Tensor field aligned hex-dominant meshing

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Abstract. The finite element method (FEM) can be used to simulate physical loads on a 3d mesh. We investigate the following idea: from a FEM simulation can be extracted a tensor, giving the main stress directions. We want to produce a hexahedral mesh aligned with these directions. Such a mesh could be the basis for the visualization of a tensor field. It could also be used as a truss structure, that could be further optimized, resistant to the initial loads.

Keywords: Finite element method, hexahedral-dominant meshing, global parameterization

This reports refers to the work realized at the Technical University of Munich, under the supervision of Rudiger Westerman, from the 19/09/2016 to the 16/12/2016.

1 Introduction

The first problem studied during this internship is topology optimization. This domain is interested in an efficient use of materials. More precisely, topology optimization consists in finding the optimal distribution of a given amount of material, under design constraints and a physical model. The applications are numerous; for example, anything that has to endure loads, like parts of a vehicle, wind turbines or medical implants, can benefit from it. Topology optimization works by simulating the removal and redistribution of material in an object, seeking a given volume reduction and the minimization of a performance measure. An extensive review of this type of problems have been done by Bendse and Sigmund [1].

A domain that benefits from topology optimization is 3d printing. Printed objects often have to meet two antagonistic goals: being cheap to print (using as few material as possible) and meet physical constraints (solidity, equilibrium, etc, that require material). Topology optimization can be used to optimize the distribution of material inside a given body, thus achieving these two goals. However, to be practical, a fine resolution of the meshes is required. Time and memory consumption issues quickly arise on a desktop computer, limiting the spread of these techniques. GPU-based implementations are a remedy, and our first goal was to extend an already existing solver[8] with heat diffusion constraints.

The second and main problem studied during this internhsip is the search of tensor-field aligned hexahedral meshes. When running physical simulations on a body, the infinitesimal strain tensor can be extracted. From this tensor, the main stress directions can be computed. The visualization of the tensor would provide informations on how the body reacts to the external forces applied on it. The visualization can be done by computing and representing the remarkable parts of its topology. This method works well in 2d [9], but is incomplete in 3d [10]. Another idea is to plot streamlines following the stress directions [2]; this methods gives good visual results, but focuses only on the first stress direction, forgetting the secondary and the tertiary one.

The idea explored here is to visualize the three main directions (in the case of 3d meshes) by finding a hexahedral mesh aligned with the tensor field. The alignement of the hexahedral elements would follow the three directions. The individual sizes of the hexahedral elements would follow the norm of the tensor field. To achieve this, we looked into automatic hexahedral remeshing [3,4], and tried to modify the process to guide the hexahedral meshing process.

2 Topology Optimisation

2.1 The topology optimization problem

Topology optimization consists in finding the optimal distribution of a given amount of material in a body, such that it optimizes its behavior relatively to external constraints. A common example is trying to minimize the compliance, given a set of external loads.

Topology optimization is similar, albeit disctinct, from sizing optimization and shape optimization (see Figure 1, borrowed from [1] for a comparison). Example of sizing problems are finding the optimal thickness of a linearly elastic plate, or finding the optimal thicknesses of the members of a truss structure. Shape optimization consists in finding the optimal shape of a body, given a topology. Topology optimization demands finding the shape but also the distribution of holes and the connectivity of the domain.

2.2 Illustration with heat diffusion and resolution technique

We studied the following problem. A domain $\Omega \in \mathbf{R}^3$ of thermal conductivity $\underline{\kappa}$, uniformly heated by heat sources of density d, is given. Its boundary is made of two distinct parts: Ω_s, held at constant temperature T_s, and Ω_i, isolated. The domain can be filled with a material of conductivity $\overline{\kappa}$, up to a fraction r of its volume. The goal is to maximize the average temperature (at thermal equilibrium) of the domain. The density of material (a function $f : \Omega \to \{0,1\}$ giving the presence, or absence, of material) is to be determined. It is an optimization problem, that can be formulated this way:

