École Normale Supérieure de Rennes

Nansen Environmental and Remote Sensing Center

Stability of a travelling wave

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Introduction

This internship report focuses on some geometrical aspects of partial differential equations (PDEs). Specifically, we studied the stability of a travelling wave in a steady medium, described as a solution of a reaction-diffusion equation. This problem is of particular interest in the case of initial condition uncertainty, brought by measurement errors for example. A broad question tackled in geophysics – e.g., climate prediction – is *can a little initial condition error bring a huge error when propagating through an equation ?* This question motivates the work presented in this report.

The initial reaction-diffusion equation is

$$\partial_t U = \partial_{xx} U + f(U) \tag{1}$$

with f smooth but not linear, $U \in \mathbb{C}^n,\ x \in \mathbb{R}$ and $t \in \mathbb{R}$.
In this report, we will assume the existence of a travelling wave, solution of (1), and the goal is to know how this solution is stable or unstable. Travelling waves appear naturally in many various domains in physics such as fluid mechanics, electromagnetic theory...
Intuitively, a travelling wave is a recognizable shape (of energy for instance) which is transferred from one part of the medium to another part with a constant speed of propagation.
Mathematically, a travelling wave is a solution of a PDE of the form $u(x,t) = f(x - ct)$. At the time $t = 0$, the wave has the spatial form $f(x)$ and then, at the time $t > 0$, $f(x - ct)$ is a translation of the initial shape, translated to the right (if $c > 0$) by ct spatial units. Thus the constant c represents the speed of the wave.
Firstly we will announce a clear definition of the problem, then we will present the geometric tools and finally we will show how this tools can be used to solve our problem. Some numerical results will be also presented.

Contents

Chapter 1

Introduction to the problem :

1.1 Definitions and hypotheses :

We have an equation

$$\partial_t U = \partial_{xx} U + f(U) \tag{1.1}$$

where $U(t,x) \in \mathbb{C}^n;\ x,t \in \mathbb{R}$ and f is a C^2 function with derivative through order 2 bounded on $\mathbb{C}^n$.

From now, we assume that there is a travelling wave solution $U_*(\xi)$.

Definition 1.1. *We call* travelling wave *a solution u of equation (1.1) which depends only on one variable $\xi = x - ct$ with $c \in \mathbb{R}$ and we further require it to decay exponentially to constant states $\lim\limits_{\xi \to \pm\infty} u(\xi) = u_\pm$ where $f(u_\pm) = 0$ and $\lim\limits_{\xi \to \pm\infty} u'(\xi) = 0$ exponentially.*

These conditions are developed in the following hypotheses : our considered travelling wave U_* satisfies the following hypothesis :

Hypothesis 1. *There exists $C, a > 0$ and $u_\pm \in \mathbb{C}^n$ such that :*

1. $f(u_\pm) = 0$,
2. $\forall \xi > 0,\ |U_*(\xi) - u_+| \leq Ce^{-a\xi}$,
3. $\forall \xi < 0,\ |U_*(\xi) - u_-| \leq Ce^{a\xi}$,
4. $\forall \xi \in \mathbb{R},\ |U_*'(\xi)| \leq Ce^{-a|\xi|}$.

Let us transform the equation (1.1) about the travelling wave : $U_*(\xi)$ satisfies the equation (called moving equation) :

$$U'' + cU' + f(U) = 0 \tag{1.2}$$

Indeed, if we write $U_*(\xi) = U_*(x - ct) = V(x,t)$ then

$$\partial_t V(x,t) = \partial_{xx} V(x,t) + f\left(V(x,t)\right) \Leftrightarrow -cU_*'(x-ct) = U_*''(x-ct) + f\left(U_*(x-ct)\right)$$

And we can write it as

$$\varphi(U_*) = 0 \qquad \text{where} : \varphi(u) = u'' + cu' + f(u) \tag{1.3}$$

As usual, to see how U_* is stable, we will differentiate φ at the point U_* and look at the differential and more precisely look at the sign of its eigenvalues' real part.
Let $\mathbb{B}(\mathbb{R}, \mathbb{C}^n)$ be the set of all bounded and uniformly continuous function from $\mathbb{R}$ to $\mathbb{C}^n$, with the L^∞ norm. Then we will consider $\varphi : \mathbb{B}(\mathbb{R}, \mathbb{C}^n) \longrightarrow \mathbb{B}(\mathbb{R}, \mathbb{C}^n)$. But before doing this, let us define clearly what a stable travelling wave does mean :

Definition 1.2. *A travelling wave U_* is said to be* asymptotically stable *regarding (1.2) if there is a neighborhood $\mathcal{U}$ of U_* in $(\mathbb{B}(\mathbb{R}, \mathbb{C}^n), \|.\|_\infty)$ such that if $U_0 \in \mathcal{U}$ and $U(\xi, t)$ satisfies (1.2) with the condition $U(\xi, 0) = U_0(\xi)$, then asymptotically, U is a translation of the travelling wave,* i.e., *there exists $r \in \mathbb{R}$ for which :*

$$\lim_{t\to\infty} \|U(\xi, t) - U_*(\xi + r)\|_\infty = 0$$

1.2 The eigenvalue problem :

Let us differentiate the operator φ, according to the Fréchet derivative :

Definition 1.3. *Let V and W be Banach spaces, and $\mathcal{U} \subset V$ be an open subset of V. A function $\Phi : \mathcal{U} \longrightarrow W$ is called Fréchet differentiable at $x \in \mathcal{U}$ if there exists a bounded linear operator $L : V \longrightarrow W$ such that*

$$\lim_{\|h\|_V \to 0} \frac{\|\Phi(x + h) - \Phi(x) - L(h)\|_W}{\|h\|_V} = 0$$

In our case, $V = W = \mathbb{B}(\mathbb{R}, \mathbb{C}^n)$ and $\Phi = \varphi$. Let $p \in \mathbb{B}(\mathbb{R}, \mathbb{C}^n)$ be a function and $\xi \in \mathbb{R}$. Then, because f is differentiable with Jacobian denoted F :

$$\begin{aligned}
\big(\varphi\,(U_* + p)\,\big)(\xi) &= (U_* + p)''(\xi) + c(U_* + p)'(\xi) + f\Big((U_* + p)(\xi)\Big) \\
&= U_*''(\xi) + p''(\xi) + cU_*'(\xi) + cp'(\xi) + f\big(U_*(\xi) + p(\xi)\big) \\
&= U_*''(\xi) + p''(\xi) + cU_*'(\xi) + cp'(\xi) + f\big(U_*(\xi)\big) + F(U_*(\xi)) \cdot p(\xi) + o(p(\xi)) \\
&= \big(\varphi(U_*)\big)(\xi) + \big(\underbrace{p'' + cp' + F(U_*) \cdot p}_{\mathcal{L}(p) := D\varphi(U_*)\cdot p}\big)(\xi) + o\left(\|p\|_\infty\right)
\end{aligned}$$

This calculation defines the Fréchet derivative of φ at the point U_* as the linear operator $\mathcal{L}$:

$$\mathcal{L} : \left| \begin{array}{rcl} \mathbb{B}(\mathbb{R}, \mathbb{C}^n) & \longrightarrow & \mathbb{B}(\mathbb{R}, \mathbb{C}^n) \\ p & \longmapsto & p'' + cp' + F(U_*) \cdot p \end{array} \right.$$

To determine if our travelling wave is asymptotically stable, we will use this very useful proposition, which make a link between the stability of U_* and its differential $\mathcal{L}$. For the proof, please see Alexander, Gardner and Jones [1] and Bates and Jones [2]:

Proposition 1.1. *If $\sigma(\mathcal{L})$ satisfies the following properties :*

1. *0 is a simple eigenvalue.*
2. *there is $\beta < 0$ such that $\sigma(\mathcal{L})\backslash\{0\} \subset \{\lambda : Re(\lambda) < \beta\}$*

Then U_* is asymptotically stable.

Because of this proposition, we will be interested in the spectrum of $\mathcal{L}$ and consider the eigenvalue problem for $\mathcal{L}$:

$$(\mathcal{L} - \lambda \,\mathrm{Id})p = 0 \quad \textit{iff} \quad p'' + cp' + F(U_*) \cdot p = \lambda p \tag{1.4}$$

Which is a linear ODE : write $Y = \begin{pmatrix} p \\ p' \end{pmatrix}$ and the equation becomes :

$$Y'(\xi) = \underbrace{\begin{pmatrix} 0_n & I_n \\ \lambda I_n - F(U_*(\xi)) & -cI_n \end{pmatrix}}_{:=A(\lambda,\xi)} Y(\xi) \tag{1.5}$$

Moreover, as $\xi \longrightarrow \pm\infty$, the matrix $A(\lambda, \xi)$ tends to $A_\pm(\lambda) := \begin{pmatrix} 0_n & I_n \\ \lambda I_n - F(u_\pm) & -cI_n \end{pmatrix}$ (well defined thanks to hypotheses 1).
We want to consider the asymptotic, autonomous system $Y' = A_\pm(\lambda)Y$, from which we will have a straightforward solution : let us compactify the system : let τ be, for $0 < \kappa < \frac{a}{2}$ (where a is defined in hypothesis 1):

$$\tau(\xi) := \frac{e^{2\kappa\xi} - 1}{e^{2\kappa\xi} + 1} \in (-1, 1)$$

Thus, problem (1.5) becomes :

$$\begin{cases} Y' = A(\lambda, \tau)Y \\ \tau' = \kappa(1 - \tau^2) \end{cases} \quad \text{where} \;\; A(\lambda, \tau) = \begin{cases} A(\lambda, \xi(\tau)) & \text{for } \;\tau \neq \pm 1 \\ A_\pm(\lambda) & \text{for } \;\tau = \pm 1 \end{cases} \tag{1.6}$$

From this expression of the equation, we have a solution for the asymptotic, autonomous systems :

$$\Big(Y = \exp\big(A_\pm(\lambda)\xi\big)Y_0,\; \tau = \pm 1\Big)$$

So we have now two fixed points of the systems, with which we will construct the stable and unstable manifolds.

The main goal is to know if λ is an eigenvalue of $\mathcal{L}$. And we will search some eigenvalue in a domain Ω which satisfies this property :

Definition 1.4. *The system is said to split into a domain $\Omega \subset \mathbb{C}$ if there exists a $k \in \mathbb{N}$ with $0 < k < n$ such as for every $\lambda \in \Omega$, the spectrum of $A_{\pm}(\lambda)$ does not contain 0 but contains exactly $n-k$ eigenvalues with negative real part and k eigenvalues with positive real part (count with multiplicities).*

Tanks to this property and with geometric tools, we will draw a contour $\lambda(s) \subset \Omega$ and be able to count the total multiplicity of the eigenvalues encircled by $\lambda(s)$.

1.3 The unstable manifolds and the bundle structure :

Let $K \subset \Omega$ be a contour, a simple closed curve, describing a path for the spectral parameter λ. Assume K does not intersect the spectrum of $\mathcal{L}$ and is parametrized by a smooth map $\lambda : [0,1] \longrightarrow K$. Then $K \times \{\tau \in [-1,1]\}$ is homeomorphic to a cylinder. Let K° be the region enclosed by K, homeomorphic to the disk.
Finally, let $M := K \times \{\tau \in [-1,1]\} \cup K^\circ \times \{\tau \in \{-1,1\}\}$ be homeomorphic to the sphere S^2 (see figure 1.3).

