
Sheet 3: Heat equation

Exercise 1 (Heat kernel). Let $d \geq 1$ and $E_d \in \mathscr{S}'(\mathbb{R}_t \times \mathbb{R}^d_x)$ be the tempered distribution defined by

$$E_d(t,x) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}} \mathbf{1}_{]0,+\infty[}(t).$$

Prove that E_d is a fundamental solution of the heat operator, that is, satisfies

$$\Big(\partial_t - \frac{1}{2}\Delta\Big) E_d = \delta_{(t,x)=(0,0)} \quad \text{in } \mathscr{S}'(\mathbb{R}_t \times \mathbb{R}^d_x).$$

Check that E_d is unique under the condition $\operatorname{Supp} E_d \subset \mathbb{R}_+ \times \mathbb{R}^d$.

Exercise 2 (Heat equation on $\mathbb{R}^d$). Let $u_0 \in L^2(\mathbb{R}^d)$. We consider the homogeneous heat equation posed on the whole space $\mathbb{R}^d$:

$$\begin{cases} \partial_t u - \frac{1}{2}\Delta u = 0 & \text{on } (0,+\infty) \times \mathbb{R}^d, \\ u(0,\cdot) = u_0 & \text{on } \mathbb{R}^d. \end{cases} \tag{1}$$

1. (Regularity) Compute explicitly the solution of the equation (1). What is its regularity ?
2. (Energy estimate) Show that for all $t \geq 0$,
$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \|\nabla u(s,\cdot)\|^2_{L^2(\mathbb{R}^d)}\, \mathrm{d}s = \|u_0\|^2_{L^2(\mathbb{R}^d)}.$$
3. (Maximum principle) Show that if $u_0 \in L^\infty(\mathbb{R}^d)$, then $u(t,\cdot) \in L^\infty(\mathbb{R}^d)$ for all $t \geq 0$ and
$$\sup_{t\geq 0} \|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}.$$
4. (Infinite speed of propagation) Prove that if $u_0 \geq 0$ is a function not identically equal to zero and non-negative, then $u > 0$ in $\mathbb{R}_+ \times \mathbb{R}^d$.

Exercise 3 (Spectral theory). Let Ω be a bounded open subset of $\mathbb{R}^d$.

1. Explain why the operator $\Delta^{-1} : L^2(\Omega) \to H^1_0(\Omega)$ is a continuous isomorphism.
2. Let $\iota : H^1_0(\Omega) \to L^2(\Omega)$ be the canonical injection. Check that the operator $T = -\Delta^{-1} \circ \iota : H^1_0(\Omega) \to H^1_0(\Omega)$ is non-negative, selfadjoint, one to one and compact.
3. Deduce that the spectrum of the Laplacian operator $-\Delta$ with Dirichlet boundary condition is a sequence $(\lambda_n)_{n\geq 0}$ of positive real numbers which is increasing and diverges to $+\infty$, and also that there exists a Hilbert basis $(e_n)_{n\geq 0}$ of $H^1_0(\Omega)$ composed of eigenfunctions of $-\Delta$ and such that
$$\forall n \geq 0, \quad -\Delta e_n = \lambda_n e_n.$$
4. Compute explicitly those eigenvalues and those eigenfunctions when $d = 1$ and $\Omega = (0,1)$.

Exercise 4 (Heat equation on bounded domains). Let Ω be a bounded open subset of $\mathbb{R}^d$ with regular boundary, $T > 0$ be a final time, $u_0 \in L^2(\Omega)$ be an initial datum and $f \in L^2((0,T), L^2(\Omega))$ be a source term. We aim at proving that there exists a unique solution $u \in L^2((0,T), H_0^1(\Omega)) \cap C^0([0,T], L^2(\Omega))$ to the following heat equation with Dirichlet boundary conditions

$$\begin{cases} \partial_t u - \Delta u = f & \text{a.e. in } (0,T) \times \Omega, \\ u = 0 & \text{a.e. on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = u_0 & \text{a.e. in } \Omega. \end{cases} \tag{2}$$

We will also check that this solution satisfies the following energy estimate for all $0 \le t \le T$,

$$\|u(t,\cdot)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u(s,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}s \le C\bigg(\|u_0\|^2_{L^2(\Omega)} + \int_0^t \|f(s,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}s\bigg), \tag{3}$$

where $C > 0$ is a positive constant only depending on Ω. In the following, we consider $(e_n)_{n\ge 0}$ a Hilbert basis of $L^2(\Omega)$ composed of eigenfunctions of the operator $-\Delta$. Moreover, we set λ_n the eigenvalue associated with the eigenfunction e_n.

1. We first prove that there exists a unique $u \in L^2((0,T), H_0^1(\Omega)) \cap C^0([0,T], L^2(\Omega))$ satisfying

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\langle u(t,\cdot), v\rangle_{L^2(\Omega)} + \displaystyle\int_\Omega \nabla u(t,\cdot)\cdot\nabla v = \langle f(t,\cdot), v\rangle_{L^2(\Omega)} & \forall v \in H_0^1(\Omega), \forall t \in (0,T), \\ u(0,\cdot) = u_0. \end{cases}$$

   a) Define properly this variational formulation.
   b) Give the expansion in the Hilbert basis $(e_n)_{n\ge 0}$ of such a solution.
   c) Prove that this expansion converges in $L^2((0,T), H_0^1(\Omega))$ and also in $C^0([0,T], L^2(\Omega))$.
   d) Conclude.

2. We now want to prove that this weak solution u is a strong solution, that is, is solution of the problem (2).

   a) Check that the boundary condition and the initial value condition hold.
   b) * Prove that $\partial_t u - \Delta u = f$ a.e. in $(0,T) \times \Omega$.

3. When $f = 0$, check that

$$\forall t \ge 0, \quad \|u(t,\cdot) - \langle u_0, e_0\rangle_{L^2(\Omega)} e^{-\lambda_0 t} e_0\|_{L^2(\Omega)} \le e^{-\lambda_1 t}\|u_0\|_{L^2(\Omega)}.$$

Exercise 5 (Maximum principle). Let Ω be a bounded open subset of $\mathbb{R}^d$ with smooth boundary, $T > 0$ be a final time, $u_0 \in H_0^1(\Omega)$ be an initial datum and $f \in L^2((0,T), L^2(\Omega))$ be a a source term. We consider $u \in L^2((0,T), H_0^1(\Omega)) \cap C^0([0,T], L^2(\Omega))$ the unique solution of the problem (2). Prove that when $f \ge 0$ a.e. in $(0,T) \times \Omega$ and $u_0 \ge 0$ a.e. in Ω, then $u \ge 0$ a.e. on $(0,T) \times \Omega$. *Hint*: Admit that $\partial_t u \in L^2((0,T), L^2(\Omega))$ and $u \in L^2((0,T), H^2(\Omega)) \cap C^0([0,T], H_0^1(\Omega))$.

Application *: Assume now that $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $f \in L^\infty([0,+\infty) \times \Omega)$. Show that

$$\sup_{t\ge 0}\|u(t,\cdot)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + \frac{\mathrm{diam}(\Omega)^2}{2d}\sup_{t\ge 0}\|f(t,\cdot)\|_{L^\infty(\Omega)}.$$