École Normale Supérieure de Lyon | Année 2020 - 2021

Département de Mathématiques | M1 - EDP

Sheet 7: Travelling waves

In all this sheet, we consider the one-dimensional Fisher-KPP equation posed on the whole space

$$\partial_t u - \partial_{xx} u = u(1-u), \quad t > 0,\ x \in \mathbb{R}. \tag{1}$$

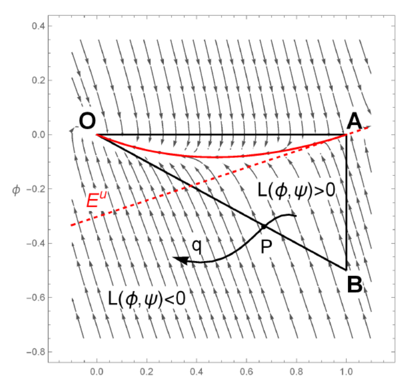
Exercise 1. First, we aim at proving that there are traveling waves solutions of the equation (1), that is, solutions of the form $u(t,x) = \phi(x-ct)$ for some function $\phi : \mathbb{R} \to [0,1]$ and $c \in \mathbb{R}$. Precisely, we are interested in traveling wavefronts, that is, satisfying $\lim_{+\infty} \phi = 0$ and $\lim_{-\infty} \phi = 1$.

1. Check that a traveling wave is solution of the equation (1) if and only if the wave profile ϕ satisfies the following ordinary equation,

$$\phi''(z) + c\phi'(z) + \phi(z)(1-\phi(z)) = 0, \quad z \in \mathbb{R}, \tag{2}$$

where $z = x - ct$ denotes the co-moving frame.

2. Write this equation as a two-dimensional system of first order equations.
3. Study the stationary points of this system.
4. Explain why such a traveling wave does not exist when $0 < c < 2$.
5. Assuming that $c \geq 2$, the purpose is to prove the existence of such a traveling wave with velocity c. We denote the origin of the phase space by O, the point $(1,0)$ by A and the point $(1,-b)$ by B as represented in the following draw (with $c = 3$), where $b > 0$ is to be chosen.



a) What is E^u in the above picture ?

b) Check that $b > 0$ can be chosen so that no orbit can leave the triangle OAB.
Hint: For the side OB, introduce the function $L(\phi, \psi) = b\phi + \psi$.

c) * By using the Poincaré-Bendixon theorem, prove that the equation (2) has a unique solution ϕ satisfying $\lim_{+\infty} \phi = 0$ and $\lim_{-\infty} \phi = 1$.

Exercise 2. We keep the notations introduced in Exercice 1 and assume that $c \geq 2$. In this exercice, we aim at determining the profile of the wave front ϕ. We make the change of variable $\xi = z/c$, so that ϕ satisfies the following ordinary equation:

$$\varepsilon \phi''(\xi) + \phi'(\xi) + \phi(\xi)(1 - \phi(\xi)) = 0, \quad \xi \in \mathbb{R}, \tag{3}$$

with $\varepsilon = 1/c^2$. We can expand ϕ in powers of ε:

$$\phi(\xi, \varepsilon) = \phi_0(\xi) + \varepsilon \phi_1(\xi) + \varepsilon^2 \phi_2(\xi) + \dots.$$

1. By substituting this expansion in (3) and splitting the different powers of ε, give the equations satisfied by the functions ϕ_0 and ϕ_1. We recall that $\lim_{+\infty} \phi = 0$ and $\lim_{-\infty} \phi = 1$.
2. Why can we choose $\phi(0) = 1/2$?
3. Solve the equations satisfied by ϕ_0 and ϕ_1, and deduce that

$$\phi(z, c) = \frac{1}{1 + e^{z/c}} + \frac{1}{c^2} \frac{e^{z/c}}{(1 + e^{z/c})^2} \ln\left(\frac{4e^{z/c}}{(1 + e^{z/c})^2}\right) + \mathcal{O}(c^{-4}).$$

Exercise 3. We keep the previous notations. The purpose is now to deal with the appearance of propagation speeds in the reality. Assume that the initial condition of the equation (1) is given by

$$u(0, x) = e^{-a|x|}, \quad x \in \mathbb{R},$$

where $a > 0$ is a positive constant.

1. By considering supersolutions of the form

$$\overline{u}(t, x) = e^{\pm s_a(x \pm c_a t)}, \qquad t > 0,\ x \geq 0,$$

   where $c_a > 0$ and $s_a > 0$ are positive constants depending on a, establish an estimate of the form

$$\forall t \geq 0, \forall x \in \mathbb{R}, \quad |u(t, x)| \leq e^{-s_a(|x| - c_a t)}.$$

   Hint: Consider the leading edge of the evolving wave where, since u is small, we can neglect u^2 in comparison with u.

2. Deduce that

$$\forall c > a + \frac{1}{a}, \quad \lim_{t \to +\infty} \sup_{|x| \geq ct} |u(t, x)| = 0, \quad \text{when } 0 < a < 1,$$

$$\forall c > 2, \qquad \lim_{t \to +\infty} \sup_{|x| \geq ct} |u(t, x)| = 0, \quad \text{when } a \geq 1.$$

3. Draw a picture, admitting that

$$\forall 0 < c < a + \frac{1}{a}, \quad \lim_{t \to +\infty} \sup_{|x| \leq ct} |1 - u(t, x)| = 0, \quad \text{when } 0 < a < 1,$$

$$\forall 0 < c < 2, \qquad \lim_{t \to +\infty} \sup_{|x| \leq ct} |1 - u(t, x)| = 0, \quad \text{when } a \geq 1.$$

   Remark: Those limits can be obtained by constructing adapted subsolutions.

4. Comment.