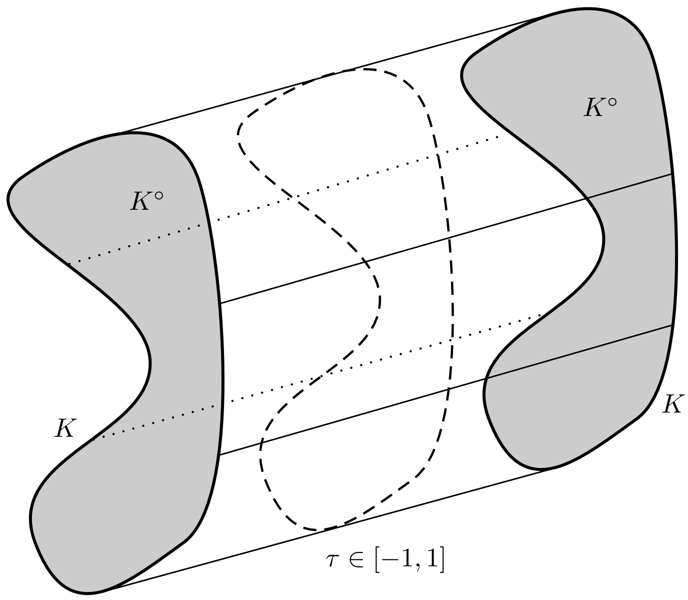

Figure 1.1: The set M, homeomorphic to the sphere S^2.

Now we will define a bundle structure above M. For the definition of a fibre bundle, please see John M.Lee [3] or the chapter 2.

Definition 1.5. *For each point $(\lambda_0, \tau_0) \in M$, the* unstable manifold *of the critical point $(0,-1) \in \mathbb{C}^n \times [-1,1]$ at the point (λ_0, τ_0) is defined by the set of initial data which are transported by the* *flow to* 0 *exponentially in backward time. Formally, it is this set :*

$$W^u(\lambda_0,\tau_0) := \left\{ Y \in \mathbb{C}^n \text{ such that } \lim_{\xi \to -\infty} \Phi^{\lambda_0}_{(Y,\tau_0)}(\xi) = 0 \right\}$$

where Φ *defined the flow of the equation (1.6), i.e.* $\Phi^{\lambda_0}_{(Y,\tau_0)}(\xi) \in \mathbb{C}^n$ *is the solution of* $Z' = A(\lambda_0, \tau(\xi))Z$ *at the point* ξ *with initial data* $Z(\xi(\tau_0)) = Y$.
We can also define an stable manifold of the critical point $(0,+1)$ *as the set of initial data which are transported by the flow at* 0 *exponentially in forward time.*

Remark 1.6. *Often, for a vector* $Y \in W^u(\lambda, \tau(\xi))$*, we will no make notation differentiation between the vector* Y *and the flow : we will keep (abusively) the notation* Y *to design the function, solution of (1.6) with initial condition* $Z(\xi(\tau)) = Y$.

Proposition 1.2. *For each* $(\lambda_0, \tau_0) \in M$*, the unstable manifold* $W^u(\lambda_0,\tau_0)$ *is a vector space over* $\mathbb{C}$.

Proof. Let Y_1, Y_2 be vectors in $W^u(\lambda_0,\tau_0)$ and $\alpha, \beta \in \mathbb{C}$ be scalar. Then let define Z_1, Z_2, Z as

$$\begin{cases} Z_1' = A(\lambda_0,\tau_0)Z_1 \\ Z_1(\xi(\tau_0)) = Y_1 \end{cases}, \begin{cases} Z_2' = A(\lambda_0,\tau_0)Z_2 \\ Z_2(\xi(\tau_0)) = Y_2 \end{cases} \quad \text{and } Z = \alpha Z_1 + \beta Z_2$$

Because of the linearity of the equation, we have

$$\begin{aligned} Z' &= \alpha Z_1' + \beta Z_2' \\ &= \alpha A(\lambda_0,\tau_0) Z_1 + \beta A(\lambda_0,\tau_0) Z_2 \\ Z' &= A(\lambda_0,\tau_0) Z \end{aligned}$$

and the initial data

$$Z(\xi(\tau_0)) = \alpha Y_1 + \beta Y_2.$$

Furthermore, Z decreases to 0 as $\xi \longrightarrow -\infty$. In conclusion, $\alpha Y_1 + \beta Y_2 \in W^u(\lambda_0,\tau_0)$ and the unstable manifold is a vector space. □

With these constructions, we have a natural bundle structure (see definition 2.3 and chapter 3) with total space

$$E := \{(\lambda,\tau,Y) \text{ for } (\lambda,\tau) \in M \text{ and } Y \in W^u(\lambda,\tau)\}$$

above M with fibres $W^u(\lambda,\tau)$. Please see figure 1.3 but keep in mind that only one direction is represented on the figure : the unstable manifold can in the general case be of complex dimension k with $0 < k < n$.

The analysis to find the eigenvalues will be based on this proposition, proved by Alexander, Gardner and Jones in [1], in Lemma 3.6.

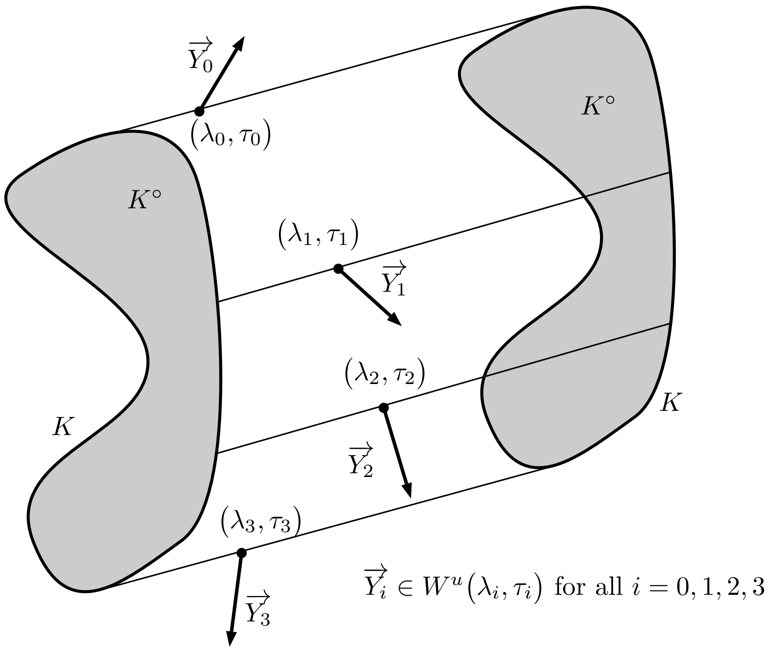

Figure 1.2: The vector bundle : for each $(\lambda_i, \tau_i) \in M$, we associate a space $W^u(\lambda_i, \tau_i)$.

Proposition 1.3. *Under the hypotheses made on the initial traveling wave solution and the set Ω, a solution to the system (1.6) is an eigenfunction for $\mathcal{L}$ corresponding to $\lambda \in \Omega$* if and only if *it is in the unstable manifold for $A_-(\lambda)$ and the stable manifold of $A_+(\lambda)$.*

The idea to prove this proposition is not difficult : due to the splitting of the spectrum of $A_\pm$, all solutions to the dynamical system (1.6) can be decomposed into a basis with vectors in the unstable manifolds of the fixed points at $\pm\infty$. We know that solutions in the stable manifold at $-\infty$ must explode in backward time, while solutions in the unstable manifold at $+\infty$ must explode in forward time. Therefore, a solution is bounded (remind we are looking for solutions in $\mathbb{B}(\mathbb{R}, \mathbb{C}^n)$) if and only if it is in the unstable manifold at $-\infty$ and the stable manifold at $+\infty$. Such solutions are precisely the eigenfunctions of $\mathcal{L}$.

Because of this proposition, we will track the solution of (1.6) which are on the unstable manifold for $A_-(\lambda)$ and then see if there are on the stable manifold of $A_+(\lambda)$. But because a solution in the unstable manifold reaches 0 as $\xi \longrightarrow -\infty$ and we will project these solution from $\mathbb{C}^n$ to S^{2n-1}, let us introduce the B-system (1.7) and the centre-unstable manifold.

1.4 Centre-unstable manifold :

From now, we will focus on the (easier) case where the system splits into Ω with $k = 1$ unstable direction for every $\lambda \in \Omega$ (see definition 1.4) and where the system is asymptotically symmetric, *i.e.* $A_-(\lambda) = A_+(\lambda)$ for every $\lambda \in \Omega$. This formulation of the problem gives a good idea of the general method. With these hypotheses on the system, let us consider $\mu(\lambda)$ the

only eigenvalue of $A_-(\lambda)$, such that *Re*$(\mu(\lambda)) > 0$ (the other eigenvalues have all non positive real parts).
Now, take $(\lambda, \tau) \in M$ and $Y \in W^u(\lambda, \tau)$ (remind the remark 1.6 for the abusive notation : here Y design the unique solution to the A-system (3.1) with initial data $Z(\xi(\tau)) = Y$ which decrease exponentially to 0 as $\xi \longrightarrow -\infty$). Then, because it is a vector space (see proposition 1.2), the function defined by

$$X(\xi) := e^{-\mu(\lambda)\xi} Y(\xi)$$

is in the unstable manifold $X \in W^u(\lambda, \tau)$. Then

$$X'(\xi) = -\mu(\lambda) e^{-\mu(\lambda)\xi} Y(\xi) + e^{-\mu(\lambda)\xi} Y'(\xi)$$
$$X'(\xi) = \Big(A\big(\lambda, \tau(\xi)\big) - \mu(\lambda)\ \mathrm{Id}\Big) X(\xi)$$

This motivates the following system on $\mathbb{C}^n$:

$$\begin{cases} X' = B(\lambda, \tau) Y \\ \tau' = \kappa(1 - \tau^2) \end{cases} \quad \text{where} \;\; B(\lambda, \tau) = A(\lambda, \tau) - \mu(\lambda)\ \mathrm{Id} \tag{1.7}$$

The transformation from the A-system (1.6) to the B-system (1.7) is made because it will be more convenient to work with trajectories in the unstable manifold produced by the B-system. I will explain later why.

Definition 1.7 (Notation)**.** *Solutions to the A-system (1.6) will be denoted with a $\sharp$, whereas solutions to the B-system (1.7) without. That is, if $Z^\sharp$ denote the solution to the A-system (3.1) where $Z^\sharp(\xi_0) \in W^u(\lambda, \tau(\xi_0))$, then $Z(\xi) := e^{-\mu(\lambda)(\xi - \xi_0)} Z^\sharp$ is the unique solution to the B-system (1.7) that agrees with $Z^\sharp$ at (λ, ξ_0).*
Similarly, if Z is a solution to the B-system, then $Z^\sharp(\xi) := e^{\mu(\lambda)(\xi - \xi_0)} Z(\xi)$ is the unique solution to the A-system that agrees with Z at ξ_0.