$$\text{minimize: } f \to \int_\Omega T(v)dv \tag{1}$$

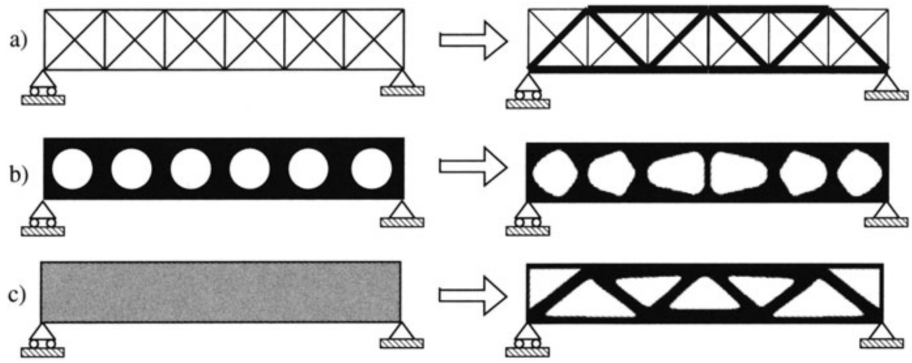


Fig. 1: Comparison between different structural optimization problems a) Example of a size optimization problem b) Example of a shape optimization problem c) Example of a topology optimization problem

with: - $f : \Omega \rightarrow \{0,1\}$ (2)

- $\int_{\Omega} f(v)dv \leq r \cdot \int_{\Omega} 1dv$ (3)

- $\kappa = \underline{\kappa} + f \cdot (\overline{\kappa} - \underline{\kappa})$ (4)

- $-\Delta \cdot (\kappa \Delta T) = d$ (5)

- $T = T_s$ on Ω_s (6)

- $(\kappa \Delta T) \cdot u = 0$ on Ω_i, u being the unit vector normal to the surface (7)

A first approximation used to solve this problem is to use a finite element method (FEM) discretization. The space is discretized in smaller elements (typically: using a regular grid). The smaller the resolution, the better the approximation.

The temperature is then supposed to be piecewise bilinear (trilinear in 3d) in each element. Shape functions are used to interpolate the temperature at any position in the body from the temperature at each node. For example, given a cubic axis-aligned element of size 1 (represented in figure 2), the temperature is computed this way:

$$T(x,y,z) = \sum_{0 \leq i < n} T_i \cdot N_i \quad (8)$$

with:

$N_0 : x,y,z \rightarrow (1-x)z(1-y)$ $\quad N_4 : x,y,z \rightarrow (1-x)zy$

$N_1 : x,y,z \rightarrow (1-x)(1-z)(1-y)$ $\quad N_5 : x,y,z \rightarrow (1-x)(1-z)y$

$N_2 : x,y,z \rightarrow xz(1-y)$ $\quad N_6 : x,y,z \rightarrow xzy$

$N_3 : x,y,z \rightarrow x(1-z)(1-y)$ $\quad N_7 : x,y,z \rightarrow x(1-z)y$

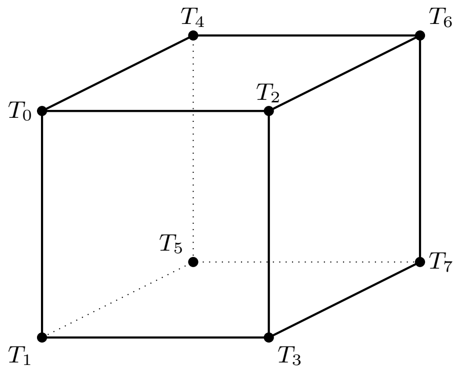


Fig. 2: A cubic FEM element

Then, a second approximation is to give the design variable continuous values ($f : \Omega \to [0,1]$). Having the integer constraint relaxed, this optimization problem becomes convex.

Intermediate values have to be dealt with later, for example by choosing a cutoff to get only 1 and 0. Also, to favor extreme values, the Solid Isotropic Material with Penalization (SIMP) model is commonly used. The density of material is taken int account differently when computing the thermal conductivity : $\kappa = \underline{\kappa} + f^p \cdot (\overline{\kappa} - \underline{\kappa})$). Intermediate values for f are penalized when $p > 1$, as they contribute less to the conductivity. The convexity of the problem is lost, however.

