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# De Rham Cohomology of a Compact Connected Lie Group

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## Abstract

In this paper we study Lie groups, Lie algebras and De Rham cohomology to obtain an exterior algebra structure for the cohomology ring of a compact connected Lie group.

We will use maximal tori theory, as well as invariant theory, and invariant differential forms in order to prove the statement about the structure.



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## Contents

## 1. Introduction

Here is presented the report of my two-month summer internship made at the Dipartimento di Matematica, Sapienza Università di Roma, under Dr. Paolo Papi. This internship was about Lie theory. After a reading course of some chapters of [3], as an introduction to Lie theory, and particularly root theory, the final subject was about a proof of a computation of the De Rham cohomology ring of the compact connected Lie group made by Mark Reeder in [9] *"On the cohomology of compact Lie groups"* published in *L'Enseignement Mathématiques* in 1995. The following theorem

**Theorem 1.1.** *For a compact connected Lie group $G$, the De Rham cohomology ring $H(G)$ is an exterior graded algebra with generators of known degree.*

is known since the first half of the 20th century. The paper studied give an alternative proof of this statement.

Lie theory was introduced by the Norwegian mathematician Sophus Lie, with help from Felix Klein and later Friedrich Engel. The theory was then developed mainly by Whilhelm Killing, Élie Cartan, Hermann Weyl and Claude Chevalley. Lie groups are an important part of mathematics, especially in geometry. They also play a major role in physics, as geometry groups are often used. There is a strong link between a Lie group and its Lie algebra, therefore the later is often used to study Lie groups. The De Rham cohomology (after Georges De Rham) of a manifold is defined with differential forms. It is an algebraic tool, topologically invariant (note that this is not trivial, and not made in this report), which helps to study the topology of the manifold.

The proof goes as follows : if $T$ is a maximal tori and $W$ its Weyl group acting on $G$, the map $\Psi : G/T \times T \to G,\ (gT, t) \mapsto gtg^{-1}$ gives rise to an isomorphism $\Psi^* : H(G) \to [H(G/T) \otimes H(T)]^W$, but $H(T)$ is easily computable as the exterior algebra of $\mathfrak{t}^*$ where $\mathfrak{t}$ is the Lie algebra associated to $T$, and Borel's theorem gives an isomorphism between $H(G/T)$ and the graded algebra of harmonic polynomials with degrees doubled. Then we use a corollary of Solomon's theorem, which comes from Chevalley's theorem, which states that what we got is an exterior algebra, and gives the degrees of its generators.

After discussing some generalities about algebras, groups and representation theory that are needed for the report in the second part, we introduce Lie groups, their Lie algebras along with fundamental concepts and some properties in the third one. In the fourth part, we discuss about maximal tori, while in the fifth one we introduce the De Rham cohomology on $\mathbb{R}^n$ and its subsets, and then on a manifold, and study some properties of it. Finally, we proceed of the proof of the theorem 1.1 in the sixth part. We start by introducing notations, and some concepts about root theory, then we discuss about invariant theory. We begin with Chevalley's theorem, fundamental for the proof, then introduce harmonic polynomials, gives results about it, and prove Solomon's theorem, whose corollary gives the basis of the exterior algebra $H(G)$. We continue by identify invariant differential forms with alternating linear forms on Lie algebras, and conclude by proving Borel's theorem and proceeding with the main proof.

The report mainly relies on the paper from M. Reeder [9], and the references given by the author. We give the reference used for each part, and every statement unproven is also referenced. Unfortunately, by a lack of time, I have not been able to study all the paper, and some proofs given in it are not made here. Thus, some results made in the report may seem useless, but are used in those unmade proofs. In the same way, some results needed are just stated as their proofs are too long to understand and to make in the report.

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## 2. Generalities

### 2.1. **Algebras, groups.** The concepts of this part are taken from Lang's book [7].

**Definition 2.1.** A *Lie algebra* $\mathfrak{g}$ is a vector space, with a bilinear product $[\ ,\ ]$, verifying the following rules :

i. $\forall X \in \mathfrak{g},\ [X,X]=0;$
ii. $\forall X,Y,Z \in \mathfrak{g},\ [X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ (Jacobi identity).