Let $X^-(\lambda)$ be an unstable eigenvector for $A_-(\lambda)$, *i.e.* $A_-(\lambda) X^-(\lambda) = \mu(\lambda) X^-(\lambda)$. Then for every vector $Z \in \mathrm{Span}_{\mathbb{C}}\left(X^-(\lambda)\right)$, then $(Z, -1)$ is a fixed point in the B-system.

Definition 1.8. *For a non-zero path of eigenvectors $X^-(\lambda)$ for $A_-(\lambda)$, in $\tau = -1$, there is exactly $n - 1$ complex stable direction, one complex centre direction corresponding to the line of fixed points, and the real unstable τ direction. This is called the* centre-unstable manifold.

The next section is made for introduce the geometrical tools we will use at the chapter 3.

Chapter 2

Paths on a principal fibre bundle : example of the Hopf bundle.

In this chapter, we will introduce some tools which will be useful later.

2.1 Generalities :

This first section follow the definitions given by John M. Lee in his book on smooth manifolds [3] and the thesis of Rupert Way [4].

2.1.1 Principal fibre bundle :

Definition 2.1. *A n-dimensional complex manifold P is a topological space which satisfies :*

- *P is a separated space,*
- *There exists a countable basis for the topology of P,*
- *Each point $p \in P$ has a neighborhood $\mathcal{U}$ that is homeomorphic to an open subset of $\mathbb{C}^n$: there exists $\mathcal{V}$ an open subset of $\mathbb{C}^n$ and a homeomorphism $\phi_p : \mathcal{U} \longrightarrow \mathcal{V}$.*

Remark 2.2. *A manifold is said to be of class C^k, with $k \in \mathbb{N} \cup \{\infty\}$ if all the homeomorphisms ϕ_p of the definition above are C^k-diffeomorphisms.*

Definition 2.3. *Let P, M be a complex smooth manifold and $\pi : P \longrightarrow M$. (P, π, M) is said to be an n-dimensional complex vector bundle over M if π is smooth and for every $x \in M$,*

- *$\pi^{-1}(\{x\})$ (called the fibre above x) is isomorphic to $\mathbb{C}^n$,*
- *there is a neighborhood $\mathcal{U}$ of x, and a diffeomorphism $\phi_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times \mathbb{C}^n$.*

If a topological group G acts (on the right or on the left) on a manifold P, there exists a natural fibre bundle, called principal fibre bundle.

Definition 2.4. *The manifold M above the complex vector bundle lives is $M := P/G$. To be a principal fibre bundle, G has to preserve the fibres and act freely $(g \cdot x = x \Rightarrow g = e)$ and transitively $(\forall x, y,\ \exists g \in G,\ x = g \cdot y)$ on them.*

This implies that, for a principal fibre bundle, each fibres is homeomorphic to the group G itself.
If, for a $p \in P$, we denote its conjugate class by $\bar{p} = \{g \cdot p, g \in G\}$, then the projection for a principal fibre bundle is given by $\pi : p \in P \longmapsto \bar{p} \in M$. Henceforth, we will only consider principal fibre bundles.

2.1.2 Decomposition of the tangent space :

In a C^1-manifold P, let define the tangent space on a point $p \in P$: for a coordinate chart $\phi_p : \mathcal{U} \longrightarrow \mathcal{V} \subset \mathbb{C}^n$. Consider also two curves $\gamma_1, \gamma_2 : [-1, 1] \longrightarrow P$ with $\gamma_1(0) = \gamma_2(0) = p$ and such that $\phi_p \circ \gamma_i : [-1, 1] \longrightarrow \mathbb{R}$ are differentiable at 0 for $i = 1, 2$. Then γ_1 and γ_2 are said to be equivalent at 0 if and only if $\left(\phi_p \circ \gamma_1\right)'(0) = \left(\phi_p \circ \gamma_2\right)'(0)$. This defines a equivalence relation $\simeq$.

Definition 2.5. *For a point $p \in P$, the tangent space is defined by :*

$$T_pP := \{[\gamma] \textit{ for } \gamma : [-1, 1] \longrightarrow P \textit{ smooth such that } \gamma(0) = p\}$$

where $[\gamma]$ is the equivalent class of γ for the equivalent relation $\simeq$. When the manifold P is a sub-manifold, included in $\mathbb{C}^n$, we have the relation (and the tangent vector space may be imagine as this expression, more intuitive) :

$$T_pP := \{\gamma'(0) \textit{ for } \gamma : [-1, 1] \longrightarrow P \textit{ smooth such that } \gamma(0) = p\}$$

It is easy to prove that it is a vector space of dimension the same dimension of P.

There are many ways to define the tangent vector space and John M. Lee [3] explain some. This one has a simple geometric interpretation as the set of all “velocities” at the point p for a path which pass through p.

At each point $p \in P$, we would like to separate the tangent vector space into a vertical and a horizontal subspace, according to the fibration, as in the figure 2.1.

The vertical subspace is easy to define :

Definition 2.6. *The vertical subspace at a point $p \in P$ is defined by*

$$V_pP = \pi_*^{-1}(\{0\})$$

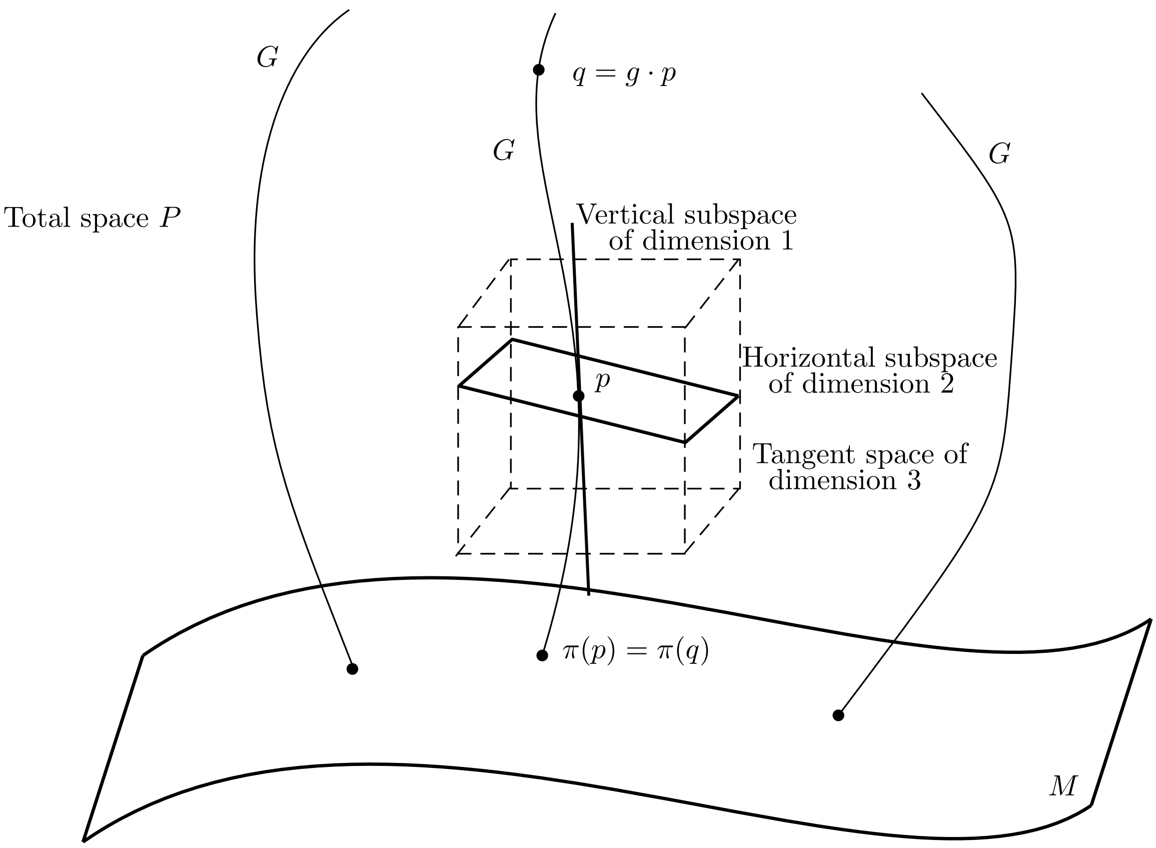

Figure 2.1: The separation of the tangent space into two subspaces

where $\pi_* : T_pP \longrightarrow T_{\pi(p)}M$ is the push-forward of π, i.e. if $v = \gamma'(0) \in T_pP$ where $\gamma(0) = p$, then

$$\pi_*(v) = \left.\frac{d}{dt}\right|_{t=0} (\pi \circ \gamma)(t)$$

Remark 2.7. *The geometric intuition comes from the local cross product decomposition, where if a path moves only along the fibre in the cross product, then under the projection map the path is trivial; therefore, such a path has a tangent vector which vanishes under the projection map derivative.*
Thus, the vertical subspace is homeomorphic to the tangent space at p according to G.

Now, we would like to define the horizontal subspace H_pP such that $T_pP = V_pP \oplus H_pP$. But it has to respect the bundle :

Definition 2.8. *At each point $p \in P$, we smoothly associate a horizontal subspace $H_pP \subset T_pP$ such that :*

- $\forall p \in P,\ T_pP = V_pP \oplus H_pP,$
- $\forall g \in G,\ \forall p \in P,\ \left(\Phi_g^p\right)_* (H_pP) = H_{\Phi_g(p)}P$ *where $\Phi_g : P \longrightarrow P$ denote the action of*

$g \in G$ on P and $(\Phi_g^p)_*$ the push-forward of the action :

$$\left(\Phi_g^p\right)_* : \left| \begin{array}{ccc} T_pP & \longrightarrow & T_{\Phi_g(p)}P \\ \gamma'(0) & \longmapsto & \frac{d}{dt}\Big|_{t=0} (\Phi_g \circ \gamma)(t) \end{array} \right.$$

Figure 2.1 illustrates the first condition : H_pP must be transverse to the fibre (in order to span the rest of the tangent space). The second condition says that if we use $(\Phi_g^p)_*$ to push H_pP along the fibre, then the result is the same as if we first push p along the fibre using Φ_g and then form the horizontal subspace at the point $\Phi_g(p)$, as described in the figure 2.2.

