Mathematics Internship

Pseudo-Holomorphic Curves in Symplectic Geometry

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1 Introduction

The study of symplectic structures was to a large extent motivated by classical mechanics. More precisely, by the Hamiltonian formalism of the classical mechanics. Take for exemple our solar system : the solutions of the equations ruling the movement of the planets are very hard to compute in an exact way. But by writing these equations in a Hamiltonian way, it appears that the maps of the associated flow preserve a specific geometric structure on the phase space, a structure that the mathematicians now call symplectic. Then, one can hope to get qualitative results on the systems coming from classical mechanics by studying the transformations which preserve the symplectic form on a space provided with such a form (called symplectic manifold) : this is the aim of the symplectic geometry. The first part of this work is dedicated to give a brief introduction to this subject, by given some motivations and questions for it.

In his paper ([Gro85]) of 1985, M. Gromov introduced the pseudo-holomorphic curves in symplectic manifolds. Since, this theory has revolutionized the study of symplectic geometry, making it possible to study in a very efficient way the global invariants of the theory. One of the key point about these curves is that they somehow connect the usual "Riemannian" area and what we could call the "symplectic area" (see Proposition 5 of this work). In order to get some interesting results in symplectic geometry and especially in order to answer the questions asked in the first part of this work, our strategy is to study the compactness properties of the specific space in which these curves occur, called the universal moduli spaces : this is what we do in the last part of this work. We will see that some important results can be stated about these spaces without never talking about symplectic (or even almost symplectic) structures ; however, the symplectic context will play an essential role in the compactness results.

Before studying the universal moduli spaces and get symplectic results from its compactness properties, we shall investigate the holomorphic properties held by pseudo-holomorphic curves : this is the purpose of the seond part of this work. We successively prove a result similar to the usual Schwarz lemma, a theorem about the removal of singularities which recalls the removable singularity of the holomorphic functions, and a generalized version of the Weierstrass theorem. Eventually, we prove Gromov's compactness theorem which sharply describe how some sequences of pseudo-holomorphic curves can converge : this may be the most important result proved in this work. Unfortunately, we would have to assume some very interesting results about the compactness properties of the hyperbolic structures.

I would like to thank professor Ko Honda, who guided me in a very interesting and pleasant way into this fascinating subject. I also thank Austin Christian, who read with me and more generally helped me whenever I had some issues in my understanding. It was a great pleasure to work with both of them.

2 Symplectic geometry

The aim of this part is to give a brief introduction to the symplectic geometry, by giving the reasons of its first introduction and two important questions in the theory.

2.1 Some motivations

Roughly speaking, the purpose of this section is to prove that each map ϕ_t of the flow generated by some Hamiltonian equations (which are the classical equations in mechanics) on a smooth manifold M preserves a certain 2-form ω on M, called symplectic form.

2.1.1 Classical mechanics

Consider a physical system whose configurations are described by points q in Euclidean space $\mathbb{R}^n$ which move along trajectories $q(t)$. Denote by

$$L : (t, q, v) \mapsto L_t(q, v) \in \mathbb{R}$$

the Lagrangian of the system, where we think of the element v of $\mathbb{R}^n$ as a tangent vector at the point q. A trajectory $\mathfrak{q}$ of the system satisfies the Hamilton's principle :

$$A_{t_1,t_2}(\mathfrak{q}) = \inf_{q(t_1),q(t_2)=\mathfrak{q}(t_1),\mathfrak{q}(t_2)} A_{t_1,t_2}(q)$$

where A denotes the action functional of the system :

$$A_{t_1,t_2}(q) = \int_{t_1}^{t_2} L_t(q(t), \dot{q}(t))dt$$

Proposition 1 (Euler-Lagrange equations)**.** *If $\mathfrak{q}$ satisfies the Hamilton's principle associated to a Lagrangian L, then $\mathfrak{q}$ satisfies the Euler-Lagrange equations :*

$$\frac{d}{dt}\frac{\partial L_t}{\partial v}(\mathfrak{q}(t), \dot{\mathfrak{q}}(t)) = \frac{\partial L_t}{\partial q}(\mathfrak{q}(t), \dot{\mathfrak{q}}(t))$$

where $\frac{\partial L_t}{\partial v}$ and $\frac{\partial L_t}{\partial q}$ stand for $(\frac{\partial L_t}{\partial v_1}, ..., \frac{\partial L_t}{\partial v_n})$ and $(\frac{\partial L_t}{\partial q_1}, ..., \frac{\partial L_t}{\partial q_n})$ respectively.

Proof. The idea is to write the fact that all the directional derivatives of A_{t_1,t_2} at $\mathfrak{q}$ must vanish. One could find a detailed proof of this result in [SM98] Lemma 1.1. □

Let L be the Lagrangian of a system satisfying the Legendre condition :

$$\det(\frac{\partial L}{\partial v_i \partial v_j}) \neq 0$$

We will see that the Legendre transformation produces a system of first-order differential equations in the variables :

$$q \in \mathbb{R}^n \text{ and } p = \frac{\partial L_t}{\partial v}(q, v) \in \mathbb{R}^n$$

According to the implicit function theorem, there exists a function :

$$\mathbf{v} : (t, q, p) \mapsto \mathbf{v}_t(q, p) \in \mathbb{R}^n$$

such that :

$$(p = \frac{\partial L_t}{\partial v}(q, v)) \Leftrightarrow (v = \mathbf{v}_t(q, p))$$

Then the Hamiltonian associated to the Lagrangian L is the function :

$$H : (t, q, p) \mapsto H_t(q, p) = \langle p, \mathbf{v}_t(q, p)\rangle - L_t(q, \mathbf{v}_t(q, p))$$

One can then easily check the following proposition.

Proposition 2 (Hamiltonian differential equations)**.** *Let L be the Lagrangian of a system satisfying the Legendre condition. Let H be the Hamiltonian associated to L. If $\mathfrak{q}$ satisfies the Euler-Lagrange equations, then the functions :*

$$\mathfrak{q} : t \mapsto \mathfrak{q}(t) \text{ and } \mathfrak{p} : t \mapsto \frac{\partial L_t}{\partial v}(\mathfrak{q}(t), \dot{\mathfrak{q}}(t))$$

satisfy the Hamiltonian differential equations :

$$\dot{\mathfrak{q}}(t) = \frac{\partial H_t}{\partial p}(\mathfrak{q}(t), \mathfrak{p}(t)), \quad \dot{\mathfrak{p}}(t) = -\frac{\partial H_t}{\partial q}(\mathfrak{q}(t), \mathfrak{p}(t))$$

Example (Kepler's problem)**.** *Consider a planet orbiting around the sun, or a (classical) point charge orbiting around a fixe centre. Then the Lagrangian is given by the difference between the kinetic and potential energies :*

$$L(q, v) = \frac{1}{2}|v|^2 + \frac{1}{|q|}$$

while the Hamiltonian is given by their difference :

$$H(q, p) = \frac{1}{2}|p|^2 - \frac{1}{|q|}$$

The corresponding Euler-Lagrange and Hamiltonian equations are nothing but these given by the Newton's law :

$$\frac{d^2q}{dt^2} = -\frac{q(t)}{|q(t)|^3}$$

In the coordinates :

$$a = (q, p) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$$

the Hamiltonian equations can be written in the form :

$$\dot{\mathfrak{a}}(t) = -J_0 \nabla H_t(\mathfrak{a}(t))$$

where ∇H_t denotes the gradient of H_t with respect to the standard inner product μ_0 on $\mathbb{R}^{2n}$, and J_0 denotes the standard complex structure on $\mathbb{R}^{2n}$ defined by :

$$J_0 x = J_0(x_1, x_2) = (-x_2, x_1)$$

where we think of :

$$x = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$$

as a tangent vector at some point a of $\mathbb{R}^{2n}$.

Observe that we can define a non-degenerate skew symmetric bilinear form ω_0 on $\mathbb{R}^{2n}$ by the formula :

$$\omega_0(x, y) = \mu_0(J_0 x, y)$$

The following theorem shows the crucial importance of ω_0 in the study of the Hamiltionan equations.

Theorem 1 (Poincaré's theorem). *If $(\phi_t)_t$ is a smooth family of diffeomorphisms of $\mathbb{R}^{2n}$ generated by some Hamiltonian differential equations via :*

$$\frac{d}{dt}\phi_t = -J_0 \nabla H_t \circ \phi_t, \quad \phi_0 = id$$

then ϕ_t preserves ω_0 for every t, i.e :

$$\phi_t^* \omega_0 = \omega_0$$

for every t, where $\phi_t^* \omega_0$ denotes the pullback of ω_0 by ϕ_t.

2.1.2 Generalization

The purpose of this paragraph is to state and prove a generalized version of the Poincaré theorem. First, we generalize the key structures which appeared in the last paragraph.

Definition. *Let M be a smooth manifold. We give the following definitions :*

1. *A Riemannian structure on M is a family $\mu = (\mu_a)_{a \in M}$, depending smoothly on a, where μ_a is an inner product on $T_a M$*
2. *An almost complex structure on M is a family $J = (J_a)_{a \in M}$, depending smoothly on a, where J_a is an endomorphism of $T_a M$ satisfying :*

$$J_a^2 = -\mathbb{1}$$

3. *An almost symplectic structure on M is a family $\omega = (\omega_a)_{a \in M}$, depending smoothly on a, where ω_a is a nondegenerate skew symmetric bilinear form on T_aM ; in other words, ω is a nondegenerate 2-form on M*

Observe that the dimension of an almost symplectic manifold is necessarily even because of the non-degeneracy condition.

We will say that :

1. An almost symplectic structure ω and an almost complex structure J are compatible if the family μ defined by :

$$\mu_a(x, y) = \omega_a(x, J_a y)$$

   is a Riemannian structure

2. An almost complex structure J and a Riemannian structure μ are compatible if the family ω defined by :

$$\omega_a(x, y) = \mu_a(J_a x, y)$$

   is an almost symplectic structure

3. An almost symplectic structure ω and a Riemannian structure μ are compatible if the family J defined by :

$$\omega_a(x, y) = \mu_a(J_a x, y)$$

   is an almost complex structure

This gives rise to the following important proposition.

Proposition 3. *If (M, ω) is an almost symplectic manifold, then there exists an almost complex structure on M compatible with ω.*

Proof. Let μ be any Riemannian metric on M, and choose an arbitrary point a in M. Let f be the endomorphism of T_aM defined by :

$$\mu_a(f(x), y) = \omega_a(x, y)$$

for x and y in T_aM. Since ω is non-degenerate, f is an automorphism of T_aM. By the skew symmetry of ω we get :

$$f^* = -f$$

for the μ-adjoint map f^* of f. Hence $(-f^2)$ is positive definite and symmetric with respect to μ ; let g be its unique positive square root. Then denote by J_a the endormorphism of T_aM defined by :

$$J_a = f \circ g^{-1}$$

and observe that the family $(J_a)_{a \in M}$ has the desired properties. $\square$

Now we generalize the Hamiltonian formalism.

Definition (Hamiltonian vector field). *Let (M, ω) be an almost symplectic manifold. A vector field X on M is said to be Hamiltonian if there exists a smooth function H on M and some compatible almost complex and Riemannian structures J, μ on (M, ω) such that :*

$$X(a) = -J_a \nabla H(a)$$

for all a in M, where ∇H is the gradient of H with respect to the Riemannian structure μ.

Using Proposition 3, one can easily check the following result, which gives an interesting characterization of the Hamiltonian vector fields.

Proposition 4. *Let (M, ω) be an almost symplectic manifold. A vector field X on M is Hamiltonian if and only if there exists a smooth function H on M such that :*

$$\iota_X \omega = dH$$

where ι_X denotes the interior product.

Consider an almost symplectic manifold (M, ω), and let $(\phi_t)_t$ be a smooth family of diffeomorphisms of M generated by a family $(X_t)_t$ of Hamiltonian vector fields on M. At this point, one could hope that ϕ_t will preserve ω for every t, but this is actually false. The key point to understand this is to write the relation :

$$\frac{d}{dt}(\phi_t^* \omega) = \phi_t^*(\mathcal{L}_{X_t} \omega)$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to X. Using the Cartan's formula :

$$\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X$$

and the previous proposition we get :

$$\frac{d}{dt}(\phi_t^* \omega) = \phi_t^*(\iota_{X_t} d\omega)$$

Then, it will be easy to show that the Poincaré theorem is generally false in an almost symplectic manifold as soon as we will give an example of an almost symplectic form which is not closed.

