)\|_{L^2(0,L)} \le e^{(1-\pi^2/L^2)t}\, \|u_0\|_{L^2(0,L)},$$

and deduce that $u(t) \to 0$ in $L^2(0,L)$ as $t \to +\infty$.

2. We now aim at proving that $u(t,x) \to 0$ as $t \to +\infty$ for all $x \in [0,L]$.
   (a) Find a subsolution $\underline{u}$ of the equation (2).
   (b) We consider $\overline{u}$ the solution of the equation

$$\begin{cases} \partial_t \overline{u} - \partial_{xx} \overline{u} = \overline{u} & t > 0,\ x \in (0,L), \\ \overline{u}(t,0) = \overline{u}(t,L) = 0 & t > 0, \\ \overline{u}(0,x) = u_0(x) & x \in (0,L), \end{cases}$$

   Check that $\overline{u}$ is a supersolution of the equation (2).
   (c) Prove that $\overline{u}(t,x) \to 0$ for all $x \in [0,L]$ as $t \to +\infty$ and conclude.

Exercise 5. We still consider the Fisher-KPP equation (2). Assuming this time that $L > \pi$, we aim at proving that there exists a supersolution $\overline{u}$ of the equation (2) such that $u(t,x) \le \overline{u}(t,x)$ for all $t \ge 0$ and $x \in (0,L)$, and satisfying $\overline{u}(t,x) \to_{t\to+\infty} q(x)$ for all $x \in [0,L]$, where q is the non-trivial non-negative steady state given by Exercice 3.

1. Let $\overline{u}$ be the solution of the equation

$$\begin{cases} \partial_t \overline{u} - \partial_{xx} \overline{u} = \overline{u}(1-\overline{u}), & t > 0,\ x \in (0,L), \\ \overline{u}(t,0) = \overline{u}(t,L) = 0, & t > 0, \\ \overline{u}(0,x) = M, & x \in (0,L), \end{cases}$$

   with $M = \max(1, \sup_{(0,L)} u_0)$. Prove that $\overline{u}$ is a supersolution of the equation (2) which dominates the function u.
2. By comparing $\overline{u}(t+h,x)$ and $\overline{u}(t,x)$, prove that for all $x \in [0,L]$, the limit $w(x) = \lim_{t\to+\infty} \overline{u}(t,x)$ exists and satisfies the estimate $0 \le w(x) \le M$.
3. Admit that w is a solution of the equation (1). Deduce then that $w = q$ and conclude.

Remark: One can also prove that there exists a subsolution $\underline{u}$ converging pointwise to q and bounding the function u from below. As a consequence, $u(t,x) \to_{t\to+\infty} q(x)$ for all $x \in [0,L]$.