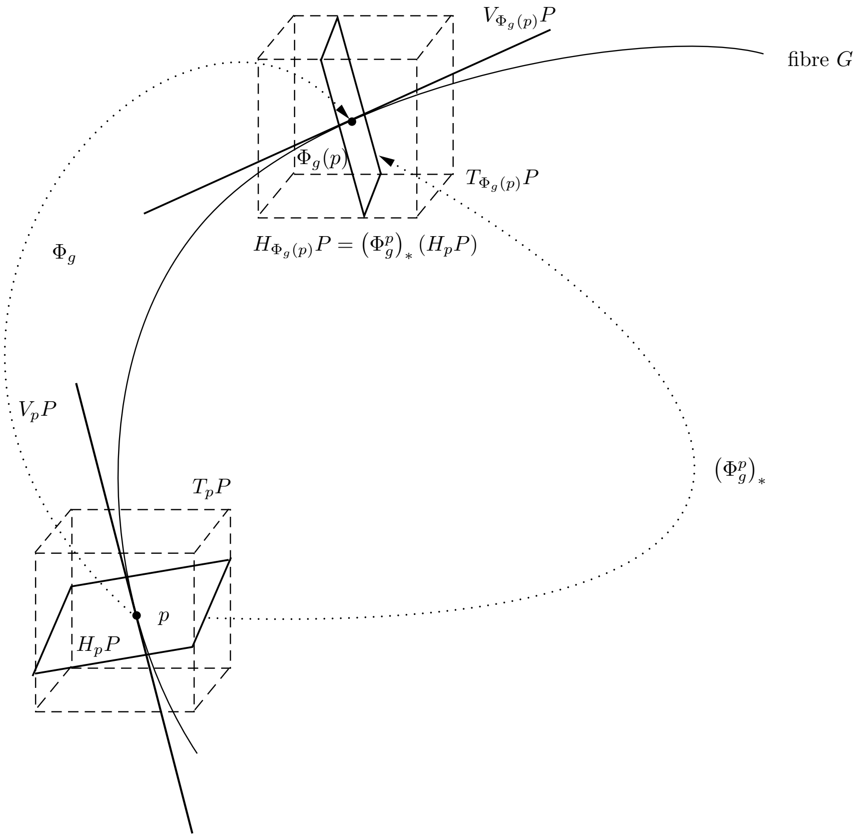

Figure 2.2: The mechanics of the Horizontal subspace.

2.1.3 Paths on a principal fibre bundle :

The goal of the decomposition of the tangent space above is to project a tangent vector in the vertical subspace according with the chosen decomposition $T_pP = V_pP \oplus H_pP$. Take a differentiable path v on the bundle, we would like to know how the path is with reference to the fibres.

Definition 2.9. *The choice of a set of horizontal subspaces defines a projecting map, for each*

$p \in P$:

$$\omega_p : \left| \begin{array}{rcl} T_pP = V_pP \oplus H_pP & \longrightarrow & V_pP \\ x = v + h & \longmapsto & h \end{array} \right.$$

where $x = v + h$ is the unique decomposition of x.
The map ω define a connection *and is entirely determine by the choice of the horizontal subspaces.*

Definition 2.10. *A path $v : I \longrightarrow P$ (where I is a interval of $\mathbb{R}$ and (P, π, M, G) a principal fibre bundle) is said to be* horizontal *if $\forall s \in I,\ v'(s) \in H_pP,$ i.e. $\forall s \in I,\ \omega_{v(s)}(v'(s)) = 0.$*

Definition 2.11. *At each point $v(s)$ of a path $v : [0,1] \longrightarrow P$ in the principal bundle, we can consider $w(s) = g(s) \cdot v(s)$ where $g(s) \in G$ and $w(s) \in P$. The path $w(s)$ is on the same fibre as $v(s)$ at each time s. w is said to be a* lift *for v.*
If w is furthermore horizontal and $w(0) = v(0)$, then w is said to be a horizontal lift *(see figure 2.6 for the horizontal lift on the Hopf bundle). Indeed, it satisfies :*

- $w(0) = v(0),$
- $\forall s,\ \pi(w(s)) = \pi(v(s))$ *i.e.* $\exists g : [0,1] \longrightarrow G,\ \forall s \in [0,1],\ w(s) = g(s) \cdot v(s)$
- $\forall s,\ \omega_{w(s)}(w'(s)) = 0.$

Remark 2.12. *We can think of $g(s)$ as an expression describing how much v deviates from being horizontal. This is the information we would like to collect later.*

2.2 An example : The Hopf bundle :

The Hopf bundle is the bundle we will use later. The Hopf bundle is the local decomposition of the $(2n-1)$-sphere onto the S^1 sphere times $\mathbb{C}P^{n-1}$. This structure will appear naturally in the computation of the eigenvalues of $\mathcal{L}$ (see [5] and [1]).

2.2.1 Definition :

Here we have $P = S^{2n-1} := \{z \in \mathbb{C}^n, |z_1|^2 + \cdots + |z_n|^2 = 1\}$ the $(2n-1)$-sphere and M is the $(n-1)$-dimensional projective space $\mathbb{C}P^{n-1}$.
For an element $e^{i\theta} \in S^1$ and $z \in S^{2n-1}$, then, $e^{i\theta}z \in S^{2n-1}$ and, using the projective coordinates, $[e^{i\theta}z] = [z]$. So S^1 acts on S^{2n-1} and the fibre bundle is given by the projection map $\pi : z \in S^{2n-1} \longmapsto [z] \in \mathbb{C}P^{n-1}$. (See figure 2.3).

In the case of $n = 2$ and with the identification via the Riemann sphere of $\mathbb{C}P^1$ with S^2, we can write explicitly the projection π :

$$\pi : \left| \begin{array}{rcl} S^3 \subset \mathbb{C}^2 & \longrightarrow & S^2 \subset \mathbb{R}^3 \\ (z_1, z_2) & \longmapsto & (\overline{z_1}z_2 + z_1\overline{z_2}\ ,\ i(\overline{z_1}z_2 - z_1\overline{z_2})\ ,\ |z_1|^2 - |z_2|^2) \end{array} \right.$$

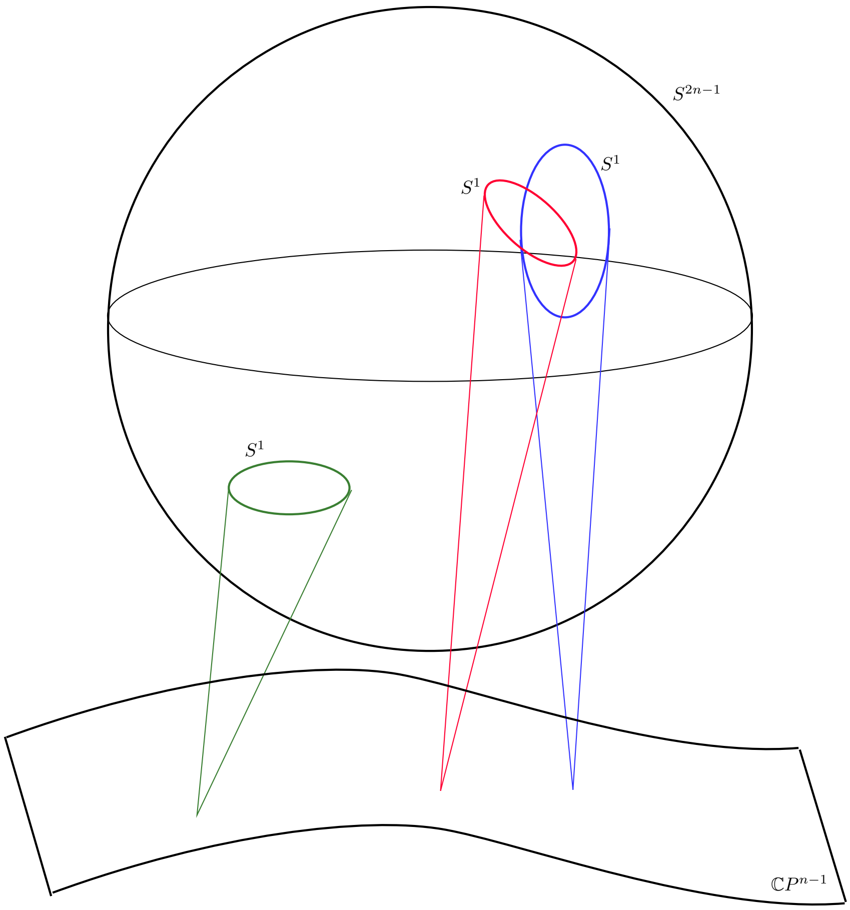

Figure 2.3: The Hopf bundle

And we can easily verify that if $z \in S^3$ and $e^{i\theta} \in S^1$ then $\pi\left(e^{i\theta} z\right) = \pi(z)$.
So, S^3 is viewed as follow : for each point in S^2, we associate a circle S^1 (see figure 2.4, taken from Wikipdia).

2.2.2 Natural connection :

For the Hopf bundle, there is a natural way to construct all the objects we have construct in general in the previous section. We will calculate them step by step.

The tangent vector space : For the Hopf bundle case, the tangent vector space can be calculated explicitly. For $p \in P = S^{2n-1}$, $T_p S^{2n-1}$ is a $\mathbb{R}$ vector space with dimension $2n-1$ and we can prove that :

$$T_p S^{2n-1} = \{z \in \mathbb{C}^n, Re\left(\langle z, p\rangle\right) = 0\} \tag{2.1}$$

where $\langle z, p\rangle = p^* z = \sum_{j=1}^{n} = z_j \overline{p_j}$ is the canonical hermitian product on $\mathbb{C}^n$.

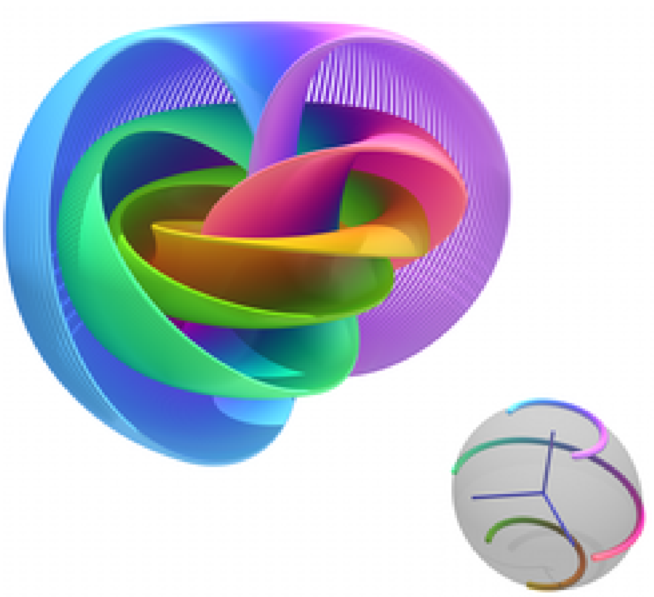

Figure 2.4: The Hopf bundle for $n = 2$

The vertical subspace : For a point $p \in S^{2n-1}$, let's calculate the vertical subspace $V_pS^{2n-1} \subset T_pS^{2n-1}$ as the kernel of the derivative of the projection map $\pi_p^* : T_pS^{2n-1} \longrightarrow T_{\pi(p)}\mathbb{C}P^{n-1}$:

$$V_pS^{2n-1} = \mathrm{Ker}\,(\pi_p^*)$$

Let's remind (remark 2.7) that we can see the vertical subspace as the tangent space with reference to the fibre, here S^1. With this point of view, it is easy to calculate the vertical subspace at a point $p \in S^{2n-1}$:

If $\gamma'(0) \in V_pP$, then there exists a neighborhood $[-\varepsilon, \varepsilon]$ of 0 such that $\gamma(t) = pe^{i\theta(t)}$ with $\theta(t) \in \mathbb{R}$ (because the projection of γ is trivial, in a neighborhood of 0) and then $\gamma'(0) = \theta'(0)ip$.