Example (A non-closed almost symplectic form). *The 6-sphere S^6, when considered as the set of unit norm imaginary octonions, inherits an almost complex structure J from the octonion multiplication. Using the proof of Proposition 3, we can easily show that there exists an almost symplectic structure ω on S^6 compatible with J. Assume that ω is closed in order to get a contradiction. Then it represents a cohomology class $[\omega]$ in $H^2(S^6)$. But $H^2(S^6)$ is trivial, and the non-degeneracy property of ω implies that the*

3-fold exterior power ω^3 of ω is a volume form on S^6. Because the integral of a closed differential form on a closed manifold only depends on its cohomology class, we have :

$$\mathrm{Vol}(S^6) = \int_{S^6} \omega^3 = \langle [\omega^3], [S^6] \rangle = \langle [\omega]^3, [S^6] \rangle = 0$$

where $\langle ., . \rangle$ denotes the duality bracket between $H^6(S^6)$ and $H_6(S^6)$, $[S^6]$ denotes the element of $H_6(S^6)$ representing by S^6, and $[\omega]^3$ denotes the 3-fold cup power of $[\omega]$. This is contradictory since the volume of S^6 is non-zero, and then ω is not closed.

The above discussion motivates the following definitions, which will eventually serve to generalize the previous section.

Definition (Symplectic structure)**.** *A symplectic structure on a smooth manifold M is an almost symplectic structure ω on M which is closed.*

Observe that the standard 2-form ω_0 defined in the previous paragraph is a symplectic structure on $\mathbb{R}^{2n}$. Indeed, it is easy to check that ω_0 is exact, given by :

$$\omega_0 = d\lambda_0 \text{ with } \lambda_0 = \sum_{i=1}^{n} x_i dy_i$$

Definition (Symplectic vector field)**.** *Let (M, ω) be a symplectic manifold. A vector field X on M is called symplectic if the 1-form $\iota_X \omega$ is closed.*

Observe that Proposition 4 implies that a Hamiltonian vector field is necessarily symplectic. It is now easy (using the discussion above) to state and prove a correct generalized version of the Poincaré theorem.

Theorem 2 (Generalized Poincaré's theorem)**.** *Let (M, ω) be a symplectic manifold. If $(\phi_t)_t$ is a smooth family of diffeomorphisms of M generated by a family $(X_t)_t$ of symplectic vector fields on M then ϕ_t preserves ω for every t.*

2.2 Some questions

In this section, we have choosen to expose two important questions in symplectic geometry, which could be related to qualitative issues in classical mechanics.

2.2.1 Uncertainty principle

Consider a Hamiltonian system which state is described by position coordinates q_i and momentum coordinates p_i ; by taking a measurement at a time t, we might find out that the state of the system lies somewhere in a subset of $\mathbb{R}^{2n}$, where n is the number of position coordinates. This subset certainly contains some r-ball, so that we have the following uncertainty relations :

$$\Delta q_i(t)\Delta p_i(t) \geq \frac{r^2}{2}$$

for all i. The question here is about knowing if the system could evolved so that one of the previous uncertainty relation vanish, making the precision on one of the couples (q_i, p_i) better. M. Gromov answers by the negative to this question in 1985, by proving its famous non-squeezing theorem. Indeed, the non-squeezing theorem can be thought of as saying that no matter how the system is transformed, the uncertainty relations will still hold for all i.

Now, we introduce the objects in order to state the non-squeezing theorem.

Definition (Symplectomorphism)**.** *A symplectomorphism from a symplectic manifold* (M_1, ω_1) *to another* (M_2, ω_2) *is a diffeomorphism* ϕ *from* M_1 *to* M_2 *such that :*

$$\phi^*\omega_2 = \omega_1$$

We denote by $B^{2n}(r)$ the Euclidean ball with centre 0 and radius r, and by :

$$Z^{2n}(R) = \mathbb{R}^{2n-2} \times B^2(R)$$

the symplectic cylinder with centre 0 and radius R. We can now state the non-squeezing theorem.

Theorem 3 (Non-squeezing theorem)**.** *If there exists a symplectic embeeding of* $B^{2n}(r)$ *into* $Z^{2n}(R)$*, then :*

$$r \leq R$$

This is a fundamental theorem in symplectic geometry which was first proved by M. Gromov by using the theory of pseudo-holomorphic curves. In the last part of this work, we will give a proof of a slightly generalized version of this theorem.

2.2.2 Invariants

Another basic question in geometry is about knowing what are the invariants of the theory. In classical mechanics, this question can be thought of as a wish to know what kind of objects may the maps of the flow of a Hamiltonian system preserves. The first result in this direction is the following Darboux's theorem (see [SM98] Theorem 3.15 for a proof) which claims that there is no local invariants in symplectic geometry.

Theorem 4 (Darboux's theorem)**.** *Any symplectic manifold* (M, ω) *is locally symplectomorphic to* $(\mathbb{R}^m, \omega_0)$*, where* m *denotes the dimension of* M*. In other words, for each point* a *in* M*, there exists a symplectomorphism between an open neighbourhood of* a *in* (M, ω) *and an open subset of* $(\mathbb{R}^m, \omega_0)$*.*

This theorem led the mathematicians to look for global invariants in symplectic geometry. We will see in the last part of this work that the non-squeezing theorem that we stated in the previous paragraph will precisely allow us to define a non-trivial and global symplectic invariant.

3 Pseudo-holomorphic curves

The aim of this part is to study some of the important properties of the pseudo-holomorphic curves.

Definition (Pseudo-holomorphic curve). *Let (M, J) be an almost complex manifold. A J-holomorphic curve in M is a smooth map σ from a Riemann surface (i.e a surface with a complex structure) (S, j) to (M, J) such that :*

$$T\sigma \circ j = J \circ T\sigma$$

where $T\sigma$ denotes the differential of σ ; we will say that σ is a J-holomorphic curve in M parametrized by (S, j). A pseudo-holomorphic curve in M is a J-holomorphic curve in M for some almost complex structure J on M.

We will see in the following that the area of these curves will play an essential role to generalize some holomorphic properties.

Definition. *Let S be an oriented surface. If ν is a Riemannian metric on S then we denote by α_ν the corresponding area form on S defined by :*

$$\alpha_\nu(\zeta, \eta) = \sqrt{\nu(\zeta, \zeta)\nu(\eta, \eta) - \nu(\zeta, \eta)^2}$$

for positively oriented couple (ζ, η) of vectors in some tangent space to S. Now if σ is a smooth map from S to a Riemannian manifold (M, μ) then we denote by $\mathcal{A}(\sigma)$ its area defined by :

$$\mathcal{A}(\sigma) = \int_S \alpha_{\sigma^*\mu}$$

If U is a open or closed subset of M then we will write $\mathcal{A}(\sigma \cap U)$ for $\mathcal{A}(\sigma_{|\sigma^{-1}(U)})$.

3.1 Holomorphic properties

The purpose of this section is to investigate the holomorphic properties of the pseudo-holomorphic curves, and then to prove Gromov's compactness theorem which will be crucial to find a non-trivial symplectic invariant in the last part of this work. In this section, we will consider an almost complex compact manifold (M, J) provided with a J-compatible Riemannian structure μ. We denote by ω the almost symplectic structure on M induced by J and μ, and by m the dimension of M.

3.1.1 Gromov-Schwarz lemma

The purpose of this paragraph is to generalized the Schwarz lemma to the case of pseudo-holomorphic curves.

Theorem 5 (Gromov-Schwarz lemma)**.** *There are non-negative constants ϵ_0, C_0 such that the following holds. If σ is a J-holomorphic curve in M parametrized by the Poincaré half-plane $\mathbb{H}$ and contained in some ϵ_0-ball, then :*

$$\|T\sigma\| = \sup_{a \in M} \|T_a \sigma\| \leq C_0$$

In order to prove this theorem, we will use some of the area properties of the pseudo-holomorphic curves, together with the following lemma which states a relation between isoperimetric inequalies and derivatives bounds.

Lemma 1. *Let $\mathcal{I}$ be a continuous map from $(0, \infty)$ to $(0, \infty)$ satisfying :*

$$\int_0^1 \left(\frac{1}{\mathcal{I}(t)^2} - \frac{1}{4\pi t}\right) dt < \infty \text{ and } \int_1^\infty \frac{dt}{\mathcal{I}(t)^2} < \infty$$

There exists a constant c such that the following holds. If σ is a conformal map from the unit disc D provided with some metric ν to (M, μ) such that :

$$\mathcal{I}(\mathcal{A}(\sigma_{|\bar{D}_r})) \leq l(\partial\sigma_{|\bar{D}_r})$$

for all r in $(0,1)$, then $T_0\sigma$ is bounded by c.

Proof. Let σ be as in the lemma and assume that $T_0\sigma$ is not zero. Denote by α, l, and ϵ the functions defined by :

$$\alpha(r) = \mathcal{A}(\sigma_{|\bar{D}_r}),\ l(r) = l(\partial\sigma_{|\bar{D}_r}) \text{ and } \epsilon(t) = \frac{1}{\mathcal{I}(t)^2} - \frac{1}{4\pi t}$$

for r in $(0,1)$ and t in $(0,\infty)$. The important idea of the proof is that the area of σ and its derivative are linked because σ is conformal. Indeed, saying that σ is conformal means that there exists a function λ from D to $\mathbb{R}$ such that :

$$\sigma^*\mu = \lambda\nu$$

and it is not difficult to see that λ equals $\|T\sigma\|^2$. In the usual polar coordinates, we then have :

$$\alpha(r) = \int\int_{\bar{D}_r} \|T\sigma\|^2 \rho d\rho d\theta$$

Under the assumptions of the lemma, the derivative of α can be estimated using the Cauchy-Schwarz inequality :

$$\alpha'(r) = \int_0^{2\pi} r\|T\sigma\|^2 d\theta \geq \frac{1}{2\pi r}\left(\int_0^{2\pi} \|T\sigma\| r d\theta\right)^2 = \frac{1}{2\pi r} l(r)^2 \geq \frac{1}{2\pi r}\mathcal{I}(\alpha(r))^2$$

Dividing by $\mathcal{I}(\alpha(r))^2$ and integrating this inequality, we get :

$$\phi(\alpha(r_2)) - \phi(\alpha(r_1)) \geq \frac{1}{2\pi}\log\left(\frac{r_2}{r_1}\right) \text{ for } 0 < r_1 < r_2 < 1$$

where ϕ is the primitive of $1/\mathcal{I}^2$ defined by :

$$\phi(s) = \int_1^s \frac{dt}{\mathcal{I}(t)^2} = \frac{1}{4\pi}\log(s) + \int_1^s \epsilon(t)dt$$

We now estimate :

$$\int_1^{\alpha(r_2)} \frac{dt}{\mathcal{I}(t)^2} = \int_1^{\alpha(r_1)} \frac{dt}{\mathcal{I}(t)^2} + \int_{\alpha(r_1)}^{\alpha(r_2)} \frac{dt}{\mathcal{I}(t)^2} \geq \frac{1}{4\pi}\log(\alpha(r_1)) + \int_1^{\alpha(r_1)} \epsilon(t)dt + \frac{1}{2\pi}\log(\frac{r_2}{r_1})$$

Then we have the following inequality, denoted in the following by $(*)$:

$$\int_1^{\alpha(r_2)} \frac{dt}{\mathcal{I}(t)^2} \geq \frac{1}{4\pi}\log(\frac{\alpha(r_1)}{\pi r_1^2}) + \frac{1}{2\pi}\log(r_2) + \frac{1}{4\pi}\log(\pi) + \int_1^{\alpha(r_1)} \epsilon(t)dt$$

Now observe that :

$$|\alpha(r) - \|T_0\sigma\|^2\pi r^2| = |\int\int_{D_r} (\|T\sigma\|^2 - \|T_0\sigma\|^2)\rho d\rho d\theta| \leq \sup_{z\in D_r} |\|T_z\sigma\|^2 - \|T_0\sigma\|^2|\pi r^2$$

so $(\alpha(r)/\pi r^2)$ converges to $\|T_0\sigma\|^2$ as r converges to 0. Taking the limit when r_1 converges to 0 and when r_2 converges to 1 in $(*)$, the assumptions on $\mathcal{I}$ imply the result. $\square$

The aim of the following lemmas is then to investigate the area properties of the pseudo-holomorhic curves. The key point of this study might be sum up in the following proposition.

