Sheet 0: Sobolev spaces and elliptic problems in one dimension

Exercise 1. Let $\Omega = (0,1)$.

1. Prove that the following continuous embeddings hold
$$W^{1,1}(\Omega) \hookrightarrow C^0(\bar{\Omega}) \quad \text{and} \quad W^{1,p}(\Omega) \hookrightarrow C^{0,1-1/p}(\bar{\Omega}) \quad \text{when } p \in (1,\infty],$$
with the convention $1/\infty = 0$.

2. Prove that for all $1 \le p < \infty$, the space $W_0^{1,p}(\Omega)$ is given by
$$W_0^{1,p}(\Omega) = \big\{ u \in W^{1,p}(\Omega) : u(0) = u(1) = 0 \big\}.$$

Exercise 2. Let $0 < \alpha < 1$ and $p > 1$ be positive real numbers. Show that there exists a positive constant $C_{\alpha,p} > 0$ such that for all $u \in C_0^\infty(\mathbb{R})$,
$$\left(\iint_{\mathbb{R}\times\mathbb{R}} \left(\frac{|u(x)-u(y)|}{|x-y|^\alpha} \right)^p \frac{\mathrm{d}x\mathrm{d}y}{|x-y|} \right)^{1/p} \le C_{\alpha,p} \|u\|_{L^p(\mathbb{R})}^{1-\alpha} \|\nabla u\|_{L^p(\mathbb{R})}^{\alpha}.$$
Hint: Consider the two regions $\{|x-y| > R\}$ and $\{|x-y| \le R\}$, where $R > 0$ is to be chosen.

Exercise 3. Let $\Omega = (0,1)$. Establish the following Poincaré inequality
$$\forall f \in H_0^1(\Omega), \quad \|f\|_{L^2(\Omega)} \le \frac{1}{\pi} \|f'\|_{L^2(\Omega)},$$
and prove that the constant $1/\pi$ is optimal. *Hint*: Use Fourier series.

Exercise 4. Let $\Omega = (0,1)$. We consider $f \in L^2(\Omega)$ and $\alpha \in L^\infty(\Omega)$ satisfying $0 < \alpha_{\min} \le \alpha(x)$ a.e. in Ω.

1. By using the Riesz representation theorem, prove that there exists a unique $u \in H_0^1(\Omega)$ satisfying
$$-(\alpha u')' + u = f \quad \text{in } \mathscr{D}'(\Omega).$$
When in addition $\alpha \in C^\infty(\Omega)$, check that $u \in H^2(\Omega)$ and that the above equality holds in $L^2(\Omega)$. What about the case $\alpha = 1$?

2. We consider moreover $\beta \in C^1(\Omega)$ a function satisfying $\beta' \le 2$ on Ω. By using the Lax-Milgram theorem, prove that there exists a unique $u \in H_0^1(\Omega)$ satisfying
$$-(\alpha u')' + \beta u' + u = f \quad \text{in } \mathscr{D}'(\Omega).$$

Exercise 5. Let $\Omega = (-1, 1)$. We consider $f \in C^0(\bar{\Omega})$ and g a function defined on $\partial\Omega$. Solve explicitly in $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ the following elliptic problem with Neumann boundary conditions

$$\begin{cases} -u'' = f & \text{on } \Omega, \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega. \end{cases}$$

Can the Lax-Milgram theorem be used to study the above boundary value equation ?

Exercise 6. Let $\Omega = (0, 1)$. We consider $f \in L^2(\Omega)$ and $\phi : \mathbb{R} \to \mathbb{R}_+$ a strictly convex C^1 function. The purpose is to prove with a variational method that there exists a unique function $u \in H^2(\Omega) \cap H^1_0(\Omega)$ satisfying

$$-u'' + \phi'(u) = f \quad \text{in } L^2(\Omega). \tag{1}$$

1. Preliminaries: Let H be a real Hilbert space and $J : H \to \mathbb{R}$ be a continuous convex functional. We assume that J is coercive, that is, $J(x) \to +\infty$ when $\|x\| \to +\infty$. Prove then that there exists $x_\star$ in H such that $J(x_\star) = \inf_{x \in H} J(x)$.

2. In this question, we prove that there exists a unique $u \in H^1_0(\Omega)$ such that

$$\forall v \in H^1_0(\Omega), \quad \int_0^1 (u'(x)v'(x) + \phi'(u(x))v(x) - f(x)v(x))\, \mathrm{d}x = 0. \tag{2}$$

   To that end, we introduce the functional $J : H^1_0(\Omega) \to \mathbb{R}$ defined for all $v \in H^1_0(\Omega)$ by

$$J(v) = \int_0^1 \left(\frac{1}{2}|v'(x)|^2 + \phi(v(x)) - f(x)v(x) \right) \mathrm{d}x.$$

   a) Check that the functional J is well-defined, strictly convex and coercive.
   b) Prove that the functional J is differentiable on $H^1_0(\Omega)$ and give the expression of its derivative.
   c) Deduce from the preliminary question that the variational problem (2) admits a unique solution $u \in H^1_0(\Omega)$.

3. Prove that the unique function $u \in H^1_0(\Omega)$ satisfying (2) belongs to $H^2(\Omega)$ and is also the unique function that satisfies (1).

4. When the function f is moreover continuous on $[0, 1]$, check that $u \in C^2(\bar{\Omega})$ is a strong solution of (1), that is

$$\forall x \in [0, 1], \quad -u''(x) + \phi'(u(x)) = f(x).$$

Exercise 7. Let $\Omega = (0, 1)$. Prove that there exists a unique function $u \in C^\infty(\bar{\Omega})$ satisfying

$$\begin{cases} -u'' + u = \cos(u), \\ u(0) = u(1) = 0. \end{cases}$$

Hint: Use the Banach-Picard fixed point theorem on the space $L^2(\Omega)$.