Reciprocally, let be $\alpha \in \mathbb{R}$, then the path

$$\gamma : \left| \begin{array}{rcl} [-1,1] & \longrightarrow & S^{2n-1} \\ t & \longmapsto & pe^{\alpha i t} \end{array} \right.$$

satisfies $\gamma(0) = p$, $\pi(\gamma(t)) = \pi(p)$ for every t and $\gamma'(0) = \alpha ip$. To conclude :

$$V_pS^{2n-1} = \mathrm{Span}_{\mathbb{R}}(ip) := \{\alpha ip,\ \alpha \in \mathbb{R}\} \subset T_pP$$

The horizontal subspace : As seen in the previous section, there is an infinity choices for the horizontal subspace. But there is a “natural” one. *Natural* for at least two reasons : firstly because we will consider the orthogonal subspace of the vertical subspace in T_pP with reference to the natural inner product. And secondly because in the case of $n = 2$, Rupert Way proves that this choice is unique.

This natural horizontal subspace (see figure 2.5) is given by

$$H_p(S^{2n-1}) := V_p(S^{2n-1})^{\perp_{\mathbb{R}}} = \{z \in \mathbb{C}^n, \langle p, z\rangle = 0\}$$

We will verify this choice is actually a horizontal subspace : let's verify it satisfies the assertions enumerate in the definition 2.8 :

- By definition we have $T_p(S^{2n-1}) = V_p(S^{2n-1}) \oplus H_p(S^{2n-1})$.
- Let be $g = e^{i\theta} \in S^1$ and $p \in S^{2n-1}$, then for every $z = \gamma'(0) \in H_p(S^{2n-1})$, we have

$$\left(\Phi_g^p\right)_*(z) = \left.\frac{d}{dt}\right|_{t=0} (e^{i\theta}\gamma(t)) = e^{i\theta} z$$

  Then $\left\langle e^{i\theta} z, e^{i\theta} p\right\rangle = \langle z, p\rangle = 0$ and $\left(\Phi_g^p\right)_* (H_p(S^{2n-1})) \subset H_{\Phi_g(p)}(S^{2n-1})$. By a dimension argument, we have the equality

$$\left(\Phi_g^p\right)_* \left(H_p\left(S^{2n-1}\right)\right) = H_{\Phi_g(p)}\left(S^{2n-1}\right)$$

  and the "natural" choice is coherent with our definition of horizontal subspace.

The connection 1-form : This "natural" choice of a horizontal subspace defines a linear projection : $\omega_p : T_pS^{2n-1} \longrightarrow V_pS^{2n-1}$, parallel to H_pS^{2n-1}. In the natural case for the Hopf bundle, the natural connection 1-form is given by the hermitian product on $\mathbb{C}^n$:

$$\omega_p(z) = \langle z, p\rangle = \sum_{j=1}^{n} z_j \overline{p_j}$$

2.3 Paths in the Hopf bundle and geometric phase :

In this section we will describe the paths on the Hopf bundle, with reference to the natural connection seen above.

2.3.1 Horizontal lift :

Definition 2.13. *A differentiable path $v : [0,1] \longrightarrow S^{2n-1}$ is a horizontal path if for each $s \in [0,1],\ v'(s) \in H_{v(s)}S^{2n-1}$, i.e. $\omega_{v(s)}v'(s) = \langle v'(s), v(s)\rangle = 0$*

For a path in a bundle, we can consider other paths which have the same projection about π. Some of these path are particularly interesting : the horizontal ones.

Definition 2.14. *For a differentiable path $v : [0,1] \longrightarrow S^{2n-1}$ in the Hopf bundle. The* horizontal lift *of v is a path $w : [0,1] \longrightarrow S^{2n-1}$ for which the following holds (see figure 2.6) :*

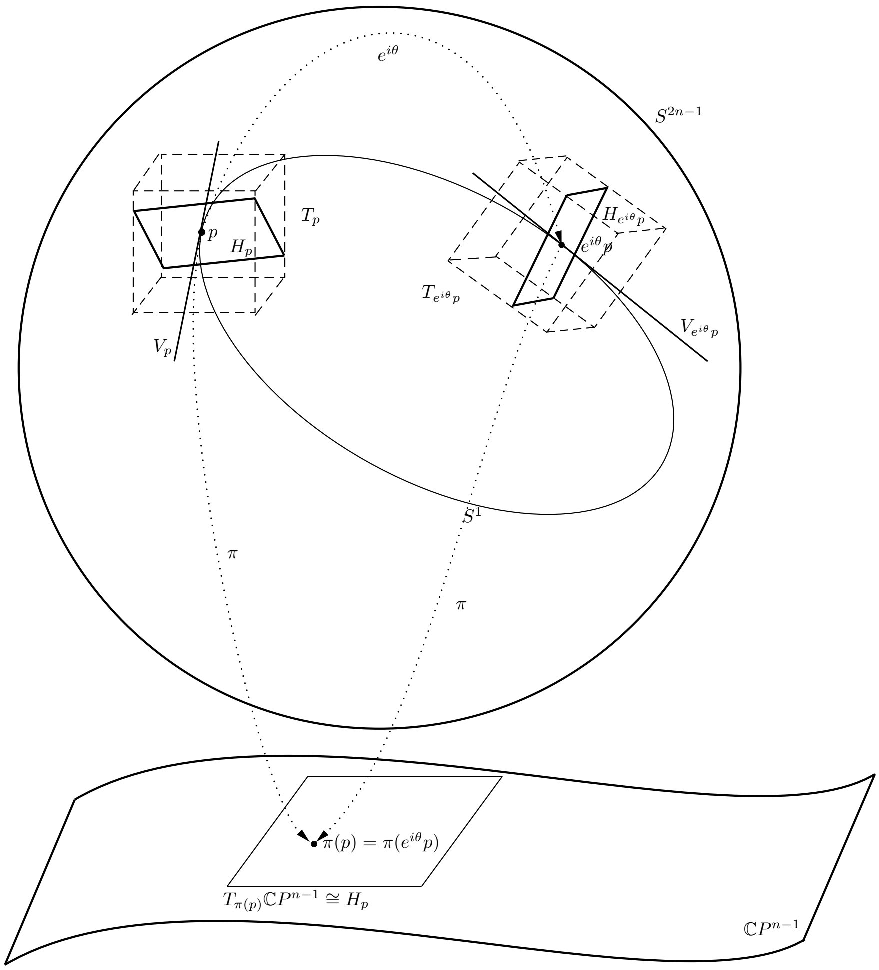

Figure 2.5: The natural decomposition on the Hopf bundle

- $w(0) = v(0)$
- $\forall s \in [0,1],\ \pi\big(w(s)\big) = \pi\big(v(s)\big)$
- $\forall s \in [0,1],\ w'(s) \in H_{w(s)}S^{2n-1}$

Proposition 2.1. *For a differentiable path $v(s)$ on the Hopf bundle, there exists a unique horizontal lift $w(s)$.*

Remark 2.15. *This result is true for every principal fibre bundle (see Way [4], page 16) but require the differential point of view for the connection, and we have not develop this notion.*

Proof. w is a lift for v so we have a path $e^{i\theta(s)} \in S^1$ such that for every $s,\ v(s) = e^{i\theta(s)}w(s)$. Let's derivate the expression $w(s) = e^{-i\theta(s)}v(s)$:

$$w'(s) = -i\theta'(s)e^{-i\theta(s)}v(s) + e^{-i\theta(s)}v'(s)$$

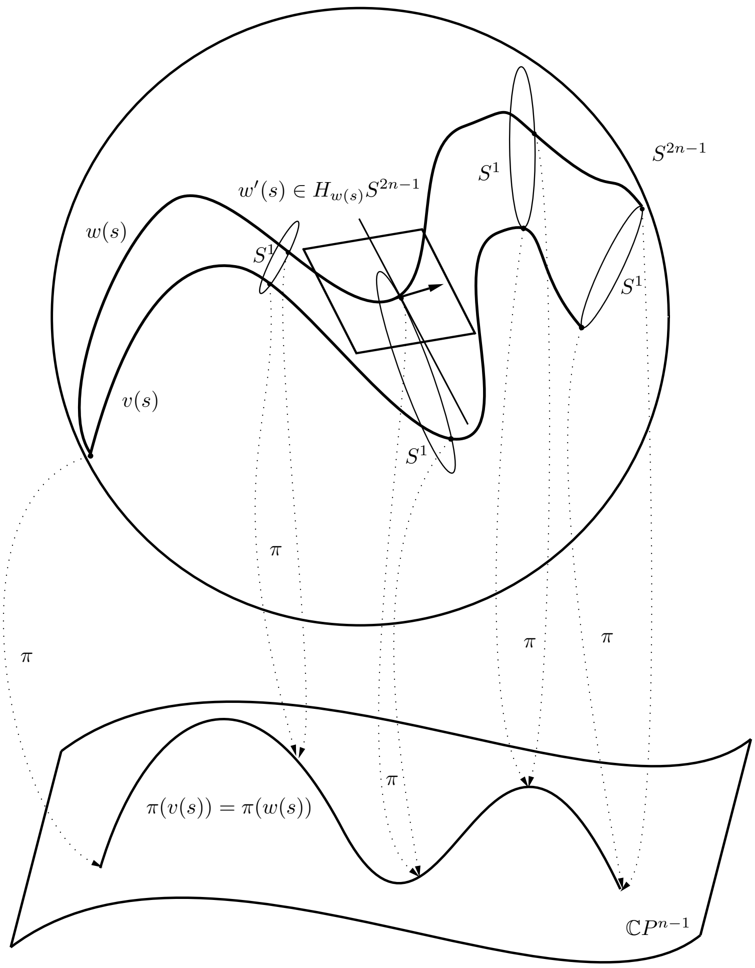

Figure 2.6: The natural horizontal lift for a path $v(s)$ in the Hopf bundle.

