Sheet 6: Lotka-Volterra system

Exercise 1. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a function of class C^1. Assume that the ODE

$$x'(t) = f(x(t)), \tag{1}$$

admits a Lyapunov function $V : \mathbb{R}^d \to \mathbb{R}$ of class C^1. Let us recall that by definition, V satisfies

$$\forall x \in \mathbb{R}^d, \quad \mathrm{d}V(x) \cdot f(x) \leq 0.$$

1. Let (x, I) be a solution of the equation (1). Check that the function $t \mapsto V(x(t))$ is non-increasing on I.
2. Let $x_0 \in \mathbb{R}^d$ be an equilibrium state of the equation (1), that is, satisfying $f(x_0) = 0$. We assume moreover that x_0 is a strict local minimum of the function V:

$$\exists r_1 > 0, \forall r \in]0, r_1[, \quad \alpha_r = \min_{|x-x_0|=r} V(x) > V(x_0).$$

   (a) Let $0 < r < r_1$. Prove that the set $U_r = \{x \in \mathbb{R}^d : V(x) < \alpha_r\} \cap B(x_0, r)$ is open and contains x_0, and that any trajectory starting from U_r stays in $B(x_0, r)$.
   (b) Prove that x_0 is a stable stationary point.

3. Keeping the same hypothesis as in the previous question, we assume moreover that

$$\forall x \in \mathbb{R}^d \setminus \{x_0\}, \quad \mathrm{d}V(x) \cdot f(x) < 0.$$

   We aim at proving that the point x_0 is asymptotically stable.

   (a) Let $0 < r < r_1$. Check that the flow $\phi_t(x)$ of (1) is well-defined on $\mathbb{R}_+ \times U_r$.
   (b) Let $x \in U_r$. We define the ω-limit set $\omega(x)$ of the point x as

$$\omega(x) = \big\{y \in \mathbb{R}^d : \exists (t_n)_{n\geq 0} \in \mathbb{R}^{\mathbb{N}},\ t_n \underset{n\to+\infty}{\to} +\infty,\ \phi_{t_n}(x) \underset{n\to+\infty}{\to} y\big\}.$$

   Check that for all $y \in \omega(x)$, the function $t \mapsto V(\phi_t(y))$ is constant on $\mathbb{R}_+$.
   (c) Check that $\omega(x) \subset \{y \in \mathbb{R}^d : \mathrm{d}V(y) \cdot f(y) = 0\}$ for all $x \in U_r$.
   (d) Prove that $\omega(x) = \{x_0\}$ for all $x \in U_r$, and conclude.

Exercise 2. Let $a, b, c, d > 0$ be positive real numbers and $x_0, y_0 \geq 0$ be non-negative real numbers. We consider the Lotka-Volterra system

$$\left\{ \begin{array}{rcl} x'(t) & = & x(a - by), \\ y'(t) & = & y(-c + dx), \end{array} \right. \tag{2}$$

with initial conditions $x(0) = x_0$ and $y(0) = y_0$.

1. Give an interpretation of this system in terms of sharks and sardines.
2. Prove that there exists a unique maximal solution defined on an open interval I of $\mathbb{R}$.
3. When $x_0 > 0$, check that $x(t) > 0$ for all $t \in I$. Similarly, prove that $y(t) > 0$ for all $t \in I$ under the assumption $y_0 > 0$.
4. We now assume that $x_0 > 0$ and $y_0 > 0$. By considering the function $H : \mathbb{R}^2 \to \mathbb{R}$ defined by

   $$H(x, y) = dx - c \ln x + by - a \ln y, \quad (x, y) \in (\mathbb{R}_+^*)^2,$$

   prove that the maximal solution of the system (2) is bounded. What to conclude?
5. What are the stationary points of the system (2) ? Study their stability.
6. Draw the phase portrait of this system.
7. How to model the influence of fishing?

Exercise 3. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. We also consider $a, b, c, d, \lambda > 0$ some positive real numbers and $u_0, v_0 \in L^2(\mathbb{R}^d)$ be smooth initial data satisfying $0 < u_0 < c_1$ and $0 < v_0 < c_2$, with $c_1, c_2 > 0$. We consider (u, v) the solution of the following Lotka-Volterra system

$$\left\{\begin{array}{rcll} \partial_t u - \lambda \Delta u & = & u(a - bv) & \text{in } (0, +\infty) \times \Omega, \\ \partial_t v - \lambda \Delta v & = & v(-c + du) & \text{in } (0, +\infty) \times \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} & = & 0 & \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) & = & u_0 & \text{in } \Omega, \\ v(0, \cdot) & = & v_0 & \text{in } \Omega. \end{array}\right.$$

We aim at proving that (u, v) tends to a spatially uniform state as $t \to +\infty$. To that end, we consider the energy s of the system without diffusion

$$s(t, x) = du(t, x) - c \ln u(t, x) + bv(t, x) - a \ln v(t, x), \quad (t, x) \in (0, +\infty) \times \Omega.$$

1. What is the equation satisfied by the energy s ?
2. We define the total energy S of the system at time t by

   $$S(t) = \int_\Omega s(t, x) \, \mathrm{d}x.$$

   Check that S is non-increasing.
3. Conclude.
4. How to interpret this result ?
5. By the way, does such a solution (u, v) exist?