Proposition 5. *If σ is a curve in M parametrized by a Riemann surface (S, j), then :*

$$\int_S \sigma^*\omega \leq \mathcal{A}(\sigma)$$

and the equality holds when σ is J-holomorphic.

Proof. Let (ζ, η) be a positively oriented couple of vectors in some tangent space of S. If $T\sigma(\zeta)$ equals 0 then we have :

$$\sigma^*\omega(\zeta, \eta) = 0 = \alpha_{\sigma^*\mu}(\zeta, \eta)$$

Thus assume that $T\sigma(\zeta)$ is not zero and denote by ζ', η' the vectors defined by :

$$\zeta' = \zeta \text{ and } \eta' = \eta - \frac{\sigma^*\mu(\zeta, \eta)}{\sigma^*\mu(\zeta, \zeta)}\zeta$$

Then ζ' and η' are orthogonal with respect to $\sigma^*\mu$ and we have the following equalities, denoted by $(*)$ in the following :

$$\sigma^*\omega(\zeta, \eta) = \sigma^*\omega(\zeta', \eta') \text{ and } \alpha_{\sigma^*\mu}(\zeta, \eta) = \sqrt{\sigma^*\mu(\zeta', \zeta')\sigma^*\mu(\eta', \eta')}$$

But since ω, J and μ are compatible, and by the Cauchy-Schwarz inequality, we have :

$$\begin{aligned}\sigma^*\omega(\zeta',\eta') &= \omega(T\sigma(\zeta'),T\sigma(\eta')) \\ &= \mu(JT\sigma(\zeta'),T\sigma(\eta')) \\ &\leq \sqrt{\mu(JT\sigma(\zeta'),JT\sigma(\zeta'))\mu(T\sigma(\eta'),T\sigma(\zeta'))} \\ &= \sqrt{\mu(T\sigma(\zeta'),T\sigma(\zeta'))\mu(T\sigma(\eta'),T\sigma(\eta'))} \\ &= \sqrt{\sigma^*\mu(\zeta',\zeta')\sigma^*\mu(\eta',\eta')}\end{aligned}$$

and it is easy to show that the equality holds if σ is J-holomorphic (use Cauchy-Schwarz inequality together with the fact that $j\zeta'$ and η' are colinear). Then the proposition follows from $(*)$. $\square$

We can now begin to prove the lemmas for the proof of the theorem.

Lemma 2 (Isoperimetric inequality). *There are non-negative constants ϵ_0, C such that the following holds. If σ is a compact J-holomorphic curve with boundary contained in a r-ball of M for some r in $(0,\epsilon_0)$, then :*

$$4\pi\mathcal{A}(\sigma) \leq (1+Cr)l(\partial\sigma)^2$$

The idea of the proof is the following. Assume that we are in the Euclidean case, i.e that (M,ω,J,μ) is $(\mathbb{R}^m,\omega_0,J_0,\mu_0)$. If σ is a compact J-holomorphic curve with boundary in $\mathbb{R}^m$, it follows from the previous proposition and the Stoke's theorem (because ω_0 is exact) that σ is area minimizing among all the parametrized curves with same boundary than σ. But area minimizing curves satisfy the classical isoperimetric inequality (see [Hum97] Annex A for a proof) which is "similar" to the one claimed in the lemma. To get closer to the Euclidean case, we are going to use the exponential map of M, of which we now give a definition.

Definition (Exponential map and injectivity radius). *In local coordinates of M, geodesics are solutions of an ordinary differential equation of second order. Thus, for any tangent vector u, there is a unique geodesic γ_u defined on an interval I_u such that :*

$$0 \in I_u \text{ and } \dot{\gamma_u}(0) = u$$

Observe that the set $\mathcal{U}$ defined by :

$$\mathcal{U} = \{u \in TM \mid 1 \in I_u\}$$

is an open neighbourhood of the zero section of TM, and that we can define the exponential map exp *on $\mathcal{U}$ by :*

$$\exp(u) = \gamma_u(1)$$

Then the injectivity radius of M at a point a is the number $\mathrm{injrad}(M,a)$ *defined by :*

$$\mathrm{injrad}(M,a) = sup\{r > 0 \mid \exp_{|B_a(0,r)} \text{ defines a diffeomorphism on its image}\}$$

where $B_a(0,r)$ denotes the ball with centre 0 and radius r in T_aM.

We can now give a proof of the previous lemma.

Proof. Let R denotes the injectivity radius of M, i.e :

$$R = \inf\{\mathrm{injrad}(M, a) \mid a \in M\}$$

Since M is compact, one can show (see [Hum97] Lemma I.1.1) that R is non-negative. We will denote by $\exp_a$ the restriction of the exponential map to the R-ball $B_a(0, R)$ of T_aM, where a is an arbitrary point of M fixed from now on. One can choose a basis of T_aM such that $(T_aM, \omega_a, J_a, \mu_a)$ is isomorphic to $(\mathbb{R}^m, \omega_0, J_0, \mu_0)$ (see [SM98] Theorem 2.3) ; in particular, ω_a is an exact symplectic form on T_aM. Since $\exp_a$ is a diffeomorphism from $B_a(0, R)$ to the ball $B(a, R)$ of M, every parametrized curve σ in $B(a, R)$ induces a parametrized curve σ_a in $B_a(0, R)$ defined by :

$$\sigma_a = \exp_a^{-1} \circ \sigma$$

We will show at the end of the proof that there is an ϵ_0 in $(0, R)$ and some non-negative constants C', C'' stricly smaller than $1/\epsilon_0$ such that :

$$l(\partial\sigma_a) \leq (1 + C'r)l(\partial\sigma) \tag{1}$$

when σ is a parametrized curve in $B(a, r)$ for some r in $(0, \epsilon_0)$, and such that :

$$\int \sigma^*\omega \leq \frac{1}{1 - C''r} \int \sigma_a^*\omega_a \tag{2}$$

when σ is besides J-holomorphic. Now, suppose that σ is a compact J-holomorphic curve with boundary in $B(a, r)$ for some r in $(0, \epsilon_0)$, and denote by S the surface which parametrizes σ. By the classical isoperimetric inequality for parametrized surfaces in Euclidean space, there exists a curve τ in $B(a, r)$ parametrized by S, with same boundary than σ and such that :

$$4\pi\mathcal{A}(\tau_a) \leq l(\partial\tau_a)^2 = l(\partial\sigma_a)^2$$

Hence we can write :

$$\begin{aligned}
4\pi\mathcal{A}(\sigma) &= 4\pi \int_S \sigma^*\omega \text{ by Proposition 5} \\
&\leq \frac{4\pi}{1 - C''r} \int_S \sigma_a^*\omega_a \text{ by (2)} \\
&= \frac{4\pi}{1 - C''r} \int_S \tau_a^*\omega_a \text{ by Stoke's theorem} \\
&\leq \frac{4\pi}{1 - C''r}\mathcal{A}(\tau_a) \text{ by Proposition 5} \\
&\leq \frac{1}{1 - C''r} l(\partial\sigma_a)^2 \text{ by the classical isoperimetric inequality} \\
&\leq \frac{(1 + C'r)^2}{1 - C''r} l(\partial\sigma)^2 \text{ by (1)}
\end{aligned}$$

and the lemma follows easily.

To conclude the proof of the lemma, it remains to show the inequalities (1) and (2). By a compactness argument which we postpone to the end of the proof, there is an ϵ_0 in $(0, R)$ and some non-negative constant C' such that the following holds. For any x in T_aM and any u in $B_a(0, \epsilon_0)$ we have :

$$\big|\|x\| - \|T_u \exp_a(x)\|\big| \leq C' d(a, \exp_a(u)) \|T_u \exp_a(x)\| \tag{3}$$

$$\big|\|x\|^2 - \|T_u \exp_a(x)\|^2\big| \leq C' d(a, \exp_a(u)) \|T_u \exp_a(x)\|^2 \tag{4}$$

$$\|Jx - (T_u \exp_a)^{-1} J T_u \exp_a(x)\| \leq C' d(a, \exp_a(u)) \|x\| \tag{5}$$

After making ϵ_0 smaller, we may addiitionally assume that $(1 - C'\epsilon_0)$ is non-negative. Using this, we get a non-negative constant C'' depending only on (M, J, μ) and ϵ_0 such that :

$$1 - C'' d(a, \exp_a(u)) \leq (\exp_a^{-1})^* \omega_a(x, Jx) \tag{6}$$

for any unitary vector x in T_aM and any u in $B_a(0, \epsilon_0)$. Namely, we have :

$$\begin{aligned}
(\exp_a^{-1})^* \omega_a(x, Jx) &= \omega_a((T_u \exp_a)^{-1} x, (T_u \exp_a)^{-1} Jx) \\
&= \mu(J (T_u \exp_a)^{-1} x, (T_u \exp_a)^{-1} Jx) \\
&= \mu(Jx, Jx) + \mu(J (T_u \exp_a)^{-1} x - (T_u \exp_a)^{-1} Jx, (T_u \exp_a)^{-1} Jx) \\
&+ \mu((T_u \exp_a)^{-1} Jx, (T_u \exp_a)^{-1} Jx) - \mu(Jx, Jx)
\end{aligned}$$

Using the inequalities (3) and (5), we estimate :

$$\begin{aligned}
|\mu(J(T_u \exp_a)^{-1} x - &(T_u \exp_a)^{-1} Jx, (T_u \exp_a)^{-1} Jx)| \\
&\leq C' d(a, \exp_a(u)) \|(T_u \exp_a)^{-1} x\| \|(T_u \exp_a)^{-1} Jx\| \\
&\leq C' d(a, \exp_a(u)) (1 + C')^2 d(a, \exp_a(u))^2
\end{aligned}$$

Furthermore, we deduce with (4) that :

$$\begin{aligned}
|\mu((T_u \exp_a)^{-1} Jx, &(T_u \exp_a)^{-1} Jx) - \mu(Jx, Jx)| \\
&= \left| \|(T_u \exp_a)^{-1} Jx\|^2 - \|Jx\|^2 \right| \\
&\leq C' d(a, \exp_a(u))
\end{aligned}$$

Then (6) follows together with :

$$\mu(Jx, Jx) = 1$$

We may assume that ϵ_0 is small enough for $(1 - C''\epsilon_0)$ to be non-negative. Now, inequality (1) follows from (3) and (2) follows from (6).

Eventually, it remains to show the inequalities (3), (4) and (5). Let U be a ball in M with radius $R/2$, and let U' be the ball in M with same center than U and radius $R/4$. We trivialize TU with respect to an orthonormal basis field on U. Let V be the closed ball in $\mathbb{R}^m$ with center 0 and radius $R/4$, and let E denote the restriction to $U' \times V$ of the exponential map with respect to the chosen trivialization. Then, for (a, u) in $U' \times V$ and x in $\mathbb{R}^m$, we see that :

$$\begin{aligned} |\|x\| - \|D_2E(a,u)(x)\|| &\leq \|\mathbb{1} - D_2E(a,u)(x)\|\|x\| \\ &\leq C'\|u\|\|x\| = C'd(a, E(a,u))\|x\| \end{aligned}$$

for some non-negative constant C' since $D_2E(a, 0)(x)$ is the identity (here D_2 denotes the differential with respect to the second factor). Now, cover M with finetely many open $R/4$-balls in order to deduce that :

$$|\|x\| - \|T_u \exp_a(x)\|| \leq C'd(a, \exp_a(u))\|x\| \tag{7}$$

In the same way, one shows (3), (4) and (5). To that end, note that there exists some constant c greater than 1 such that :

$$\frac{1}{c}\|x\| \leq \|D_2E(b,u)(x)\| \leq c\|x\|$$

Then (3) is an immediate consequence of (7), and (4) follows together with :

$$\left|\|x\|^2 - \|D_2E(a,u)(x)\|^2\right| = |\|x\| - \|D_2E(a,u)(x)\|| \left(\|x\| + \|D_2E(a,u)(x)\|\right)$$

Arguing as in the inequality before (7), one proves (5). □

The next lemma can be seen as a consequence of the previous ; it will be usefull in the following.