And take the connection at the point $w(s)$ (equal zero because of the horizontality of w) :

$$\begin{aligned}
\omega_{w(s)}(w'(s)) &= \langle w'(s), w(s)\rangle \\
&= \left\langle -i\theta'(s)e^{-i\theta(s)}v(s) + e^{-i\theta(s)}v'(s), e^{-i\theta(s)}v(s)\right\rangle \\
&= -i\theta'(s)e^{-i\theta(s)}e^{i\theta(s)}\langle v(s), v(s)\rangle + e^{-i\theta(s)}e^{i\theta(s)}\langle v'(s), v(s)\rangle \\
0 &= -i\theta'(s)\underbrace{\|v(s)\|}_{=1} + \langle v'(s), v(s)\rangle
\end{aligned}$$

And so θ is the unique solution of the Cauchy's problem

$$\begin{cases} \theta'(s) = -i\,\langle v'(s), v(s)\rangle \\ \theta(0) = 0 \end{cases} \tag{2.2}$$

□

2.3.2 The geometric phase : definition :

Definition 2.16. *Let $v : [0,1] \longrightarrow S^{2n-1}$ be a path and let w be its horizontal lift. The* phase curve *$\theta : [0,1] \longrightarrow \mathbb{R}$ is defined by the equation*

$$v(s) = e^{i\theta(s)} w(s)$$

And the geometric phase *is the change in the phase curve, i.e. :*

$$\mathrm{GP}\left(v\big([0,1]\big)\right) := \frac{\theta(1) - \theta(0)}{2\pi} \tag{2.3}$$

θ must be understood as a path in the fibre describing the displacement along v between v and its horizontal lift w. The geometric phase represents the lack of horizontality of the path v (see figure 2.7)

Remark 2.17. *Even if the phase curve is not unique (chosen modulo 2π, usually $\theta(0) = 0$), the* geometric phase *is unique, because of the renormalisation about $\theta(0)$.*

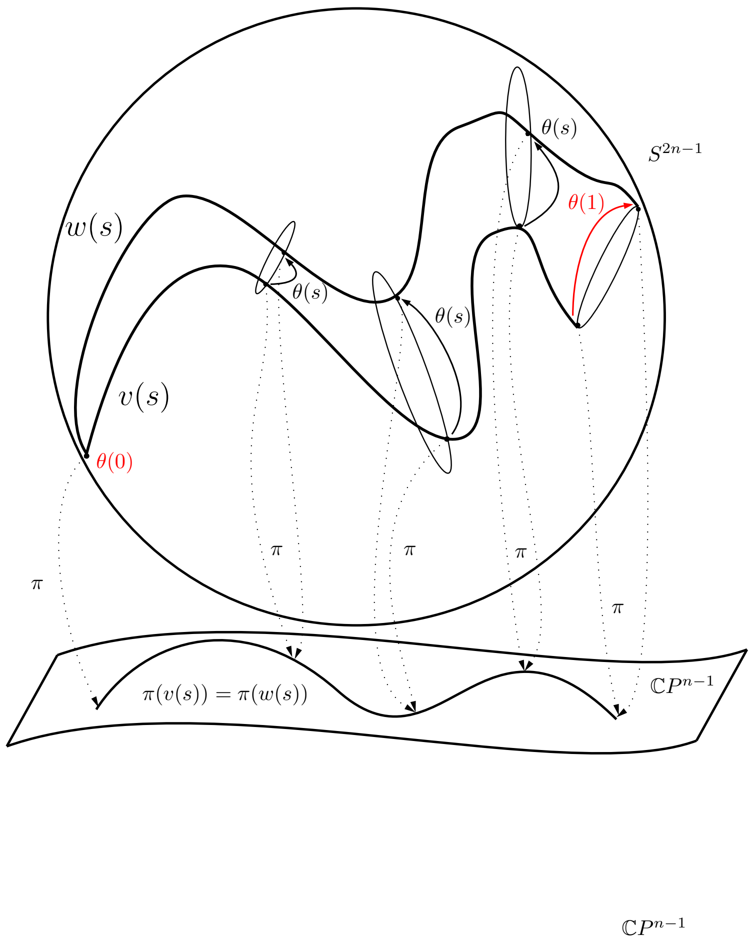

Figure 2.7: The phase curve along a path v with horizontal lift w and the geometric phase.

2.3.3 The geometric phase : calculation :

In the proof of proposition 2.1, we have found a Cauchy problem (2.2) for the phase curve $\theta(s)$. We can integrate this equation and it gives us this formula :

Proposition 2.2. *The geometric phase of a differentiable path $v : [0,1] \longrightarrow S^{2n-1}$ is equal to :*

$$\mathrm{GP}\left(v\big([0,1]\big)\right) = \frac{1}{2i\pi}\int_0^1 \langle v'(s), v(s)\rangle\, ds \tag{2.4}$$

Because for every $s \in [0,1],\ v'(s) \in T_{v(s)}S^{2n-1} \underset{(2.1)}{=} \{z \in \mathbb{C}^n, Re\left(\langle z, v(s)\rangle\right) = 0\}$, we have $\langle v'(s), v(s)\rangle = i\, Im\left(\langle v'(s), v(s)\rangle\right)$ and the following proposition holds :

Proposition 2.3. *The geometric phase of a differentiable path $v : [0,1] \longrightarrow S^{2n-1}$ is equal to :*

$$\mathrm{GP}\left(v\big([0,1]\big)\right) = \frac{1}{2\pi}\int_0^1 Im\left(\langle v'(s), v(s)\rangle\right) ds \tag{2.5}$$

2.4 Geometric phase for a path in $\mathbb{C}^n\backslash\{0\}$:

In fact, given a path $v : [0,1] \longrightarrow \mathbb{C}^n\backslash\{0\}$, we will firstly project it in a path on the sphere $\widehat{v}(s) := \frac{v(s)}{\|v(s)\|} \in S^{2n-1}$ and then calculate the geometric phase of $\widehat{v}([0,1])$. But the following proposition gives formulas for calculate it, without calculate explicitly $\widehat{v}(s)$.

Proposition 2.4. *For a differentiable non-zero path $v(s) \in \mathbb{C}^n\backslash\{0\}$ and its spherical projection $\widehat{v}(s) \in S^{2n-1}$, we have the following relations :*

- *The natural connection of $\widehat{v}'(s)$ is*

$$\omega_{\widehat{v}(s)}\left(\widehat{v}'(s)\right) = i\,\frac{Im\big(\, \langle v'(s), v(s)\rangle\,\big)}{\|v(s)\|^2} \tag{2.6}$$

- *The geometric phase along $\widehat{v}(s)$ can be computed as :*

$$\mathrm{GP}\left(\widehat{v}\big([0,1]\big)\right) = \frac{1}{2\pi}\int_0^1 \frac{Im\big(\, \langle v'(s), v(s)\rangle\,\big)}{\|v(s)\|^2} ds \tag{2.7}$$

- *If v is also a closed curve (it will be the case later), then the real part of the integral above is zero and the geometric phase can be computed as :*

$$\mathrm{GP}\left(\widehat{v}\big([0,1]\big)\right) = \frac{1}{2i\pi}\int_0^1 \frac{\langle v'(s), v(s)\rangle}{\langle v(s), v(s)\rangle} ds \tag{2.8}$$

This is the formula we will use in computation.

Proof. Because $\widehat{v}(s)$ is the spherical projection of $v(s)$, we have the relation

$$\widehat{v}(s) = \frac{v(s)}{\|v(s)\|} = \frac{v(s)}{\langle v(s), v(s)\rangle^{\frac{1}{2}}}$$

so

$$\widehat{v}'(s) = \frac{\langle v(s), v(s)\rangle^{\frac{1}{2}}\ v'(s) - \frac{\langle v'(s), v(s)\rangle + \langle v(s), v'(s)\rangle}{2\langle v(s), v(s)\rangle^{\frac{1}{2}}}\, v(s)}{\langle v(s), v(s)\rangle} \tag{2.9}$$

$$\tag{2.10}$$

$$= \frac{v'(s)}{\langle v(s), v(s)\rangle^{\frac{1}{2}}} - \frac{Re\left(\langle v'(s), v(s)\rangle\right)}{\langle v(s), v(s)\rangle^{\frac{3}{2}}}\, v(s) \tag{2.11}$$

$$\tag{2.12}$$

$$\widehat{v}'(s) = \frac{v'(s)}{\|v(s)\|} - \frac{Re\left(\langle v'(s), v(s)\rangle\right)}{\|v(s)\|^3}\, v(s) \tag{2.13}$$

With the relation (2.13), we are able to prove the above proposition :

- The natural connection for $\widehat{v}'(s)$ is :

$$\omega_{\widehat{v}(s)}\left(\widehat{v}'(s)\right) = \langle \widehat{v}'(s), \widehat{v}(s)\rangle$$

$$= \frac{\langle v'(s), v(s)\rangle}{\|v(s)\|^2} - \frac{Re\left(\langle v'(s), v(s)\rangle\right)}{\|v(s)\|^4}\ \langle v(s), v(s)\rangle$$

$$= \frac{\langle v'(s), v(s)\rangle - Re\left(\langle v'(s), v(s)\rangle\right)}{\|v(s)\|^2}$$

$$\omega_{\widehat{v}(s)}\left(\widehat{v}'(s)\right) = i\,\frac{Im\big(\,\langle v'(s), v(s)\rangle\,\big)}{\|v(s)\|^2}$$

- The geometric phase :

$$\mathrm{GP}\left(\widehat{v}\big([0,1]\big)\right) = \frac{1}{2\pi}\int_0^1 \textit{Im}\left(\omega_{\widehat{v}(s)}\left(\widehat{v}'(s)\right)\right) ds$$

$$= \frac{1}{2\pi}\int_0^1 \textit{Im}\left(i\,\frac{\textit{Im}\big(\,\langle v'(s), v(s)\rangle\,\big)}{\|v(s)\|^2}\right) ds$$

$$\mathrm{GP}\left(\widehat{v}\big([0,1]\big)\right) = \frac{1}{2\pi}\int_0^1 \frac{\textit{Im}\big(\,\langle v'(s), v(s)\rangle\,\big)}{\|v(s)\|^2} ds$$

- If v is a closed curve, then $v(1) = v(0)$, or $\ln\big(\|v(1)\|\big) - \ln\big(\|v(0)\|\big) = 0$ and

$$\ln\big(\|v(1)\|\big) - \ln\big(\|v(0)\|\big) = \int_0^1 \frac{d}{ds}\Big(\ln\big(\|v(s)\|\big)\Big) ds$$

$$= \int_0^1 \frac{\frac{d}{ds}\big(\|v(s)\|\big)}{\|v(s)\|} ds$$

$$0 = \int_0^1 \frac{2\,\textit{Re}\big(\,\langle v'(s), v(s)\rangle\,\big)}{\|v(s)\|} ds$$

And, considering also the second point, we can conclude :

$$\mathrm{GP}\left(\widehat{v}\big([0,1]\big)\right) = \frac{1}{2i\pi}\int_0^1 \frac{\langle v'(s), v(s)\rangle}{\langle v(s), v(s)\rangle} ds$$

□

Chapter 3

The method in the Hopf bundle for the symmetric case and $k = 1$:

This chapter is largely inspired from the PhD thesis of Colin Grudzien [5].
Go back to our eigenvalue problem (1.6). Reminder the proposition 1.3 : we would like to track the solution of (1.6) which are on the unstable manifold for $A_{-}(\lambda)$. Remember we will focus on the case where the system splits into Ω with $k = 1$ unstable direction for every $\lambda \in \Omega$ and where the system is asymptotically symmetric. This formulation of the problem gives a good idea of the general method : in fact, for $k > 1$, we will transpose the problem in the wedge product $\bigwedge^k(\mathbb{C}^n) \cong \mathbb{C}^{\binom{n}{k}}$ where, seen in this space, the unstable direction is 1-dimensional. But for $k \simeq \frac{n}{2}$ and $n \gg 1$, the main difficulty is a computational difficulty because

$$\binom{n}{k} \underset{n\to\infty}{\sim} \sqrt{\frac{2}{\pi}} \frac{2^n}{\sqrt{n}}$$

for $k \sim \frac{n}{2}$. So one of the aims of this internship was to understand an other way to compute the problem. But on this chapter, we will follow this plan :

Step 1 : Choose a contour $K \subset \Omega$ that does not intersect the spectrum of $\mathcal{L}$. K is parametrized by $\lambda : [0, 1] \longrightarrow K$ (usually a circle).

