École Normale Supérieure de Lyon **Année 2020 - 2021**

Département de Mathématiques **M1 - EDP**

Sheet 1: Weak formulation of elliptic equations

Exercise 1 (Ellipticity). For each of the following linear differential operator L, give the symbol, the principal symbol of L, and discuss the ellipticity and uniform ellipticity.

1. $Lu(x) = -\sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i} + c(x)u, \quad x \in \Omega \subset \mathbb{R}^d,$
2. $Lf(x,v) = v \cdot \nabla_x f + F(x) \cdot \nabla_v f, \quad x, v \in \mathbb{R}^d,\ F\colon \mathbb{R}^d \to \mathbb{R}^d,$
3. $Lu(t,x) = \partial_t u - \Delta u, \quad t > 0,\ x \in \mathbb{R}^d,$
4. $Lu(t,x) = \partial_t u - i\Delta u, \quad t > 0,\ x \in \mathbb{R}^d.$

Exercise 2 (Faber-Krahn inequality). Let Ω be an open bounded subset of $\mathbb{R}^d$ with $d \geq 3$ and $V \in L^\infty(\Omega)$ such that $V \geq 0$. We consider the problem

$$(1) \qquad \begin{cases} -\Delta u = Vu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

1. Give the definition of a weak solution to (1).
2. Can you apply the Lax-Milgram theorem here?
3. Let $r > \frac{d}{2}$. Show that there is a constant $c_d > 0$ depending on d only such that, if (1) has a non-trivial weak solution, then
$$|\Omega|^{\frac{2}{d} - \frac{1}{r}} \|V\|_{L^r(\Omega)} \geq c_d.$$
Hint: Use the following Sobolev inequality
$$\|u\|_{L^{2^*}(\Omega)} \leq M_d \|\nabla u\|_{L^2(\Omega)}, \qquad \frac{1}{2^*} = \frac{1}{2} - \frac{1}{d},$$
which holds for all $u \in H_0^1(\Omega)$, where M_d depends on d only.
4. What do you obtain in the particular case $V = \lambda = \text{cst}$?

Exercise 3 (Dirichlet problem). Let Ω be an open bounded subset of $\mathbb{R}^d$, $f \in L^2(\Omega)$ and $F \in L^2(\Omega)^d$. Show that the following elliptic problem with Dirichlet boundary condition

$$\begin{cases} -\Delta u = f - \operatorname{div} F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique weak solution $u \in H_0^1(\Omega)$.

Exercise 4 (Neumann problem). Let Ω be an open bounded subset of $\mathbb{R}^d$ with smooth boundary, the exterior unit normal being denoted by n, and $f \in L^2(\Omega)$. Show that, for all $\mu > 0$, the elliptic problem with Neumann boundary condition

$$(2) \qquad \begin{cases} -\Delta u + \mu u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique weak solution $u \in H^1(\Omega)$. In the case $\mu = 0$, give a necessary condition on $\int_\Omega f$ to the existence of a weak solution to (2).

Exercise 5 (Fourier condition). Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with smooth boundary, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ and $\lambda > 0$. We consider the following elliptic problem with Fourier boundary condition

$$(3) \qquad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ \lambda u + \dfrac{\partial u}{\partial n} = g & \text{on } \partial\Omega. \end{cases}$$

1. Give the variational formulation of the problem (3).
2. Prove that there exists a positive constant $C_\Omega > 0$ only depending on Ω such that for all $u \in H^1(\Omega)$,
$$\|u\|^2_{L^2(\Omega)} \leq C_\Omega\big(\|\nabla u\|^2_{L^2(\Omega)} + \lambda\|\gamma_0 u\|^2_{L^2(\partial\Omega)}\big),$$
where γ_0 denotes the trace operator $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$.
3. Prove that (3) has a unique weak solution.
4. * Is this weak solution a strong solution ?

Exercise 6 (System of equations). Let Ω be an open bounded subset of $\mathbb{R}^d$, $f, g \in L^2(\Omega)$ and A, B, C, D be four matrices in $\mathcal{M}_d(\mathbb{R})$. We analyse the following system of equations with Dirichlet boundary counditions

$$\begin{cases} -\operatorname{div}(A\nabla u) - \operatorname{div}(B\nabla v) = f & \text{in } \Omega, \\ -\operatorname{div}(C\nabla u) - \operatorname{div}(D\nabla v) = g & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

We assume that the *Legendre-Hadamard ellipticity condition* holds:

$$\exists\theta > 0, \forall\xi \in \mathbb{R}^d, \forall\eta \in \mathbb{R}^2, \quad \sum_{i,j=1}^{2} (\mathcal{A}_{ij}\xi\cdot\xi)\eta_i\eta_j \geq \theta|\xi|^2|\eta|^2 \quad \text{where} \quad \mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

1. Check that A and D are uniformly elliptic.
2. Show that the functional $a(U, V) := \int_\Omega \sum_{i,j=1}^2 \mathcal{A}_{ij}\nabla U_i \cdot \nabla V_j \,\mathrm{d}x$ is continuous and coercive on $H^1_0(\Omega) \times H^1_0(\Omega)$. *Hint*: Use Bessel-Parseval theorem after extending U and V by 0 outside Ω.
3. Conclude.

Exercise 7 (Resolution by minimization). Let $\Omega \subset \mathbb{R}^3$ be open, bounded with smooth boundary. The purpose is to prove that the following elliptic problem has a non-trivial weak solution

$$\begin{cases} -\Delta u = u^3 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

1. Prove that there exists a solution to the following minimization problem
$$(4) \qquad \inf\big\{\|\nabla v\|_{L^2(\Omega)} : v \in H^1_0(\Omega),\ \|v\|_{L^4(\Omega)} = 1\big\}.$$
Recall: Since $d = 3$ here, the continuous embedding $H^1_0(\Omega) \hookrightarrow L^q(\Omega)$ holds for all $1 \leq q \leq 6$, and is moreover compact when $1 \leq q < 6$.
2. Prove that if the function $v \in H^1_0(\Omega)$ solves (4), there exists a positive constant $\lambda > 0$ such that $-\Delta v = \lambda v^3$ weakly in Ω.
3. Conclude.