Let n be the number of elements. Let then $\vec{\kappa} = (\kappa_1, \kappa_2, \cdots, \kappa_n)$ be the thermic conductivity of each element. Let $\vec{f} = (f_1, f_2, \cdots, f_n)$ be the density of material in each element. Let $\vec{t} = (t_1, t_2, \cdots, t_n)$ and $\vec{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_n)$ be respectively the degrees of freedom of T and the shape functions of the elements, such that $T = \sum_{1 \leq i \leq n} t_i \cdot \varphi_i$. The optimization problen can then we written:

$$\text{minimize: } \vec{f} \to \int_{\Omega} T(v)dv$$

with:

- $f \in [0,1]^n$ (9)
- $\sum_{1 \leq i \leq n} f_i \leq r \cdot n$ (10)
- $\vec{\kappa} = \underline{\kappa} + \vec{f} \cdot (\overline{\kappa} - \underline{\kappa})$ (11)
- $-\Delta \cdot (\kappa \Delta T) = d$ (12)
- $T = T_s$ on Ω_s (13)
- $(\kappa \Delta T) \cdot u = 0$ on Ω_i, u being the unit vector normal to the surface (14)

We only gave an overview of how topology optimization works, and what it can achieve. It is possible to include material constraints (like regions with a forced presence of material). Other physical constraints can be studied and combined (vibrations, mechanical stress, pressure loads, etc). Solutions to some discretization-induced issues exist, like the checkboard pattern issue (visible in figure 3). Another example of topology optimization is given in figure 4. The initial, an intermediate, and the final step of the resolution are shown. For this example, a vertical force is applied at the bottom right corner of the object, and the compliance is minimized. A 40% volume reduction is seeked; the resolution is initialized by giving a density of material of .4 for every element.

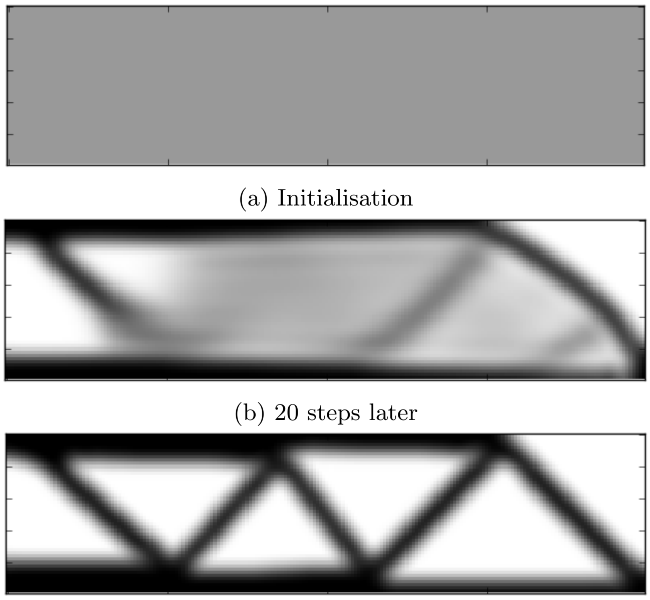
Fig. 3: Visualization of the solution for a topology optimzation problem on heat diffusion, using a 60 by 60 grid for the FEM simulation

The combination of topology optimization and heat diffusion have already been studied. One can refer to [5] for a study in 2d with a GPU implementation. Our goal was to study the problem in 3d, also targeting a GPU implementation, and eventually 3d printing some results. However, we moved to another topic after a while.

3 Tensor field aligned hexahedral mesh

3.1 Introduction

We consider here deformable bodies. Given an oject, with its physical properties, and a set of loads, the finite element method can be applied to compute the



(a) Initialisation

(b) 20 steps later

(c) Final design

Fig. 4: Topology optimization for compliance minimization

resulting displacement of the points of the object. The goal of these computation could be to minimize the compliance of the object, *i.e* the average squared displacement, by topology optimization. The result would be the most resistant object to the given load. We will try a different method to generate a resistant object.