Step 2 : For every $\lambda \in K$, define $X^{+}(\lambda)$ an analytic loop of eigenvectors for $A_{+}(\lambda)$ corresponding to the eigenvalue of positive real part.

Step 3 : For each $\lambda \in K$, resolve the A-system : $\frac{d}{d\xi} Y(\lambda, \tau(\xi)) = A(\lambda, \tau(\xi)) Y(\lambda, \tau(\xi))$ such that Y is in the unstable manifold.

Step 4 : For each $\xi \in \mathbb{R}$, calculate the geometric phase of $Y(\lambda(s), \tau(\xi))$ with respect to $X^{+}(\lambda(s))$ *i.e.* $\mathrm{GP}\left(Y\left(\lambda([0,1]), \tau(\xi)\right)\right) - \mathrm{GP}\left(X^{+}\left(\lambda([0,1])\right)\right)$.

3.1 The induced phase on the Hopf bundle :

Let remind the eigenvalue problem : $\mathcal{L}$ is the linearization of a reaction-diffusion equation about a steady state. From the equation $(\mathcal{L} - \lambda)p = 0$ for $\lambda \in \Omega$, Ω simply open and simply connected, one can derive the system :

$$\begin{cases} Y' = A(\lambda, \tau)Y \\ \tau' = \kappa(1 - \tau^2) \end{cases} \quad \text{where } A(\lambda, \tau) = \begin{cases} A(\lambda, \xi(\tau)) & \text{for } \tau \neq \pm 1 \\ A_{\pm}(\lambda) & \text{for } \tau = \pm 1 \end{cases} \tag{3.1}$$

As seen before, it will be more convenient to work with the following B-system :

$$\begin{cases} X' = B(\lambda, \tau)Y \\ \tau' = \kappa(1 - \tau^2) \end{cases} \quad \text{where } B(\lambda, \tau) = A(\lambda, \tau) - \mu(\lambda)\,\mathrm{Id} \tag{3.2}$$

More convenient because we would like to calculate the geometric phase of a path which spans the unstable manifold. Indeed we will project vectors from $\mathbb{C}^n$ into S^{2n-1}. For a well definition, the vectors must be non-zero. In finite time $\tau \in (-1, 1)$, it is not an issue. But for $\tau = -1$, a solution in the unstable manifold must go to 0 in the A-system. But thanks to the translation, in the B-system, the solution is non-zero for every time.

Lemma 3.1. *There exists an analytic choice of eigenvectors associated with the leader eigenvalue : $\exists X^{\pm}(\lambda)$ analytic for $\lambda \in K$ such that $A_{\pm}(\lambda)X^{\pm}(\lambda) = \mu(\lambda)X^{\pm}(\lambda)$.*

Proof. Humpherys, Sandstede and Zumbrun [6] proved that for a analytic dependent matrix $M(z) \in \mathbb{C}^{n \times p},\ z \in \Omega \subset \mathbb{C}$ where the dimension of the kernel of $M(z)$ is the same for every $z \in \Omega$, then there exists a uniformly analytic frame for the kernel of $M(z)$: there exists $\big(e_1(z), \ldots, e_r(z)\big)$ such that :

- $\forall i \in [\![1, r]\!],\ e_i : \Omega \longrightarrow \mathbb{C}^p$ is analytic,
- $\forall z \in \Omega,\ \ker\big(M(z)\big) = \mathit{Span}\big\{e_1(z), \ldots, e_r(z)\big\}$.

For our lemma, we apply this result to $A_{\pm} - \mu$ which is analytic and because the system splits into Ω, the dimension of the kernel is constant (in our case $k = 1$). $\square$

Consider such a eigenvector's path.

Remark 3.1. *Note that we will project the vectors $X^{\pm}(\lambda)$ into the sphere S^{2n-1} and by this projection, we will loose the $\mathbb{C}$-differentiability. But it will conserve the differentiability with reference to $s \in [0, 1]$ where $\lambda(s)$ will parametrize the contour K.*

At the and of the first chapter, we have mentioned the idea of the unstable bundle (see section 1.3). Let us define it properly.

Definition 3.2. *Remind the definition of M (see figure 1.3) :*

$$M = K \times [-1,1] \cup K^\circ \times \{-1,1\}$$

Then

$$E := \{(\lambda,\tau,Y) \text{ for } (\lambda,\tau) \in M \text{ and } Y \in W^u(\lambda,\tau)\}$$

define a natural bundle structure (dimension k) above M with fibres $W^u(\lambda,\tau)$, called unstable bundle.

More precisely, the projection map π is given by :

$$\pi : \left| \begin{array}{ccc} E & \longrightarrow & M \\ (\lambda,\tau,Y) & \longmapsto & (\lambda,\tau) \end{array} \right.$$

Proof. Let us verify that (E,π,M) is a bundle : it satisfies the points given in definition 2.3 : let be $(\lambda,\tau) \in M$. Then

- $\pi^{-1}\Big(\big\{(\lambda,\tau)\big\}\Big) = \Big\{(\lambda,\tau)\Big\} \times W^u(\lambda,\tau)$ is isomorphic to $\mathbb{C}^k$ ($k=1$ in this chapter but it is true in the general case $k \in \mathbb{N}$).
- The second point is longer and the reader is referred to the construction os Alexander, Gardner and Jones in [1].

□

Definition 3.3. *Let the contour $K \subset \Omega$ be given. A* reference path *for $\lambda \in K$ is an analytical loop $X^\pm(\lambda)$ of eigenvectors for $A_\pm(\lambda)$ that corresponds to the eigenvalue of largest, positive, real part for $A_\pm(\lambda)$.*

The reference path $X^\pm(\lambda)$ is said to be non-degenerate *if $X^\pm$ can be extended smoothly over K° without zeros and if $X^\pm(\lambda)$ defines fibres compatible with the unstable bundle construction.*

For the proof of the following lemma, the reader is referred to Colin Grudzien [5] and to Alexander, Gardner and Jones [1]. This lemma assure the the smoothness of the centre-unstable manifold, transported by the flow.

Lemma 3.2. *Let $X^-(\lambda)$ be a non-degenerate path for $A_-(\lambda)$. Let the centre-unstable manifold of this line of critical points, in the B-system (3.2), be parametrized by (λ,τ) as $Z(\lambda,\tau)$. Then $Z(\lambda,\tau)$ is non-singular and continuous in its limit $\xi \longrightarrow +\infty$, and furthermore*

$$\forall(\lambda,\tau) \in K \times [-1,1],\ \mathit{Span}_{\mathbb{C}}\big(Z(\lambda,\tau)\big) = W^u(\lambda,\tau)$$

Remark 3.4. *The above lemma is also valid for non asymptotically symmetric system. For a development of this case, please see the Grudzien's thesis [5].*

Let Z and $X^{\pm}(\lambda)$ defined as in lemma (3.2) and consider their spherical projection $\widehat{X^{\pm}}(\lambda)$ and $\widehat{Z}$ (as in proposition 2.4). Then we have defined a map

$$\widehat{Z} : \left| \begin{array}{ccc} K \times [-1,1] & \longrightarrow & S^{2n-1} \\ (\lambda, \tau) & \longmapsto & \frac{Z(\lambda,\tau)}{\|Z(\lambda,\tau)\|} \end{array} \right.$$

for which the following hold :

Remark 3.5. *We are in the symmetric case so we should consider $X^+ = X^-$. But for a good comprehension, it is good to imagine it is two different loops, equal.*

- $\forall \lambda \in K,\ \widehat{Z}(\lambda, \tau) \underset{\xi \to -\infty}{\longrightarrow} \widehat{X^-}(\lambda)$
- $\forall \lambda \in K,\ \exists \zeta(\lambda) \in S^1,\ \widehat{Z}(\lambda, \tau) \underset{\xi \to +\infty}{\longrightarrow} \zeta(\lambda)\widehat{X^+}(\lambda)$
- $\forall (\lambda, \tau) \in K \times [-1,1],\ \mathit{Span}_{\mathbb{C}}\big(\widehat{Z}(\lambda, \tau)\big) = W^u(\lambda, \tau)$

All these points are direct conclusion from the lemma 3.2.

Definition 3.6. *The path $\zeta : [0,1] \longrightarrow S^1$ is called the* induced phase*, with respect to the reference paths $X^{\pm}(\lambda)$.*

Let us admit the following proposition :

Proposition 3.3. *The induced phase $\zeta \circ \lambda$ where $\lambda : [0,1] \longrightarrow K$ is a smooth parametrization of K (often a circle) is a differentiable function.*

Lemma 3.4. *The* relative phase *defined by*

$$\mathrm{GP}\left(\widehat{Z}(K, \tau)\right) - \mathrm{GP}\left(\widehat{X^+}(K)\right) \tag{3.3}$$

tends to

$$\frac{1}{2i\pi} \int_0^1 \frac{\zeta'\big(\lambda(s)\big)}{\zeta\big(\lambda(s)\big)} \lambda'(s) ds$$

as $\tau \longrightarrow 1$.

Proof. We have the following relation, for every $s \in [0,1]$,

$$\widehat{Z}\big(\lambda(s), +1\big) = \zeta\big(\lambda(s)\big)\widehat{X^+}\big(\lambda(s)\big)$$

so :

$$\frac{d}{ds}\widehat{Z}\big(\lambda(s), +1\big) = \zeta'\big(\lambda(s)\big)\lambda'(s)\widehat{X^+}\big(\lambda(s)\big) + \zeta\big(\lambda(s)\big)\frac{d}{ds}\widehat{X^+}\big(\lambda(s)\big)$$

and the natural connection :

$$\omega_{\widehat{Z}\left(\lambda(s),+1\right)}\left(\frac{d}{ds}\widehat{Z}\left(\lambda(s),+1\right)\right) = \left\langle \frac{d}{ds}\widehat{Z}\left(\lambda(s),+1\right),\widehat{Z}\left(\lambda(s),+1\right)\right\rangle$$

$$= \left\langle \zeta'\left(\lambda(s)\right)\lambda'(s)\widehat{X^+}\left(\lambda(s)\right),\zeta\left(\lambda(s)\right)\widehat{X^+}\left(\lambda(s)\right)\right\rangle + \left\langle \zeta\left(\lambda(s)\right)\frac{d}{ds}\widehat{X^+}\left(\lambda(s)\right),\zeta\left(\lambda(s)\right)\widehat{X^+}\left(\lambda(s)\right)\right\rangle$$

$$= \overline{\zeta\left(\lambda(s)\right)}\zeta'\left(\lambda(s)\right)\lambda'(s)\left\|\widehat{X^+}\left(\lambda(s)\right)\right\| + \left|\zeta\left(\lambda(s)\right)\right|\left\langle \frac{d}{ds}\widehat{X^+}\left(\lambda(s)\right),\widehat{X^+}\left(\lambda(s)\right)\right\rangle$$

$$\omega_{\widehat{Z}\left(\lambda(s),+1\right)}\left(\frac{d}{ds}\widehat{Z}\left(\lambda(s),+1\right)\right) = \overline{\zeta\left(\lambda(s)\right)}\zeta'\left(\lambda(s)\right)\lambda'(s) + \omega_{\widehat{X^+}\left(\lambda(s)\right)}\left(\frac{d}{ds}\widehat{X^+}\left(\lambda(s)\right)\right)$$