Lemma 3 (Monotocity lemma)**.** *There are non-negative constants ϵ_0, C_M such that the following holds. If σ is a compact J-holomorphic curve in M whose boundary is contained in the complement of a r-ball $B(a, r)$, with a in σ and r in $(0, \epsilon_0)$, then :*

$$\mathcal{A}(\sigma \cap \bar{B}(a, r)) \geq C_M r^2$$

Proof. The idea of the proof is the following. Let ϵ_0 be the constant of the previous lemma, and let σ be a compact J-holomorphic curve in M whose boundary is contained in the complement of a r-ball $B(a, r)$, with a in σ and r in $(0, \epsilon_0)$. Denote by σ_t the restriction of σ to $\sigma^{-1}(B(a, t))$. Then, we will show that there exists a compact subset K_0 of $(0, r)$ of measure 0 such that the function :

$$\alpha : t \in [0, r] \mapsto \mathcal{A}(\sigma_t)$$

is differentiable on $(0, r) \setminus K_0$ and satisfies :

$$\alpha'(t) \geq l(\partial\sigma_t)$$

Using the isoperimetric inequality satisfied by the J-holomorphic curves, it follows that :

$$\alpha'(t) \geq 2\kappa\sqrt{\alpha(t)}$$

for some non-negative constant κ depending only on (M, J, μ) and ϵ_0. Since K_0 is a closed set of measure 0, this implies that :

$$\sqrt{\alpha(r)} \geq \kappa r$$

and hence the claim.

It remains to show the existence of a set K_0 as above. We may assume without loss of generality that the surface S which parametrizes σ is connected. Consider the maps :

$$\rho : b \in M \mapsto d(a, b) \text{ and } \phi = \rho \circ \sigma$$

Since :

$$\rho \circ \exp_a(u) = \|u\|$$

for u in $B_a(0, \epsilon_0)$, ϕ is smooth on a neighbourhood of $\phi^{-1}(0, r]$ in S. Notice also that $[0, r]$ is contained in the image of ϕ because S is assumed to be connected. Let K be the compact of S defined by :

$$K = \{z \in \phi^{-1}(0, r] \mid T_z\phi = 0\} \cup \phi^{-1}(0)$$

Then the set K_0 defined by :

$$K_0 = \phi(K)$$

is a compact subset of $[0, r]$ and has measure zero by Sard's theorem. If t_0 is an element of $(0, r) \setminus K_0$, then there exists an open neighbourhood I_{t_0} of t_0 in $(0, r)$ which does not contain any critical value of ϕ, since K_0 is compact. Consequently, the sets $\phi^{-1}(t)$ for t in I_{t_0} are 1-dimensional submanifolds of $S \setminus \partial S$. The differential of any J-holomorphic curve either has rank 0 or 2. Hence, there are no critical points of σ in $\phi^{-1}(I_{t_0})$. Thus $\sigma^*\mu$ is non-degenerate on the neighbourhood $\phi^{-1}(I_{t_0})$ of $\phi^{-1}(t_0)$, so the gradient $\nabla\phi$ of ϕ with respect to $\sigma^*\mu$ is well defined on $\phi^{-1}(I_{t_0})$ by the identity :

$$\iota_{\nabla\phi}(\sigma^*\mu) = T\phi$$

Note that :

$$0 < \|\nabla\phi\| \leq 1$$

since $\|T\rho\|$ equals 1. For non-negative δ so small that $[t_0 - \delta, t_0 + \delta]$ is contained in I_{t_0}, a diffeomorphism Φ is defined from $\phi^{-1}(t_0) \times [t_0 - \delta, t_0 + \delta]$ to $\phi^{-1}([t_0 - \delta, t_0 + \delta])$ as the solution of the ordinary differential equation :

$$\Phi(., t_0) = id_{\phi^{-1}(t_0)} \text{ and } \frac{\partial\Phi}{\partial t}(z, t) = \frac{\nabla\phi}{\|\nabla\phi\|^2} \circ \Phi(z, t)$$

Then Φ satisfies :

$$\Phi(\phi^{-1}(t_0) \times \{t\}) = \phi^{-1}(t)$$

for t in $[t_0 - \delta, t_0 + \delta]$. We can now prove that α is differentiable at t_0 since :

$$\alpha(t_0 + t) - \alpha(t_0) = \int_{\phi^{-1}[t_0, t_0+t]} \alpha_{\sigma^*\mu} = \int_{\phi^{-1}(t_0)\times[t_0,t_0+t]} \Phi^*\alpha_{\sigma^*\mu}$$

for t smaller than δ, and we can give the estimate :

$$\begin{aligned}
\alpha'(t_0) &= \lim_{t\to 0} \frac{1}{|t|} \int_{\phi^{-1}(t_0)\times[t_0,t_0+t]} \Phi^*\alpha_{\sigma^*\mu} \\
&\geq \lim_{t\to 0} \frac{1}{|t|} \int_{\phi^{-1}(t_0)\times[t_0,t_0+t]} (\|\nabla\phi\| \circ \Phi)\Phi^*\alpha_{\sigma^*\mu} \text{ since } \|\nabla\phi\| \leq 1 \\
&= \lim_{t\to 0} \frac{1}{t} \int_{t_0}^{t_0+t} l(\sigma_{|\phi^{-1}(t')})dt' = l(\sigma_{|\phi^{-1}(t_0)})
\end{aligned}$$

By assuption, $\sigma(\partial S)$ is contained in the complement of $\bar{B}(a, t_0)$ so that $\phi^{-1}([0, t_0])$ is a submanifold of S with smooth boundary $\phi^{-1}(t_0)$. Hence we have :

$$\sigma_{|\phi^{-1}(t_0)} = \partial\sigma_{t_0}$$

and this concludes the proof. $\square$

Lemma 2 stated an isoperimetric inequality fo compact J-holomorphic curves with boundary in sufficiently small balls. The following lemma shows that the same isoperimetric inequality holds provided only the area of the J-holomorphic curve is sufficiently small.

Lemma 4. *There are non-negative constants ϵ_1, C_1 such that the following holds. Let σ be a compact J-holomorphic curve in M with connected boundary. If the area of σ is smaller than ϵ_1, then :*

$$4\pi\mathcal{A}(\sigma) \leq (1 + C_1 l(\partial\sigma))l(\partial\sigma)^2$$

Proof. Let ϵ_0 and C_M be the constants of the monotocity lemma. Let σ be a compact J-holomorphic curve in M with connected boundary. Assume that the area of σ is smaller than

$$\epsilon_1 = C_M\epsilon_0^2$$

Then for any a in σ we have :

$$d(a, \partial\sigma) \leq \sqrt{\frac{\mathcal{A}(\sigma)}{C_M}} < \epsilon_0$$

because of the monotocity lemma. Appliying Lemma 2 we get :

$$d(a, \partial\sigma) \leq \kappa l(\partial\sigma)$$

for any a in σ and some constant κ depending only on (M, J, μ) and ϵ_0. Since $\partial\sigma$ is connected, we can give the following estimate for the diameter $\delta(\sigma)$ of σ :

$$\delta(\sigma) \leq 2(1 + \kappa)l(\partial\sigma)$$

Together with Lemma 2 we get :

$$4\pi\mathcal{A}(\sigma) \leq (1 + C_1 l(\partial\sigma))l(\partial\sigma)^2 \text{ with } C_1 = 2C(1+\kappa)$$

□

Eventually, we prove a last lemma which gives area estimates for compact J-holomorphic curves with boundary in some specific open subsets of M.

Lemma 5. *Let U be an open subset of M on which there exists a 1-form λ whose norm is bounded by some positive constant $C(\lambda)$ and whose exterior derivative satisfies :*

$$d\lambda(x, Jx) \geq \mu(x, x)$$

for any x in TU. Then :

$$\mathcal{A}(\sigma) \leq C(\lambda)l(\partial\sigma)$$

for any compact J-holomorphic curves in U with boundary.

Proof. This lemma is an immediate consequence of Stoke's theorem :

$$\mathcal{A}(\sigma) = \int_S \alpha_{\sigma^*\mu} \leq \int_S \sigma^* d\beta \leq \int_{\partial S} \sigma^*\beta \leq C(\lambda)l(\partial\sigma)$$

where S is the Riemann surface which parametrizes σ. □

Now, the proof of the Gromov-Schwarz lemma can be concluded using the previous lemmas.

Proof. Let ϵ_0 be the constant of Lemma 2, and let a be a point of M fixed from now on. We saw in the proof of Lemma 2 that there exists a basis on T_aM so that $(T_aM, \omega_a, J_a, \mu_a)$ is isomorphic to $(\mathbb{R}^m, \omega_0, J_0, \mu_0)$. This shows that ω_a is an exact symplectic form on T_aM given by :

$$\omega_a = d\lambda_a$$

On $B_a(0, \epsilon_0)$, the 1-form λ_a is bounded by a constant depending only on ϵ_0. From the proof of Lemma 2, especially the inequalities (3) and (6), we see that the 1-form $(\exp_a^{-1})^*\lambda_a$ on $B(a, \epsilon_0)$ is bounded by some constant depending only on (M, J, μ) and ϵ_0. Moreover, there is a constant κ, also depending only on (M, J, μ) and ϵ_0, such that the 1-form λ on $B(a, \epsilon_0)$ defined by :

$$\lambda = \kappa(\exp_a^{-1})^*\lambda_a$$

satisfies the assuptions of Lemma 5.

Now, let ϵ_1, C_1 be the constants of Lemma 4. Denote by ψ the strictly monotically increasing function given for l in $(0, \infty)$ by :

$$\psi(l) = \frac{(1 + C_1 l)l^2}{4\pi}$$

Then define :

$$\begin{aligned}\mathcal{I}(t) &= \psi^{-1}(t) \text{ if } t < \frac{\epsilon_1}{2} \\ &= \chi(t) \text{ if } \frac{\epsilon_1}{2} \leq t \leq \epsilon_1 \\ &= \frac{t}{C(\lambda)} \text{ if } t > \epsilon_1\end{aligned}$$

for t in $(0, \infty)$, where χ is chosen so that :

$$\chi(t) \leq \max\{\frac{t}{C(\lambda)}, \psi^{-1}(t)\}$$

and $\mathcal{I}$ becomes continuous. It is not difficult to see that $\mathcal{I}$ satisfies the assumptions of Lemma 1 for all J-holomorphic curves in $B(a, \epsilon_0)$ parametrized by a closed disc. Hence, we get an upper bound C_0 for $\|T_0\sigma\|$ depending only on $B(a, \epsilon_0)$, J, μ, and $C(\lambda)$. Moreover, with respect to the Poincaré metric on D, there is an orientation preserving isometry mapping any given z in D to 0. Thus, the same estimate holds for the differential at any point of D with respect to the Poincaré metric. Since the open unit disc with the Poincaré metric is isometric to $\mathbb{H}$, this concludes the proof of the Gromov-Schwarz lemma. □

3.1.2 Removal of singularities

The purpose of this paragraph is to prove the following theorem.

Theorem 6. *Let S be a Riemann surface, z an interior point of S and σ a relatively compact J-holomorphic curve in M parametrized by $S \setminus \{z\}$. If there exists a neighboorhood U of z such that $\sigma_{|U\setminus\{z\}}$ has finite area, then σ can be extended to a J-holomorphic curve parametrized by S.*

Observe that the statement is purely local, so it is sufficient to prove the theorem in the case where S is the open unit disc D of $\mathbb{C}$ and where z equals 0. Then, the proof has essentially two steps, which consist in the two following lemmas.