Here, we are interested in the infinitesimal strain tensor ϵ that can be extracted from the displacement vector u:

$$\epsilon = \frac{1}{2}\left(\Delta u + (\Delta u)^{\mathrm{T}}\right) \tag{15}$$

This tensor is at every point a real and symetric matrix, and can thus be diagonalized. Its three eigenvectors give the main stress directions in the object at every point. The visualization of this tensor can provide valuable informations, like finding the most stressed areas, assessing the regularity of this tensor field or adjusting some parameters to achieve some properties. A simple visualization consists in taking the first stress direction, and plotting a set of streamlines alongside this direction. As an example, it have been used to steer orthopedic implants from a reference stress field [2] (see figure 5). This method is however incomplete, as only one of the stress directions is used.

Our objective was to produce, from a tetrahedral mesh and a set of loads, a hexahedral mesh aligned with the infinitesimal strain tensor. The objective is twofold. First, we wanted to investigate the possibility of visualizing a tensor

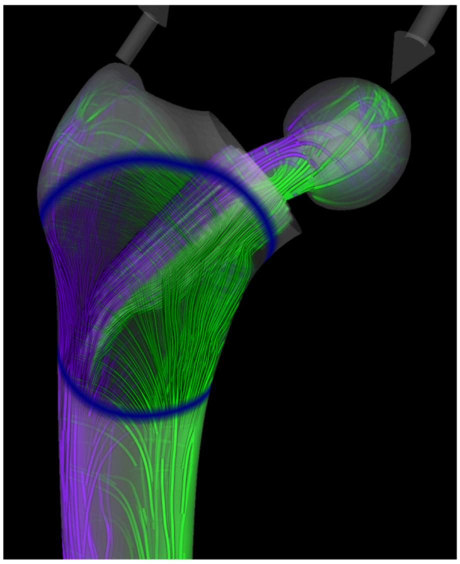


Fig. 5: Visualization of a simulated stress tensor fields for a human femur under load

through a hexahedral mesh. Second, we wanted to use the mesh as a basis for a structure resistant to the initial loads. This truss structure would be lightweight in amount of material, thus suited for 3d printing for example. Also, it could have some intersting properties, like being resistant to loads close to the original ones, something common topology optimization can not guarantee.

3.2 Parameterizations

We exploited ideas from automatic hexahedral remeshing [3] and hex-dominant remeshing [4], where a generic framework for defining hexahedral meshes is given. By defining a *parameterization* of an object, a continous function $f : \Omega \to \mathbb{R}^3$, one obtains a hexahedral meshing of the object by taking the pre-image of the integer grid $f^{-1}\left(\left\{(x,y,z) \in \mathbb{R}^3 | x \in \mathbb{Z} \vee y \in \mathbb{Z} \vee z \in \mathbb{Z}\right\}\right)$. With this vision, a hexahedral mesh is defined by iso-surfaces in the parameter space $\mathbb{R}^3$. Thus, by taking a parameterization whose gradient is close to a given tensor field, one obtains a hexadral mesh aligned with it.

This idea of finding a parameterization have to be refined. First, boundary counditions have to be enforced, so that the final mesh respects the boundaries of the initial one. For example, one can enforce the final boundary vertices to belong to the initial boundary surface. Second, this type of parameterization can not produce all hexahedral meshes, and some simple tensor field can produce distorted meshes. Figure 6 is an example of a mesh that can not be the pre-image of a continuous parameterization. The center of the object sees three lines joining on one point; such a pattern cannot occur if f is continuous.