From this expression of the connection, we are able to calculate the geometric phase. Remember $\zeta(s) \in S^1$ so $\overline{\zeta(s)} = \frac{1}{\zeta(s)}$ and :

$$\mathrm{GP}\left(Z(K,+1)\right) = \frac{1}{2i\pi}\int_0^1 \omega_{\widehat{Z}\left(\lambda(s),+1\right)}\left(\frac{d}{ds}\widehat{Z}\left(\lambda(s),+1\right)\right)ds$$

$$= \frac{1}{2i\pi}\int_0^1 \overline{\zeta\left(\lambda(s)\right)}\,\zeta'\left(\lambda(s)\right)\lambda'(s)ds + \frac{1}{2i\pi}\int_0^1 \omega_{\widehat{X^+}\left(\lambda(s)\right)}\left(\frac{d}{ds}\widehat{X^+}\left(\lambda(s)\right)\right)ds$$

$$= \frac{1}{2i\pi}\int_0^1 \frac{\zeta'\left(\lambda(s)\right)}{\zeta\left(\lambda(s)\right)}\lambda'(s)ds + \mathrm{GP}\left(X^+(K)\right)$$

□

Remind the Cauchy argument principle :

Theorem 3.5. *Let C be a contour in $\mathbb{C}$ and C° the region encircled by C. Let $f : C^\circ \longrightarrow \mathbb{C}$ be a meromorphic function such that C contains no zeros or poles of f. Then, if we denote Z the number of zeros of f included in C° and P the number of poles of f included in C° (counting the* *multiplicities),*

$$\int_C \frac{f'(z)}{f(z)} dz = 2i\pi(Z - P)$$

The two following lemmas elaborate the dependence between the relative phase and the reference paths chosen. The two lemmas' proofs are similar so we will only prove the second one (see Grudzien's thesis [5] for more details).

Lemma 3.6. *Given the contour K and its parametrization $\lambda : [0,1] \longrightarrow K$, let $V_1(\lambda)$ be a non-degenerate reference path and $V_2(\lambda)$ be a meromorphic reference path for $A_+(\lambda)$. Then*

$$\mathrm{GP}\left(V_1(K)\right) = \mathrm{GP}\left(V_2(K)\right) + \mathrm{Ind}(V_2) \tag{3.4}$$

where $\mathrm{Ind}(V_2)$ is plus or minus multiplicity of any zero or pole for V_2 in K°.

Lemma 3.7. *Let $V(\lambda)$ and $X^-(\lambda)$ be two reference paths for $A_-(\lambda)$ such that X^- is non-degenerate and $V(\lambda)$ has a pole or zero in K°. Let $V(\lambda,\tau)$ be a solution which correspond to V, . Then :*

$$\mathrm{GP}\left(V(K,+1)\right) = \mathrm{GP}\left(X^-(K)\right) + \mathrm{Ind}(V)$$

Proof. By definition $V(\lambda)$ and $X^-(\lambda)$ are eigenvectors of $A_-(\lambda)$. Because the eigenspace is 1-dimensional, there is a smooth scaling $\alpha : K \longrightarrow \mathbb{C}^*$ such that

$$V(\lambda) = \alpha(\lambda)X^-(\lambda)$$

Let V and X^- design the solutions in the centre unstable manifolds for these reference paths. Then by linearity of the flow, we have : (remind you that $\widehat{v} := \frac{v}{\|v\|}$)

$$\widehat{V}(\lambda,+1) = \widehat{\alpha}(\lambda)\ \widehat{Z}(\lambda,+1)$$

And

$$\frac{d}{ds}\widehat{V}\big(\lambda(s),+1\big) = \frac{d}{ds}\Big(\widehat{\alpha}\big(\lambda(s)\big)\Big)\widehat{Z}\big(\lambda(s),+1\big) + \widehat{\alpha}\big(\lambda(s)\big)\ \frac{d}{ds}\Big(\widehat{Z}\big(\lambda(s),+1\big)\Big)$$

And this implies the connection :

$$\begin{aligned}
\omega_{\widehat{V}(\lambda(s),+1)}\left(\frac{d}{ds}\widehat{V}\big(\lambda(s),+1\big)\right) &= \left\langle \frac{d}{ds}\widehat{V}\big(\lambda(s),+1\big), \widehat{V}\big(\lambda(s),+1\big)\right\rangle \\
&= \frac{d}{ds}\Big(\widehat{\alpha}\big(\lambda(s)\big)\Big)\overline{\widehat{\alpha}\big(\lambda(s)\big)}\ \left\langle \widehat{Z}\big(\lambda(s),+1\big), \widehat{Z}\big(\lambda(s),+1\big)\right\rangle \\
&\quad + \widehat{\alpha}\big(\lambda(s)\big)\overline{\widehat{\alpha}\big(\lambda(s)\big)}\ \omega_{\widehat{Z}\left(\lambda(s),+1\right)}\left(\frac{d}{ds}\Big(\widehat{Z}\big(\lambda(s),+1\big)\Big)\right) \\
&= \frac{\frac{d}{ds}\Big(\widehat{\alpha}\big(\lambda(s)\big)\Big)}{\widehat{\alpha}\big(\lambda(s)\big)} + \omega_{\widehat{Z}\left(\lambda(s),+1\right)}\left(\frac{d}{ds}\Big(\widehat{Z}\big(\lambda(s),+1\big)\Big)\right)
\end{aligned}$$

And then by integration, we have the result, thanks to the Cauchy argument principle. $\square$

The main theorem is the following one, proved by Colin Grudzien in [5]. The proof of this theorem is really complicated because it uses some algebraic topology notions which I am not familiar with : Chern classes and Chern numbers are some invariants between spaces which characterize the wind of this space.

Theorem 3.8. *The geometric phase for a non-degenerate reference path tends to the number of eigenvalues included in the contour K as $\xi \to \infty$.*

3.2 The method in the symmetric case for $k > 1$

This idea is to change the space studied space from $\mathbb{C}^n$ to the wedge product $\bigwedge^k \mathbb{C}^n \cong \mathbb{C}^{\binom{n}{k}}$. Thanks to this transformation, we will be sure that there is only one leading eigenvalue and with the transformation into the B-system, the centre-unstable manifold will already be 1-dimensional. For more details, please see Grudzien thesis [5], where he explains the equivalence between the system in $\mathbb{C}^n$ and the system in $\bigwedge^k \mathbb{C}^n$.

Chapter 4

The method in the Stiefel bundle

The Stiefel bundle is an extension of the Hopf bundle : in the Hopf bundle, we consider only one normal vector (in S^{2n-1}) we can decompose into two coordinates (one in S^1 and one in the projective space $\mathbb{C}P^{n-1}$). In the Stiefel bundle, we will consider a set of k orthonormal vectors at the same time. So the coordinates will be in the unitary group and the Grassmann space. Let us define all these objects.

4.1 Definition and paths in the Stiefel bundle

4.1.1 Definition

The Grassmannian $Gr(n,k)$: The Grassmannian spaces are extension of the projective space. When for the projective space we consider as a point a vectorial line in $\mathbb{C}^n$, for the Grassmannian $\mathrm{Gr}(n,k)$ we consider as a point a vectorial space of dimension k in $\mathbb{C}^n$.

The unitary group $\mathcal{U}(k)$: The unitary group is define as the set of matrices M in $\mathbb{C}^{k\times k}$ such that $M^H M = I_k$.

The space $V(n,k)$: It should be seen as the set of all k-orthonormal collection of vectors (viewed as columns of matrices) :

$$V(n,k) := \left\{W \in \mathbb{C}^{n\times k},\ W^H W = I_k\right\}$$

The unitary group acts on $V(n,k)$: by the right multiplication : for $W \in$ and $U \in \mathcal{U}(k)$,

$$(WU)^H WU = U^H W^H WU = U^H I_k U = I_k$$

and $WU \in V(n,k)$.

Definition 4.1. *The Stiefel bundle is the principal fibre bundle, as defined below the definition 2.3. The total space is $V(n,k)$, the fibers are $\mathcal{U}(k)$ and the base $\mathrm{Gr}(n,k)$. The projection*

$$\pi : \left| \begin{array}{rcl} V(n,k) & \longrightarrow & \mathrm{Gr}(n,k) \\ W & \longmapsto & Im(W) \end{array} \right.$$

can be viewed as the k-dimensional vector subspace the vectors (the columns ok W) span.

4.1.2 Paths in the Stiefel bundle.

As in the Hopf bundle, we can define the tangent space and vertical and horizontal subspaces at a point $W \in V(n,k)$ and as in the Hopf bundle, there is a natural choice for the horizontal subspace and so, a "universal" connection (for more details, see [7]). For a point $W \in V(n,k)$, we will consider the "universal" connection :

$$\omega_W : \left| \begin{array}{rcl} T_W V(n,k) & \longrightarrow & Skew(k) \\ \mathcal{V} & \longmapsto & W^H \mathcal{V} \end{array} \right.$$

where *Skew*(k) is the Lie algebra of $\mathcal{U}(k)$: the set of the skew-hermitian $k \times k$ matrices.
Considering a path $W(s) \in V(n,k)$, we can – with reference to the choice of the "universal" connection ω – calculate the horizontal lift $V(s)$ of $W(s)$ and then, calculate the equivalence of the geometric phase :
For every $s \in [0,1]$, there exists a unitary matrix $U(s)$ such that $W(s) = V(s)U(s)$. The geometric phase in the Stiefel bundle is defined as :

$$\mathrm{GP}(W([0,1])) = U(1)U(0)^{-1}$$

4.2 The induced phase in the Stiefel bundle.

The idea is to work with the Stiefel bundle and to consider the unstable space in one way. Remind that the Stiefel manifold can be viewed as the set of k-orthonormal vectors.
Instead of changing global space to have only one main unstable direction – as in the Hopf bundle method – we consider an orthonormal frame of the unstable space for $-\infty$ which, along the time ξ, will be moved by the flow. The numerical difficulties of this method is keeping the orthonormal property of the frame along the time ξ. Usually, a Gramm-Shmidt orthonormalization is computed at each step of the algorithm to preserve this orthonormalization. In the Colin Grudzien's thesis [5], it is developed how we can connect the "universal" connection in the Stiefel bundle to the "natural" connection in the $\mathbb{C}^{\binom{n}{k}}$-Hopf bundle and the reader is referred to this paper to know precisely how does it work.

Conclusion

To conclude, I would like to emphasize on the link between the geometrical phase on the Hopf bundle – from which we know the number of eigenvalues included in K – and the geometric phase in the Stiefel bundle which, until now, is not used in its own nature. So far, we are just using the "universal" connection in the Stiefel bundle to compute the "natural" connection in the Hopf bundle, but we do not use the geometric phase in the unitary group.
With my supervisor Colin Grudzien, we think that all the information we need is coded on this geometric phase in the unitary group but it is still an open question.

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