Lemma 6. *Let σ be a relatively compact J-holomorphic curve in M parametrized by $D \setminus \{0\}$. If there exists a neighboorhood U of 0 such that $\sigma_{|U\setminus\{0\}}$ has finite area, then σ has a continuous extension to D.*

Proof. Because the area of σ is finite, there exists a sequence $(z_n)_n$ in $D \setminus \{0\}$ such that :

$$\lim_{n\to\infty} z_n = 0 \text{ and } \lim_{n\to\infty} l(\partial\sigma_{|\bar{D}_{|z_n|}}) = 0$$

Otherwise, there would exists a non-negative constant ϵ and an element r_0 of $(0, 1)$ such that :

$$l(\partial\sigma_{|\bar{D}_r}) \geq \epsilon$$

for all r smaller than r_0 and this would yield that :

$$\begin{aligned}
\mathcal{A}(\sigma_{|D_{r_0}\setminus\{0\}}) &= \int\int_{D_{r_0}\setminus\{0\}} \|T\sigma\|^2 \rho d\rho d\theta \\
&\geq \int_0^{r_0} \frac{1}{2\pi\rho}\left(\int_0^{2\pi} \|T\sigma\|\rho d\theta\right)^2 d\rho \\
&\geq \frac{1}{2\pi}\int_0^{r_0} \frac{l(\partial\sigma_{|\bar{D}_\rho})^2}{\rho} d\rho \\
&\geq \frac{\epsilon^2}{2\pi}\int_0^{r_0} \frac{d\rho}{\rho} = \infty
\end{aligned}$$

Since $\sigma(D\setminus\{0\})$ is relatively compact in M, we may suppose that $(\sigma(z_n))_n$ converges to some point a in M. Assume there is another sequence $(z'_n)_n$ in $D\setminus\{0\}$ such that :

$$\lim_{n\to\infty} z'_n = 0 \text{ and } \lim_{n\to\infty} \sigma(z'_n) = a' \neq a$$

Without loss of generality, we may suppose that :

$$|z_n| > |z'_n| > |z_{n+1}|$$

for all n. Now, denote by ρ the distance between a and a' and choose an integer N such that $d(\sigma(z_n), a)$, $d(\sigma(z'_n), a')$ and $l(\partial\sigma_{|\bar{D}_{|z_n|}})$ are smaller than $d(a, a')/8$ when n is greater than N. Then for each n greater than N and each k in $\{n, n+1\}$ we have :

$$B(\sigma(z'_n), \frac{\rho}{2}) \cap \partial\sigma_{|\bar{D}_{|z_k|}} = \emptyset$$

Since $\sigma(D\setminus\{0\})$ is relatively compact in M, the monotocity lemma implies that there exists a non-negative constant c such that :

$$\mathcal{A}(\sigma_{|\bar{D}_{|z_n|}\setminus D_{|z_{n+1}|}}) \geq c$$

for each n greater than N. Hence, the area of σ is non-finite and this is in contradiction with our assumption. Thus σ can be continuously extended to D by defining :

$$\sigma(0) = a$$

□

Before stating the second lemma, we need to talk a little bit about the jets of J-holomorphic curves.

Definition (1-jets of J-holomorphic curves)**.** *Let (S, j) be a Riemann surface. We denote by $[S, M]$ the total space of a complex vector bundle over $S\times M$ defined by :*

$$[S, M] = \bigcup_{(z,a)\in S\times M} \mathrm{Hom}_{\mathbb{C}}(T_zS, T_aM)$$

Assume that σ is a J-holomorphic curve in M parametrized by an open subset V of S. Then the map $\sigma^{(1)}$ defined from V to $[S, M]$ by :

$$\sigma^{(1)}(z) = T_z\sigma$$

is called the 1-jet of σ.

The following proposition (which is proved in [Hum97] Proposition III.1.4) shows that one can force the 1-jets of pseudo-holomorphic curves to be themselves pseudo-holomorphic.

Proposition 6. *There exists an almost complex structure $J^{(1)}$ on $[S, M]$ such that the 1-jet $\sigma^{(1)}$ of any J-holomorphic curve σ is itself $J^{(1)}$-holomorphic.*

Then, the same can be done for higher order derivatives.

Definition (n-jets of J-holomorphic curves)**.** *Let (S, j) be a Riemann surface. We inductively define an almost complex manifold $([S, M]^{(n)}, J^{(n)})$ by :*

$$([S, M]^{(0)}, J^{(0)}) = (M, J) \text{ and } ([S, M]^{(n+1)}, J^{(n+1)}) = ([S, [S, M]^{(n)}], (J^{(n)})^{(1)})$$

Assume that σ is a J-holomorphic curve in M parametrized by an open subset V of S. We define inductively a $J^{(n)}$-holomorphic curve $\sigma^{(n)}$ in $[S, M]^{(n)}$, called the n-jet of σ, by :

$$\sigma^{(0)} = \sigma \text{ and } \sigma^{(n+1)} = (\sigma^{(n)})^{(1)}$$

We can now state the second lemma that we need to prove the theorem on the removal of singularities.

Lemma 7. *Let σ be a relatively compact J-holomorphic curve in M parametrized by $D \setminus \{0\}$. If there exists a neighboorhood U of 0 such that $\sigma_{|U\setminus\{0\}}$ has finite area, then $\sigma^{(1)}$ is relatively compact and there exists a neighboorhood $U^{(1)}$ of 0 such that $\sigma^{(1)}|_{U^{(1)}\setminus\{0\}}$ has finite area.*

Proof. We may assume without loss of generality that U is in fact D. Using the Gromov-Schwarz lemma, we can then show that the differential $T\sigma$ of σ is bounded on $D_{1/2} \setminus \{0\}$. This implies the existence of a compact subset K in $[D, M]_{(0,\sigma(0))}$ such that for any neighbourhood W of K in $[D, M]$ there exists a neighbourhood V of 0 in D with :

$$\sigma^{(1)}(V \setminus \{0\}) \subset W$$

We choose a metric $\mu^{(1)}$ on $[D, M]$ compatible whith $J^{(1)}$, and view $[D, M]_{(0,\sigma(0))}$ as a submanifold of $[D, M]$. Additionaly, we require that the induced metric on $[D, M]_{(0,\sigma(0))}$ comes from a scalar product on $[D, M]_{(0,\sigma(0))}$. Denote by $\mathcal{N}$ the normal bundle of $[D, M]_{(0,\sigma(0))}$, and by $\exp^{\mathcal{N}}$ the restriction of the exponential map of $([D, M], J^{(1)}, \mu^{(1)})$ to some neighbourood of the zero section of $\mathcal{N}$ in $\mathcal{N}$. Together with $T[D, M]_{(0,\sigma(0))}$, the vector bundle $\mathcal{N}$ is also a complex subbundle of $T[D,M]|_{[D,M]_{(0,\sigma(0))}}$ since $\mu^{(1)}$ is compatible with $J^{(1)}$. Hence we can choose an orthonormal complex basis field $X_1,...,X_{m+1}$ of sections of $\mathcal{N} \to [D,M]_{(0,\sigma(0))}$ since $[D,M]_{(0,\sigma(0))}$ is contractible. Now, identify $([D,M]_{(0,\sigma(0))}, J^{(1)}|_{[D,M]_{(0,\sigma(0))}}, \mu^{(1)}|_{[D,M]_{(0,\sigma(0))}})$ with $\mathbb{C}^m$ and its standard structures via the coordinates :

$$z^k = x^k + iy^k,\ k = 1,...,m$$

We define coordinates $\zeta_1,...,\zeta^{2m},\eta_1,...,\eta^{2m+2}$ on some neighbourhood W of K in $[D,M]$ in the following way :

$$\zeta^k \circ \exp^{\mathcal{N}}(\sum_{s=1}^{m+1}(c_s X_s(\tau) + e_s J^{(1)}X_s(\tau))) = x^k(\tau)$$
$$\zeta^{k+m} \circ \exp^{\mathcal{N}}(\sum_{s=1}^{m+1}(c_s X_s(\tau) + e_s J^{(1)}X_s(\tau))) = y^k(\tau)$$
$$\eta^l \circ \exp^{\mathcal{N}}(\sum_{s=1}^{m+1}(c_s X_s(\tau) + e_s J^{(1)}X_s(\tau))) = c_l$$
$$\eta^{l+m+1} \circ \exp^{\mathcal{N}}(\sum_{s=1}^{m+1}(c_s X_s(\tau) + e_s J^{(1)}X_s(\tau))) = e_l$$

for τ in $[D,M]_{(0,\sigma(0))}$, k in $\{1,...,m\}$ and l in $\{1,...,m+1\}$. Since $\exp^{\mathcal{N}}$, restricted to the zero section, is injective, and K is compact, we get coordinates on some neighbourhood W of K. With respect to these coordinates, we have for any τ in $[D,M]_{(0,\sigma(0))} \cap W$ that :

$$\frac{\partial}{\partial \eta^l}\Big|_\tau = X_l(\tau) \text{ and } \frac{\partial}{\partial \eta^{l+m+1}}\Big|_\tau = J^{(1)}X_l(\tau)$$

Let λ be the 1-form on W defined by :

$$\lambda = \sum_{k=1}^{m} \zeta_k d\zeta^{k+m} + \sum_{l=1}^{m+1} \eta^l d\eta^{l+m+1}$$

Then we have :

$$d\lambda(x, J^{(1)}x) = \mu^{(1)}(x,x)$$

for each x in $T[D,M]|_{[D,M]_{(0,\sigma(0))}}$. Since K is compact, there exists a positive constant C in $(0,1]$ and a sufficiently small neighbourhood W' of K in $[D,M]$ such that :

$$d\lambda(x, J^{(1)}x) \geq C\mu^{(1)}(x,x)$$

for each x in TW'. Moreover, λ is bounded by some constant $C(\lambda)$ on W'. Eventually, Lemma 5 implies the claim. □

It is now easy to write a proof of Theorem 6.

Proof. Under the assumptions of the theorem, we get from the lemmas 6 and 7 that σ is differentiable at 0 with :

$$T_0\sigma = \sigma^{(1)}(0)$$

Especially, $T_0\sigma$ is complex linear. Applying the lemmas 6 and 7 successively implies that the n-jets $\sigma^{(n)}$ of σ are also continuously extensible to 0 and hence the extension σ is smooth. ☐

3.1.3 Generalized Weierstrass theorem

The purpose of this paragraph is to prove the following generalized Weierstrass theorem, and to give an important corollary of it.

Theorem 7 (Generalized Weierstrass theorem)**.** *Let (S,j) be a Riemann surface without boundary. Assume that $(\sigma_n)_n$ is a sequence of J-holomorphic curves in M parametrized by S converging in the C^0-topology to a map σ from S to M. Then $(\sigma_n)_n$ converges even in the C^∞-topology and its limit σ is a J-holomorphic curve.*

Proof. Let $(\sigma_n)_n$ and σ be as in the assumptions of the lemma. It is enough to show that the corresponding sequence of 1-jets $(\sigma_n^{(1)})_n$ also converges in C^0. Choose a metric ν on S compatible with j. By the C^0-convergence of $(\sigma_n)_n$ there is, for any z in S and any neighbourhood U of $\sigma(z)$ in M, a neighbourhood V of z in S such that U contains $\sigma(V)$ for each sufficiently large n. We may assume, without loss of generality, that V is conformaly equivalent to an open disc. With respect to the Poincaré metric on V, $T\sigma_n$ is bounded on V independently of n by the Gromov-Schwarz lemma. Now, choose a metric $\mu^{(1)}$ on $[S,M]$ compatible with $J^{(1)}$. Arguing as in the proof of the last paragraph, we see that there is a compact subset K of the fibre $[S,M]_{(z,\sigma(z))}$ and some open neighbourhood W of K in $[S,M]$ with the following properties :

1. There is a bounded 1-form λ on W such that :

   $$d\lambda(x, J^{(1)}x) \geq \mu^{(1)}(x,x)$$

   for each x in TW

2. For a sufficiently small neighbourhood V' of z in S, $\sigma_n^{(1)}(V')$ is contained in V for each sufficiently large n

Again by Gromov-Schwarz lemma, $T\sigma_n^{(1)}$ is bounded on V' independently of n. This implies the claim. ☐

We can now give a proof of the following corollary, which will be usefull in the proof of Gromov's compactness theorem.

Corollary 1. *Let D be an open disk in $\mathbb{R}^2$. Assume that $(\mu_n)_n$ is a sequence of Riemannian metrics on D such that each (D, μ_n) is isometric to an open subset of $\mathbb{H}$. Suppose that $(\mu_n)_n$ converges in the C^∞-topology to a Riemannian metric μ. Denote by j_n and j the complex structures on D induced by μ_n and μ respectively. Then any C^0-convergent sequence $(\sigma_n)_n$ of J-holomorphic curves in M, where σ_n is parametrized by (D, j_n), converges in fact in C^∞ to a J-holomorphic curve in M parametrized by (D, j).*

Proof. Under the assumptions of the corollary, (D, μ) is also isometric to an open subset of $\mathbb{H}$. The statement to be proved is purely local. Fix an arbitrary point z in D and choose a non-negative constant r such that the exponential map $\exp_z$ od (D, μ) is well defined on the $3r$-ball in (T_zD, μ_z) with center 0. Let $(I_n)_n$ be a sequence of transformations of T_zD, where I_n is an orientation preserving orthogonal map from (T_zD, μ_z) to $(T_zD, (\mu_n)_z)$, converging to the identity and denote by $\exp_z^n$ the exponenetial map of (D, μ_n) at z. Then, for each sufficiently large n, the map f_n defined by :

$$f_n = \exp_z^n \circ I_n \circ \exp_z^{-1} |_{B(z,2r)}$$

is a well defined isometry of the $2r$-ball in (D, μ) around z onto the $2r$-ball in (D, μ_n) around z. This implies that for each sufficiently large n, f_n is a j_n-biholomorphic curve in D parametrized by (D, j). Since $(\mu_n)_n$ converges in C^∞ to μ, $(f_n)_n$ converges in C^∞ to the inclusion of $B(z, 2r)$ in D. One can show (see [Hum97] Lemma B.2) that $(f_n^{-1}|_{B(z,r)})$ also converges in C^∞ to the inclusion of $B(z, r)$ in D. Then the generalized Weierstrass theorem implies that the sequence $(\sigma_n \circ f_n)_n$ converges in C^∞ to a J-holomorphic curve σ in M parametrized by (D, j). Hence, $(\sigma_{n\,|B(z,r)})$ converges in C^∞ to $\sigma_{|B(z,r)}$. $\square$

3.2 Gromov's compactness theorem

The purpose of this paragraph is to state and prove the Gromov's compactness theorem about pseudo-holomorphic curves. Roughly speaking, this theorem explains that, given a sequence of pseudo-holomorphic curves of bounded area, some subsequences of it will certainly converge, but not toward a true pseudo-holomorphic curve. Before precisely stating this theorem, we give in the first paragraph of this section a compactness result about hyperbolic Riemann surfaces which will be important in the proof of the Gromov's compactness theorem.