From this point, we consider the base mesh to be a tetrahedral mesh (or "tet mesh"). To cover more final hexahedral meshes, the set of parameterizations can be extended to the set of function $\Omega \to \mathbb{R}^3$ continuous on each tetrahedron, respecting the following condition:

$$\text{for every adjacent tets } t \text{ and } s\text{: } f_{|t} = R_{t,s} \cdot f_{|s} + T_{t,s} \tag{16}$$

where $T_{t,s}$ is constant vector of $\mathbb{Z}^3$ and $R_{t,s}$ an element of the chiral cubical symmetry group (a rotation matrix mapping coordinate axes to coordinate axes). The set of transformations defined by the different $T_{t,s}$ and $R_{t,s}$ leaves the integer grid invariant, allowing a smooth transtion between every adjacent tetrahedra. In practice, we work with parameterizations that are linear on each tet (instead of being continuous).

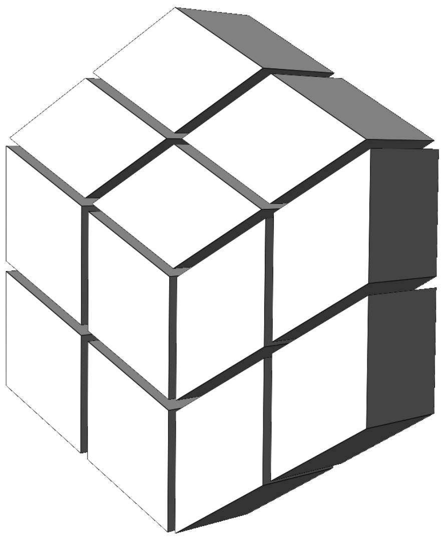


Fig. 6: Hexahedral mesh that can not correspond to a simple continuous parameterization

An example of transition is shown in figure 7 (in 2d). In each tetrahedron, the parameterization is represented. The origin (the point p verifying $f(p) = (0, 0)$) is plotted as a black dot. The integer lines parallel to (Ox) are plotted in blue (the points p verifying $f(p) \in \mathbb{R} \times \mathbb{Z}$). The integer lines parallel to (Oy) are plotted in red. The axis are plotted with arrows. Despite the parameterization being discontinuous at the frontier between the two tets, the integer grid is left continuous: each integer line on a tet joins another one of the other tet. In this example, by naming t the upper tet and s the lower one, the rotation matrix is:

$R_{t,s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$: going from s to t, (Ox) is mapped to $-(Oy)$ and (Oy) to (Ox). Also we can compute $T_{t,s} = \begin{pmatrix} \text{-3} \\ 1 \end{pmatrix}$ by identifying two points at the frontier.

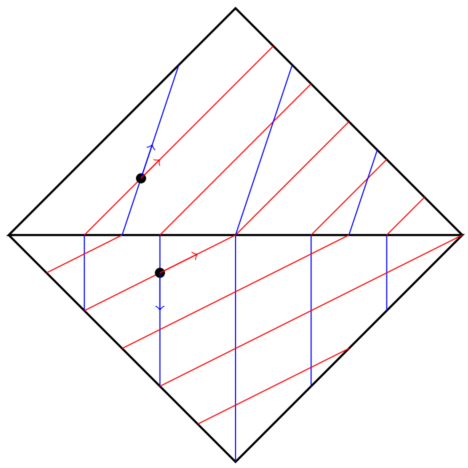


Fig. 7: Example of transition between two adjacent tets

3.3 Implementation

We tried to implement the idea described here. Our software pipeline consists of the following steps:

1. Run a finite element simulation on a given tet mesh with a given set of loads.
2. Extract the strain tensor ϵ from the simulation.
3. Run a hex-dominant remeshing program, replacing the automatically generated frame-field with the one extracted earlier.

The FEM simulation is realised using the Vega library [6]. The hex-dominant meshing algorithm, introduced in [4], is implemented by the Geogram [7] library. Geogram also includes a mesh visualization tool, run at the end of the pipeline to examine the results. Unfortunately, we didn't have enough time to get a working pipeline.

4 Conclusion

We explored two subjets during this internship. Topology optimization for heat diffusion was the first one. The goal was to get familiar with the finite element method and topology optimization. However, we didn't go to the final implementation. Then, we tried to explore the idea of producing a hexahedral mesh guided by a tensor field. However, we couldn't get a working implementation during the course of the internship, so it is not possible yet to verify the validity of this project.

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