**Definition 2.2.** A *Lie subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ invariant under product :

$$\forall X,Y \in \mathfrak{h},\ [X,Y] \in \mathfrak{h}.$$

**Definition 2.3** (Graded algebra)**.** A *graded algebra* is an algebra $\mathcal{A}$ which can be written $\mathcal{A}=\bigoplus_{i\in\mathbb{N}} A^i$ where $(A^i)_{i\in\mathbb{N}}$ is a collection of subspaces which verifies $A^iA^j \subset A^{i+j}$. We say that $a \in A$ is an *homogeneous element of degree* $i$ if $a$ is in $A^i$, and we denote $\deg(a)=i$.

**Definition 2.4** (Homogeneous ideal)**.** An *homogeneous ideal* of a graded algebra $\mathcal{A}=\bigoplus_{i\in\mathbb{N}} A^i$ is an ideal $I$ of $\mathcal{A}$ generated by homogeneous elements.

**Proposition 2.5** ([1])**.** *If $I$ is an homogeneous ideal of $\mathcal{A}=\bigoplus_{i\in\mathbb{N}} A^i$, then the algebra $\mathcal{A}/I$ is a graded algebra, with grading given by : $\mathcal{A}/I=\bigoplus_{i\in\mathbb{N}}(A^i+I)/I$.*

**Definition 2.6** (Graded tensor product)**.** If $A, B$ are two graded algebras, we define the *graded tensor product* of $A$ and $B$, denoted $A \otimes_G B$ to be the ordinary tensor product as modules, with product given by :

$$(a\otimes b)(a'\otimes b') := (-1)^{\deg(b)\deg(a')} aa' \otimes bb',\ \forall a,a' \in A,\ \forall b,b' \in B.$$

**Definition 2.7** (tensor algebra)**.** The *tensor algebra* of a vector space $V$ is the graded algebra :

$$T(V)=\bigoplus_{n\in\mathbb{N}}\bigotimes_{r=1}^{n} V.$$

**Definition 2.8** (symmetric algebra)**.** The *symmetric algebra* of a vector space $V$ is the graded algebra :

$$S(V)=T(V)/I,\ I=< x\otimes y - y\otimes x | x,y \in V > .$$

Since $I$ is homogeneous, $S(V)$ is graded, and we denote $S(V) := \bigoplus_{k\in\mathbb{N}} S^k(V)$ the grading.

**Definition 2.9** (exterior algebra)**.** The *exterior algebra* of a vector space $V$ is the graded algebra :

$$\Lambda(V) = T(V)/J,\ J =< x \otimes y + y \otimes x | x, y \in V > .$$

Again, $J$ is homogeneous and therefore $\Lambda(V)$ is graded. We denote $\wedge$ the product in $\Lambda(V)$ and $\Lambda(V) = \bigoplus_{k=0}^{n} \Lambda^k(V)$ the grading, where $\dim(V) = n$. The exterior algebra *generated* by $\{y_1, \ldots, y_n\}$ is the exterior algebra of the vector space generated by $\{y_1, \ldots, y_n\}$.

**Definition 2.10** (group ring)**.** Let $G$ be a group, and $R$ a ring. We define the *group ring of $G$ over $R$* as the set $R[G]$ of all maps $f : G \to R$ with finite support. This set has a natural linear structure, and for $f, g$ two such maps, we define $fg$ to be the map given by $fg(x) = \sum_{uv=x} f(u)g(v)$ for $x$ in $G$. With this product, we obtain a ring structure for $R[G]$.

2.2. **Representation theory.** We refer to [4] for this section.

**Definition 2.11.** A *representation* of an algebra $A$ is an algebra morphism $\rho : A \to \mathrm{Aut}(V)$ where $V$ is a vector space. Similarly, a *representation* of a Lie algebra $L$ is a Lie algebra morphism $\rho : L \to \mathrm{End}(V)$ where $V$ is a vector space and $\mathrm{End}(V)$ is the Lie algebra associated to the algebra $\mathrm{Aut}(V)$. A *representation* of a group $G$ is a group morphism $\rho : G \to \mathrm{GL}(V)$ where $V$ is a vector space. We also say that $A$, $L$ or $G$ *acts* on $V$, the action being defined for example for $a \in A$ by $a \cdot x := \rho(a)(x)$. The *dimension* of a representation is the dimension of the vector space $V$.