3.2.1 Hyperbolic Riemann surfaces

As observed before, J-holomorphic curves can be viewed as parametrized surfaces on which J induces a complex structure. In this paragraph, we give some results (without proof) about the geometry of the "non-exceptional" Riemann surfaces which will be usefull in the proof of the Gromov's compactness theorem.

According to the uniformization theorem, every simply connected Riemann surface without boundary is conformally equivalent either to $\mathbb{C}$, to the Poincaré half-plane $\mathbb{H}$ or to the Riemann sphere S^2. Hence, the universal cover of a connected Riemann surface without boundary is conformally equivalent either to $\mathbb{C}$, $\mathbb{H}$ or S^2. One can show that the connected Riemann surfaces without boundary not covered by $\mathbb{H}$ are, up to conformal equivalences, S^2, $\mathbb{C}$, $\mathbb{C}/\mathbb{Z}$ and $\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ with non-negative $\text{Im}(\tau)$. These Riemann surfaces are said to be of exceptional type. For the rest of the Riemann surfaces, we have the following result (see [Hum97], Corollary I.4.2 for a proof), which is the raison d'etre of this paragraph.

Proposition 7. *If (S, j) is a Riemann surface without boundary and with all its connected components of non-exceptional type, then there exists a unique complete metric ν on S, called the Poincaré metric of S, compatible with j and such that (S, ν) is a hyperbolic surface, i.e such that (S, ν) and $\mathbb{H}$ are locally isometric. Such Riemann surfaces will be called hyberbolic Riemann surfaces.*

Consider a connected and closed Riemann surface (S, j) of genus g. By removing d open disks and p points from S, we obtain a connected Riemann surface (S^*, j) which is certainly not of exceptional type (except when $2g + d + p$ is smaller than 2). According to the previous theorem, there exists a (unique) complete hyperbolic structure ν on S^* compatible with j. Thus, we are lead to study a hyperbolic Riemann surface (S^*, j, ν) where S^* is obtained from a surface S of genus g by removing p points and d open disks : such surfaces are called hyperbolic Riemann surfaces of signature (g, d, p). We first give an important result (see [Hum97] Lemma IV.4.1 for a proof) about the simple closed geodesics of length smaller than $2 \operatorname{arcsinh}(1)$ in hyperbolic Riemann surfaces of signature $(g, 0, p)$.

Proposition 8. *Let (S^*, j, ν) be a hyperbolic Riemann surface of signature $(g, 0, p)$ and of finite area. Then the simple closed geodesics in (S^*, ν) of length smaller than $2 \operatorname{arcsinh}(1)$ are pairwise disjoint, and their number is bounded by a constant depending only on g and p.*

Now, we state a compactness property of the hyperbolic Riemann surfaces of given signature. In order to do that, we define a notion of convergence for Riemann surfaces provided with compatible metrics.

Definition. *Let S_1 and S be two smooth surfaces. A deformation of S_1 onto S is a continuous surjective map from the closure of S_1 to the closure of S with the following properties :*

1. *ϕ induces a bijective map between the boundary components of S_1 and those of S*
2. *The restriction of ϕ to $S_1 \setminus \partial S_1$ is a diffeomorphism onto $S \setminus \partial S$; we denote by ϕ_n^{-1} its inverse map*

Now we say that a sequence $(S_n, j_n, \nu_n)_n$ of Riemann surfaces with compatible metrics converges to (S, j, ν) via a sequence $(\phi_n)_n$ if the following hold :

1. *For each n, ϕ_n is a deformation of S_n onto S*

2. *$((\phi_n^{-1})^* j_n)_n$ converges in the C^∞-topology to j*

3. *$((\phi_n^{-1})^* \nu_n)_n$ converges in the C^∞-topology to ν*

We can now state the theorem (of which a proof can be found in [Hum97] Proposition IV.5.1) about the compactness of the hyperbolic Riemann surfaces of given signature.

Theorem 8. *Let $(S_n^*, j_n, \nu_n)_n$ be a sequence of Riemann hyperbolic surfaces of signature (g, d, p) and of finite area. Then some subsequence of $(S_n^*, j_n, \nu_n)_n$ converges to a hyperbolic Riemann surface of signature $(g, p+p_0, d-p_0)$ (via some sequence $(\phi_n)_n$, where p_0 is an integer smaller than d.*

3.2.2 Statement and proof

In order to state Gromov's compactness theorem, we have to give some more definitions. Let S be a closed surface, and let $(\gamma^i)_{i\in I}$ be a possibly empty family of finetely many smooth simple closed and pairwise disjoint loops in S. Let $\widehat{S}$ be the surface obtained from $S \setminus \cup_{i\in I}\gamma^i$ by the one-point compactification at each end. For each k in I, we denote by z_k' and z_k'' the two points of $\widehat{S}$ added to $S \setminus \cup_i \gamma^i$ at the two ends which arise from removing γ^k ; identifying z_k' and z_k'', we obtain a compact topological space $\tilde{S}$. We denote by π the canonical projection of $\widehat{S}$ in $\tilde{S}$, and by $\mathrm{si}(\tilde{S})$ the set of singular points of $\tilde{S}$ defined by :

$$\mathrm{si}(\tilde{S}) = \{\pi(z_k') \mid k \in I\}$$

In the following, we may identify $\widehat{S} \setminus \{z_k', z_k'' \mid k \in I\}$ and $\tilde{S} \setminus \mathrm{si}(\tilde{S})$. Besides, we will say that $(\tilde{S}, \tilde{j})$ is a singular Riemann surface if $\tilde{j}$ is a complex structure on $\widehat{S}$; this allows us to define the curves toward which some sequences of pseudo-holomorphic curves are going to converge.

Definition (Cusp curve)**.** *A cusp curve in (M, J) is a continuous map $\tilde{\sigma}$ from a singular Riemann surface $(\tilde{S}, \tilde{j})$ to (M, J) such that $\tilde{\sigma} \circ \pi$ is J-holomorphic, where π denotes the canonical projection of $\widehat{S}$ in $\tilde{S}$.*

Now, we need to define what it means for a sequence of pseudo-holomorphic curves to converge to a cusp curve.

Definition. *Let S be a closed surface, and assume that $(\sigma_n)_n$ is a sequence of J-holomorphic curves in M parametrized by (S, j_n), where j_n is a complex structure on S. Then $(\sigma_n)_n$ is said to converge weakly to a cusp curve $\tilde{\sigma}$ parametrized by $(\tilde{S}, \tilde{j})$ if there exists a sequence $(\phi_n)_n$ with the following properties :*

1. *For each n, ϕ_n is a deformation of S onto $\tilde{S}$*

2. *$(\sigma_n \circ \phi_n^{-1})_n$ converges uniformly and in C^∞ to $\tilde{\sigma}|_{\tilde{S}\setminus \mathrm{si}(\tilde{S})}$*

If the weak limit of a sequence of closed J-holomorphic curves is not unique, we still have the important following result (see [Hum97] Proposition V.1.1 for a proof).

Proposition 9. *If $\tilde{\sigma_1}$ and $\tilde{\sigma_2}$ are two weak limits of one sequence $(\sigma_n)_n$ of closed J-holomorphic curves, then the images of $\tilde{\sigma_1}$ and $\tilde{\sigma_2}$ coincide.*

We are now able to state the Gromov's compactness theorem.

Theorem 9 (Gromov's compactness theorem)**.** *Let S be a closed surface and $(j_n)_n$ a sequence of complex structures on S. Assume that $(\sigma_n)_n$ is a sequence of J-holomorphic curves in M of bounded area, where σ_n is parametrized by (S, j_n). Then there exists a subsequence of $(\sigma_n)_n$ which converges weakly to some cusp curve $\tilde{\sigma}$ in M parametrized by a singular Riemann surface $(\tilde{S}, \tilde{j})$.*

The proof that we are going to give uses almost all the previous results of this part, together with the properties of the hyperbolic surfaces assumed in the first paragraph of this section. Neththeless, in order this proof to be complete, we need to state one more (important) lemma (see [Hum97] Corollary V.2.4 for a proof). In the following, we will only consider J-holomorphic curves in M whose areas are bounded by some constant C fixed from now on.

Lemma 8. *There exists an integer K and a constant D such that the following holds. For any closed J-holomorphic curve σ in M parametrized by a Riemann surface (S, j) and of area bounded by C, there exists a finite subset F of S whose cardinality is bounded by K and such that $T\sigma$ is bounded by D on the space S^* defined by :*

$$S^* = S \setminus F$$

with respect to its Poincaré metric (see Proposition 7).

We are now able to write a correct proof of the Gromov's compactness theorem.

Proof. Let S be a closed surface and $(j_n)_n$ a sequence of almost complex structures on S. Assume that $(\sigma_n)_n$ is a sequence of J-holomorphic curves in M of bounded area, where σ_n is parametrized by (S, j_n).

Using Lemma 8, choose for each n a finite subset F_n of S such that $\sigma_{n\,|S_n^*}$ has a Lipschitz constant independent of n with respect to the Poincaré metric ν_n on S_n^*, where S_n^* stands for $S \setminus F_n$, and such that the cardinality of F_n is bounded independently of n. After passing to some subsequence of $(\sigma_n)_n$, again denoted by $(\sigma_n)_n$, we may assume that the cardinality of F_n is independent of n, and we will denote it by p. Thus, $(S_n^*, j_n, \nu_n)_n$ is a sequence of hyperbolic Riemann surfaces of finite area and of same signature $(g, 0, p)$, where g is the genus of S. By Proposition 8, the simple closed geodesics $(\gamma_n^i)_i$ in (S_n^*, ν_n) of length smaller than $2\,\mathrm{arcsinh}(1)$ are pairwise disjoint and their number is bounded by some constant depending only on g and p. After passing to a further subsequence, we can suppose that their number is independent of n, and that for each n there exists an orientation preserving diffeomorphism χ_n from S_1^* to S_n^* mapping each γ_1^i onto γ_n^i. Moreover, it can be assumed that the sequences $(l(\gamma_n^i))_n$ converge. Let I denote the subset of indices i such that $(l(\gamma_n^i))_n$ converges to 0. As above, a singular surface $\tilde{S}$ is obtained by collapsing the loops $(\gamma_1^i)_{i\in I}$ in S. Using Theorem 8, we can show that there exists a subsequence of $(S_n)_n$ again denoted by $(S_n)_n$, a complex structure $\tilde{j}$ on $\widehat{S}$, a finite subset F of $\tilde{S}\setminus \mathrm{si}(\tilde{S})$, and a sequence $(\phi_n)_n$ such that the following hold :

1. $(S_n^* \setminus \cup_{i\in I}\gamma_1^i, j_n|_{S_n^*\setminus\cup_{i\in I}\gamma_1^i}, \nu_n|_{S_n^*\setminus\cup_{i\in I}\gamma_1^i})_n$ converges to $(\tilde{S}^*\setminus \mathrm{si}(\tilde{S}), \tilde{j}|_{\tilde{S}^*\setminus\mathrm{si}(\tilde{S})}, \tilde{\nu})$ via $(\phi_n)_n$, where $\tilde{S}^*$ stands for $\tilde{S}\setminus F$, and $\tilde{\nu}$ is the Poincaré metric on $\tilde{S}^*\setminus\mathrm{si}(\tilde{S})$
2. For each n, ϕ_n maps bijectively F_n onto F

Namely, the subsequence of $(S_n)_n$, $\tilde{j}$, F and $(\phi_n)_n$ are obtained in the following way. For each n, denote by $S_n^1,\ldots,S_n^{|I|+1}$ the closures of the components of $S_n^*\setminus\cup_{i\in I}\gamma_n^i$ in S_n^*. We may assume that, for each n and k, χ_n maps S_1^k onto S_n^k. Then for each k, $(S_n^k)_n$ is a sequence of Riemann hyperbolic surfaces of finite area and of same signature, so that we can apply Theorem 8 for each k to the original sequence $(S_n^k)_n$, passing successively to subsequences.

So let $(\sigma_n)_n$ be the corresponding subsequence. By Lemma 8, and since $(\phi_n^{-1})^*\nu_n$ converges in C^∞ to the Poincaré metric $\tilde{\nu}$ on $\tilde{S}^*\setminus\mathrm{si}(\tilde{S})$, the maps $\sigma_n\circ\phi_n^{-1}|_{\tilde{S}^*\setminus\mathrm{si}(\tilde{S})}$ are uniformly (with respect to n) Lipschitz on each compact subset in the domain. Since M is compact, some subsequence of $(\sigma_n\circ\phi_n^{-1}|_{\tilde{S}^*\setminus\mathrm{si}(\tilde{S})})_n$, again denoted $(\sigma_n\circ\phi_n^{-1}|_{\tilde{S}^*\setminus\mathrm{si}(\tilde{S})})_n$, converges in C^0 to a continuous map $\tilde{\sigma}$ by Ascoli's theorem. Since $(\phi_n^{-1})^*j_n$ converges in C^∞ on $\tilde{S}^*\setminus\mathrm{si}(\tilde{S})$, Corollary 1 implies that $(\sigma_n\circ\phi_n^{-1}|_{\tilde{S}^*\setminus\mathrm{si}(\tilde{S})})_n$ converges in C^∞ to $\tilde{\sigma}$, and thus $\tilde{\sigma}$ is a J-holomorphic curve in M parametrized by $(\tilde{S}^*\setminus\mathrm{si}(\tilde{S}), \tilde{j}|_{\tilde{S}^*\setminus\mathrm{si}(\tilde{S})})$. Observe that, for each n, $\sigma_n|_{S_n^*\setminus\cup_{i\in I}\gamma_1^i}{}^*\mu$ is a Riemannian structure on $S_n^*\setminus\cup_{i\in I}\gamma_1^i$ compatible with $j_n|_{S_n^*\setminus\cup_{i\in I}\gamma_1^i}$ and that $\tilde{\sigma}|_{\tilde{S}^*\setminus\mathrm{si}(\tilde{S})}{}^*\mu$ is a $\tilde{\nu}$-compatible metric on $\tilde{S}^*\setminus\mathrm{si}(\tilde{S})$. Hence, by Proposition 7, we have that :

$$\sigma_n|_{S_n^*\setminus\cup_{i\in I}\gamma_1^i}{}^*\mu = \nu_n|_{S_n^*\setminus\cup_{i\in I}\gamma_1^i} \text{ and } \tilde{\sigma}|_{\tilde{S}^*\setminus\mathrm{si}(\tilde{S})}{}^*\mu = \tilde{\nu}$$

Using this together with the convergence of $((\phi_n^{-1})^*\nu_n|_{S_n^*\setminus\cup_{i\in I}\gamma_1^i})_n$ to $\tilde{\nu}$, one can show that the sequence $(\mathcal{A}(\sigma_n\circ\phi_n^{-1}))_n$ converges to the area of $\tilde{\sigma}$. Since $(\sigma_n)_n$ is a sequence of curves of bounded area, this implies that the area of $\tilde{\sigma}$ is finite. By Theorem 6, $\tilde{\sigma}$ can be extended to a J-holomorphic curve in M parametrized by $\widehat{S}$. To conclude the proof of the theorem, we have to show that :

1. $\tilde{\sigma}$ is compatible with the projection π of $\widehat{S}$ in $\tilde{S}$, so that it can be viewed as a cusp curve in M parametrized by $(\tilde{S}, \tilde{j})$
2. $(\sigma_n\circ\phi_n^{-1})_n$ converges uniformly and in C^∞ to $\tilde{\sigma}|_{\tilde{S}\setminus\mathrm{si}(\tilde{S})}$

To prove 1 we argue indirectly. Assume that there exists a singular point z in $\tilde{S}$ such that $\tilde{\sigma}(z')$ and $\tilde{\sigma}(z'')$ are different for the preimages z' and z'' of z under π. Let ϵ_0 and

C_M be some constants as in the monotocity lemma. Denote by ρ the constant defined by :

$$\rho = \min(\epsilon_0, \frac{1}{4} d(\tilde{\sigma}(z'), \sigma(\tilde{z''})))$$

Then choose compact and connected neighbourhoods V' and V'' of z' and z'' respectively such that :

$$\tilde{\sigma}(V') \subset B(\tilde{\sigma}(z'), \rho) \text{ and } \tilde{\sigma}(V'') \subset B(\tilde{\sigma}(z''), \rho)$$

Denote by V the open subset of $\tilde{S} \setminus \mathrm{si}(\tilde{S})$ defined by :

$$V = \pi(V' \cup V'') \setminus \{z\}$$

After possibly making V' and V'' smaller, we may assume that :

$$\mathcal{A}(\sigma_{n\,|\phi_n^{-1}(V)}) < C_M \rho^2$$

for each n. Since $(\sigma_n \circ \phi_n^{-1}|_{\tilde{S}^* \setminus \mathrm{si}(\tilde{S})})_n$ converges in C^0, we have that :

$$\partial \sigma_{n\,|\phi_n^{-1}(V)} \subset B(\tilde{\sigma}(z'), \rho) \cup B(\tilde{\sigma}(z''), \rho)$$

for each sufficiently large n. Hence the previous inequality contradicts the monotocity lemma, and 1 is proved.

To prove 2, we also argue indirectly. Assume that $(\sigma_n \circ \phi_n^{-1})_n$ does not converge uniformly to $\tilde{\sigma}|_{\tilde{S} \setminus \mathrm{si}(\tilde{S})}$. Since we already know that $(\sigma_n \circ \phi_n^{-1})_n$ converges in C^0 on $\tilde{S}^* \setminus \mathrm{si}(\tilde{S})$, there exists some z_0 in $F \cup \mathrm{si}(\tilde{S})$, some sequence $(z_k)_k$ in $\tilde{S} \setminus \mathrm{si}(\tilde{S})$ converging to z_0, and some subsequence $(\sigma_{n_k} \circ \phi_{n_k}^{-1})_k$ of $(\sigma_n \circ \phi_n^{-1})_n$ such that :

$$d(\sigma_{n_k} \circ \phi_{n_k}^{-1}(z_k), \tilde{\sigma}(z_k) > \epsilon$$

for some ϵ in $(0, \epsilon_0)$ and all k. Now pick some compact neighbourhood V of z_0 in $\tilde{S}$ such that :

$$\begin{aligned} \partial V \cap (F \cup \mathrm{si}(\tilde{S})) &= \emptyset \\ \tilde{\sigma}(V) &\subset B(\tilde{\sigma}(z_0), \frac{\epsilon}{4}) \\ \mathcal{A}(\tilde{\sigma}_{|V}) &< C_M (\frac{\epsilon}{4})^2 \end{aligned}$$

On the other hand, by the C^0-convergence of $(\sigma_n \circ \phi_n^{-1})_n$ to $\tilde{\sigma}$ on $\tilde{S}^* \setminus \mathrm{si}(\tilde{S})$, and the non-uniform convergence, we have that :

$$\begin{aligned} \sigma_{n_k} \circ \phi_{n_k}^{-1}(z_k) &\notin B(\tilde{\sigma}(z_0), \frac{\epsilon}{2}) \\ \sigma_{n_k} \circ \phi_{n_k}^{-1}(\partial V) &\subset B(\tilde{\sigma}(z_0), \frac{\epsilon}{4}) \end{aligned}$$

for all sufficiently large k. It follows from the monotocity lemma that :

$$\mathcal{A}(\sigma_{n_k\,|\phi_{n_k}^{-1}(V)}) > C_M (\frac{\epsilon}{4})^2$$

and this leads to a contradiction since $(\mathcal{A}(\sigma_{n_k}{}_{|\phi_{n_k}^{-1}(V)}))_k$ converges to $\mathcal{A}(\tilde{\sigma}_{|V})$. Hence, $(\sigma_n \circ \phi_n^{-1})_n$ converges uniformly to $\tilde{\sigma}_{|\tilde{S}\setminus \mathrm{si}(\tilde{S})}$. Finally, the convergence in C^∞ is obtained by applying the generalized Weierstrass theorem. $\square$

Before ending this paragraph, we make the following remark.

Remark. *Assume $(J_n)_n$ is a sequence of almost complex structures on M converging in C^∞ to J. Then the compactness theorem generalizes to sequences $(\sigma_n)_n$ of pseudo-holomorphic curves in M of bounded area, where σ_n is a J_n-holomorphic curve. Any limit cusp curve is J-holomorphic where the definition of weak convergence generalizes in the obvious way.*

4 Pseudo-holomorphic curves in symplectic geometry

The aim of this part is to study the universal moduli spaces of pseudo-holomorphic curves, and especially to show that a symplectic context gives to some of these spaces compactness properties. Moreover, we will show in the last section of this work how the pseudo-holomorphic curves can be used to get important results in symplectic geometry.

In the following, we shall only consider pseudo-holomorphic curves parametrized by the 2-sphere S^2.

4.1 Universal moduli spaces

The purpose of this section is to study a universal moduli space of pseudo-holomorphic curves through the properties of an evaluation map defined on the product of this universal moduli space by S^2. To that purpose, we fix a compact manifold M without boundary, a homology class A in M and a Banach manifold $\mathcal{J}$ of almost complex structures on M.

4.1.1 First results

Roughly speaking, the universal moduli space that we are going to study is the space $\mathcal{M}(A, \mathcal{J})$ defined by :

$$\mathcal{M}(A, \mathcal{J}) = \{(J, \sigma) \in \mathcal{J} \times \Sigma \mid \sigma \text{ is } J\text{-holomorphic}\}$$

where Σ is the space of all smooth maps from S^2 to M representing the homology class A. However, for the following results to be true, we have to slightly modify this universal moduli space so that it becomes a Banach manifold. Here, we briefly describe what should be done, and refer to [SM95] for details. We may assume that M is embedeed into $\mathbb{R}^N$ for a sufficiently large integer N. For p strictly greater than 2, the Sobolev space $W^{1,p}(S^2, \mathbb{R}^{\mathbb{N}})$ is the completion of $C^\infty(S^2, \mathbb{R}^N)$ with respect to the norm $\|.\|_{1,p}$ defined for σ in $C^\infty(S^2, \mathbb{R}^N)$ by :

$$\|\sigma\|_{1,p} = (\int_{S^2} (|\sigma|^p + |T\sigma|^p)\alpha)^{\frac{1}{p}}$$

where α is an area form on S^2 induced by some given Riemannian metric on S^2. One can show that the normed vector space $(W^{1,p}(S^2, \mathbb{R}^{\mathbb{N}}), \|.\|_{1,p})$ is a Banach space which embeds naturally into $C^0(S^2, \mathbb{R}^N)$. We denote by Σ^p the subspace of $W^{1,p}(S^2, \mathbb{R}^{\mathbb{N}})$ defined by :

$$\Sigma^p = \{\sigma \in W^{1,p}(S^2, \mathbb{R}^{\mathbb{N}}) \mid \sigma(S^2) \subset M \text{ and } [\sigma] = A\}$$

where $[\sigma]$ denotes the element of $H_2(M)$ represented by σ. It can be shown that Σ^p is a Banach manifold for p strictly greater than 2, and that the corresponding universal moduli space, again denoted by $\mathcal{M}(A, \mathcal{J})$, is a Banach manifold (this result is also stated in [Hum97] Proposition VI.3.7).

The evaluation map about which we are going to state (and sometimes prove) important results is the map ev defined by :

$$\text{ev}((J,\sigma),z) = (J,\sigma(z))$$

for (J,σ) in $\mathcal{M}(A,\mathcal{J})$ and z in S^2. Its first remarkable properties are sum up in the following theorem (see [Hum97] Lemma IV.4.3 and more generally [SM95] for a proof).

Theorem 10. *The set of regular values of* ev *is dense in $\mathcal{J} \times S^2$, and all its elements are attained by* ev*.*

4.1.2 Symplectic context

In this paragraph, we assume that M is provided with a symplectic structure ω. Unfortunately, the space $\mathcal{J}_c$ of almost complex structures on M compatible with ω is not a Banach manifold, but there exists a subset $\mathcal{J}$ of $\mathcal{J}_c$ which is dense in the C^∞ topology and which can be given the structure of a Banach manifold such that the inclusion of $\mathcal{J}$ in $\mathcal{J}_c$ is continuous (see the discussion below Proposition VI.3.5 in [Hum97]).

Denote by G the group of the conformal transformations of S^2, and observe that there is an action of G on $\mathcal{M}(A,\mathcal{J}) \times S^2$ given by :

$$g * ((J,\sigma),z) = ((J,\sigma \circ g^{-1}), g(z))$$

If $((J,\sigma),z)$ is an element of $\mathcal{M}(A,\mathcal{J}) \times S^2$, we will denote by $[((J,\sigma),z)]$ the corresponding element in the quotient space $\mathcal{M}(A,\mathcal{J}) \times_G S^2$. Now observe that ev factors through the previous action, and denote by $\bar{\text{ev}}$ the corresponding map defined on $\mathcal{M}(A,\mathcal{J}) \times_G S^2$. The purpose of this paragraph is to state and prove a compactness property of $\bar{\text{ev}}$, but in order to do that we shall give one more definition.

Definition (Spherical homology class)**.** *An element B of $H_2(M)$ is said to be a spherical homology class in M if there exists a family $(n_i)_i$ of integers and a family $(\sigma_i)_i$ of smooth maps from S^2 to M such that :*

$$B = \sum_i n_i[\sigma_i]$$

We can now state and prove the following theorem, which can be viewed as a corollary of Gromov's compactness theorem.

Theorem 11. *If $\langle[\omega],A\rangle$ is the smallest positive value of $\langle[\omega],.\rangle$ on spherical homological classes in M, then the following holds. If $([(J_n,\sigma_n),z_n])_n$ is a sequence in $\mathcal{M}(A,\mathcal{J}) \times_G S^2$ whose image by $\bar{\text{ev}}$ has a convergent subsequence in $\mathcal{J} \times S^2$, then it admits a convergent subsequence itself.*

Proof. Assume that $\langle[\omega],A\rangle$ is the smallest positive value of $\langle[\omega],.\rangle$ on spherical homological classes in M, and let $([(J_n,\sigma_n),z_n])_n$ be as in the assumption of the theorem.

After passing to a subsequence, we may assume that $(J_n)_n$ converges to some $\tilde{J}$ in $\mathcal{J}$ and $(z_n)_n$ to some z in S^2. The claim is that there is a sequence $(g_n)_n$ in G such that $(\sigma_n \circ g_n)_n$ has a uniformly convergent subsequence. Since each σ_n represents A and J_n is ω-compatible, the area of σ_n equals $\langle[\omega], A\rangle$. Thus, the sequence $(\sigma_n)_n$ has bounded area. Gromov's compactness theorem implies that, after passing to a subsequence, $(\sigma_n)_n$ converges weakly to a $\tilde{J}$-cusp curve $\tilde{\sigma}$ in M parametrized by a singular Riemann surface $(\tilde{S^2}, \tilde{j})$ (see the remark at the end of the paragraph 3.2.2). Denote by S_1,..., S_k the different components of $\tilde{S^2}$ on which $\tilde{\sigma}$ is not constant. Clearly, each S_i is a sphere. If k is strictly greater than 1 then we have :

$$0 < \langle[\omega], [\tilde{\sigma}_{|S_i}]\rangle = \mathcal{A}(\tilde{\sigma}_{|S_i}) < \mathcal{A}(\tilde{\sigma}) = \langle[\omega], A\rangle$$

in contradiction to the fact that $\langle[\omega], A\rangle$ is the smallest positive value of $\langle[\omega], .\rangle$ on spherical homological classes in M. Hence k equals 1 and we may assume that $(\tilde{S}, \tilde{j})$ equals (S, j). By the definition of weak convergence, there is a sequence of diffeomorphisms $(\phi_n)_n$ of S^2 such that $((\phi_n^{-1})^* j)_n$ converges in C^∞ to j and $(\sigma_n \circ \phi_n^{-1})_n$ converges uniformly to $\tilde{\sigma}$. Choose now three points z_1, z_2 and z_3 in S^2 and let, for each n, ψ_n be the unique biholomorphic transformation from (S^2, j) to $(S^2, (\phi_n^{-1})^* j)$ such that $\psi_n(z_i)$ equals z_i for each i. For each n, denote by g_n the transformation of S^2 defined by :

$$g_n = \phi_n^{-1} \circ \psi_n$$

and observe that g_n is in G. Applying the compactness theorem, it is readily seen that $(\psi_n)_n$ converges uniformly to the identity and thus $(\sigma_n \circ g_n)_n$ converges uniformly to $\tilde{\sigma}$. □

This theorem allows us to prove the following result, which will be crucial to get a non-trivial symplectic invariant.

Corollary 2. *If $\langle[\omega], A\rangle$ is the smallest positive value of $\langle[\omega], .\rangle$ on spherical homological classes in M, then $\bar{\mathrm{ev}}$ is a surjective map.*

Proof. Assume that $\langle[\omega], A\rangle$ is the smallest positive value of $\langle[\omega], .\rangle$ on spherical homological classes in M, and let (J, z) be an element of $\mathcal{J} \times S^2$. By Theorem 10, there exists a sequence $((J_n, \sigma_n), z_n)_n$ in $\mathcal{M}(A, \mathcal{J}) \times S^2$ whose image by ev converges to (J, z). Thus, the image by $\bar{\mathrm{ev}}$ of the sequence $([(J_n, \sigma_n), z_n])_n$ is convergent. By Theorem 11, $([(J_n, \sigma_n), z_n])_n$ has a convergent subsequence, and by continuity of $\bar{\mathrm{ev}}$ the image by $\bar{\mathrm{ev}}$ of the limit of this subsequence is precisely (J, z). □

4.2 Symplectic consequences

The aim of this section is to show how the results given in the previous paragraph about the universal moduli spaces in a symplectic context can be used to prove a fundamental theorem of symplectic geometry, and then to get a global non-trivial symplectic invariant.

4.2.1 Non-squeezing theorem

The purpose of this paragraph is to prove the following generalized version of the non-squeezing theorem stated in 2.2.1.

Theorem 12 (Generalized non-squeezing theorem). *Let (M_1, ω_1) be a symplectic manifold without boundary of dimension $(m-2)$ and assume that its second fundamental group is trivial. If there exists a symplectic embedding of $B^m(r)$ into the symplectic product $M_1 \times B^2(R)$, then :*

$$r \leq R$$

Proof. Let (M_1, ω_1) be as in the assumption of the theorem and assume that there exists a symplectic embedding of $B^m(r)$ into $M_1 \times B^2(R)$ We choose ϵ in $(0, r)$ and a symplectic structure ω_2 on S^2 such that the area of S^2 with respect to ω_2 equals $(\pi R^2 + \epsilon)$. Since the area of $B^2(R)$ equals πR^2, we can view $B^2(R)$ as a subset of S^2 via a symplectic (i.e area preserving) embedding. Hence, there is a symplectic embedding ϕ of $B^m(r)$ into the symplectic manifold (M, ω) defined by :

$$(M, \omega) = (M_1 \times S^2, \omega_1 \oplus \omega_2)$$

In the following we denote by A the homology class in M represented by a standard inclusion of S^2 in M.

To continue the proof, we have to show that there exists an ω-compatible almost complex structure J on M such that :

$$\phi^* J|_{\phi(B^m(r-\epsilon))} = J_0|_{B^m(r-\epsilon)}$$

where J_0 is the standard complex structure on $\mathbb{R}^m$. To that end, use a partition of unity argument in order to construct a Riemannian metric μ on M such that :

$$\phi^* \mu|_{\phi(B^m(r-\epsilon))} = \mu_0|_{B^m(r-\epsilon)}$$

and use the construction from Proposition 3 to get an almost complex structure J on M from the data of ω and μ.

Now observe that the triviality of the second fundamental group of M_1 implies that $\langle [\omega], A \rangle$ is the smallest positive value of $\langle [\omega], . \rangle$ on spherical homological classes in M. By Corollary 2, we can thus choose a J-holomorphic curve σ in M parametrized by S^2, representing A and containing $\phi(0)$ in its image ($(J, \phi(0))$ is attained by ev). We claim that :

$$\pi(r-\epsilon)^2 \leq \mathcal{A}(\sigma \cap \phi(B^m(r-\epsilon)))$$

To that end, note that the subspace $\phi(B^m(r-\epsilon))$ of (M, ω, J, μ) is holomorphically isometric to the Euclidean ball $B^m(r-\epsilon)$ of $(\mathbb{R}^m, J_0, \mu_0)$ by the choice of J and with μ the Riemannian metric on M induced by J and ω. In particular, for each compact surface S in S^2 with boundary and wose image by σ is contained in $B^m(r-\epsilon)$, the

J-holomorphic curve $\sigma_{|S}$ is absolutely area minimizing among all smooth maps τ from S to $B^m(r-\epsilon)$ with same boundary than $\sigma_{|S}$ (see Proposition 5). Observe also that the image of σ is not contained in $\phi(B^m(r-\epsilon))$ for otherwise σ would be constant. Then the inequality claimed is the classical monotocity lemma for minimal surfaces in Euclidean space and its proof can be easily recovered from the arguments given in 3.1.1. Indeed, in order to show that the statement of the monotocity lemma holds when C_M equals π and when ϵ_0 equals ∞, just copy the proof of the monotocity lemma but use the classical isoperimetric inequality instead of Lemma 2.

In order to prove now that r is smaller than R, continue the inequality claimed as follow :

$$\pi(r-\epsilon)^2 \leq \mathcal{A}(\sigma \cap \phi(B^m(r-\epsilon))) < \mathcal{A}(\sigma) = \langle[\omega],[\sigma]\rangle = \langle[\omega],A\rangle = \int_{S^2}\omega_2 = \pi R^2 + \epsilon$$

where the first and second equality follow from the fact that σ is J-holomorphic with ω-compatible J and that σ represents A respectively. Since this inequality is true for each ϵ in $(0,r)$, it follows easily that :

$$r \leq R$$

and this concludes the proof. $\square$

4.2.2 Gromov width

The purpose of this paragraph is to show how the non-squeezing theorem proved in the previous paragraph allows us to give a non trivial symplectic invariant. More precisely, we are going to build what we call a symplectic capacity.

Definition (Symplectic capacity)**.** *A symplectic capacity is a functor c which assigns to every symplectic manifold (M,ω) a non-negative number (possibly infinite) $c(M,\omega)$ such that :*

1. *If there is a symplectic embeeding from (M_1,ω_1) to (M_2,ω_2) with M_1 and M_2 of same dimension, then :*
$$c(M_1,\omega_1) \leq c(M_2,\omega_2)$$
2. *For any symplectic manifold (M,ω) and any λ in $\mathbb{R}$:*
$$c(M,\lambda\omega) = \lambda c(M,\omega)$$
3. *c is a non-trivial 2-dimensional invariant :*
$$c(B^{2n}(1),\omega_0) > 0 \text{ and } c(Z^{2n}(1),\omega_0) < 0$$

A priori, it is not at all clear that such invariants exist. The one that we are going to build is closely linked to the non-squeezing theorem, and is usually called Gromov width.

Definition (Gromov width)**.** *The Gromov width of a symplectic manifold* (M, ω) *is the number* $w_G(M, \omega)$ *defined by :*

$$w_G(M, \omega) = sup(\{\pi r^2 \mid B^{2n}(r) \textit{ embeds symplectically in } M\})$$

Then, the aim of this section is to prove the following theorem.

Theorem 13. *The Gromov width* w_G *is a symplectic capacity such that :*

$$w_G(B^{2n}(1), \omega_0) = w_G(Z^{2n}(1), \omega_0) = \pi$$

Proof. The Gromov width clearly satisfies the first and second axioms of a symplectic capacity, and satisfies the third one by the non-squeezing theorem. □

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