**Definition 2.12.** A subspace $U \subset V$ is said to be *invariant* if the action of the algebra or the group $A$ leaves $U$ invariant : $\forall a \in A, a \cdot U \subset U$. A representation $V$ is said to be *irreducible* if there is no nontrivial invariant subspace.

**Proposition 2.13** ([4, Proposition 1.8])**.** *Consider a representation $V$ of a finite group $W$. We have a decomposition*

$$V = \bigoplus_{j=1}^{a_1} V_1 \oplus \cdots \oplus \bigoplus_{j=1}^{a_k} V_k \tag{1}$$

*where $V_1, \ldots, V_k$ are non-isomorphic $W$-invariants subspaces and $a_i = \dim(V_i)$. This decomposition is unique up to order.*

## 3. Lie groups, Lie algebras

For more details, see [10].

### 3.1. **Lie groups.**

**Definition 3.1** (smooth manifold)**.** A ($n$-dimensional) *(smooth) manifold* is a second countable (i.e. that has a countable base of open subsets) Hausdorff topological space $M$ which is locally homeomorphic to $\mathbb{R}^n$ : there is an *atlas* $(U_\alpha, \phi_\alpha)_{\alpha \in A}$ consisting of on open cover $(U_\alpha)_{\alpha \in A}$ of $M$ whose elements are called *coordinate* together with a collection of homeomorphisms $\phi_\alpha : U_\alpha \to V_\alpha$ where $V_\alpha$ is an open subset of $\mathbb{R}^n$, called *coordinate maps.* The manifold is *smooth* if for every $\alpha, \beta$ in $A$, $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^n$ is smooth. The atlas is required to be maximal : one cannot add more coordinate maps.

In all the report all the manifolds are supposed to be smooth, we will therefore call a smooth manifold simply a manifold.

**Definition 3.2** (smooth map)**.** A map $f$ between two manifold $M$ and $N$ with atlases $(U_\alpha, \phi_\alpha)_{\alpha \in A}$ and $(V_\beta, \psi_\beta)_{\beta \in B}$ is called *smooth* if for every point $m$ of $M$ there exists $\alpha, \beta$ in $A \times B$, such that $m$ is in $U_\alpha$, $f(m)$ is in $V_\beta$, $f^{-1}(V_\beta) \cap U_\alpha$ is open and the map $\psi_\beta \circ f \circ \phi_\alpha|_{\phi_\alpha(f^{-1}(V_\beta) \cap U_\alpha)} : \phi_\alpha(f^{-1}(V_\beta) \cap U_\alpha) \to \psi_\beta(V_\beta)$ is smooth.

**Definition 3.3** (Lie group)**.** A *Lie group* $G$ is a manifold which is also a group such that the group product and the inverse map are smooth. A *homomorphism of Lie groups* is a group homomorphism between two Lie groups which is smooth.

*Example* 3.4. The *torus* $T^n := S^1 \times \cdots \times S^1$ is a Lie group.

*Example* 3.5. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $n \in \mathbb{N}^*$, any closed subgroup of $GL(n, \mathbb{K})$ is a Lie group. Such a Lie group is called a *matrix Lie group.*

*Remark* 3.6. We recall that an immersion is a differentiable function whose differential is everywhere one-to-one.

**Definition 3.7.** A *Lie subgroup* $H$ of a Lie group $G$ is the image of a Lie group $H'$ under an immersion $\phi : H' \to G$ together with a Lie group structure on $H$ which makes $\phi : H' \to H$ a Lie groups homomorphism.

**Definition 3.8** (germ)**.** Two smooth maps $f, g : M \to N$ between two manifolds $M, N$ are said to have the same *germ* at a point $p \in M$ if there exists a neighbourhood $U$ of $p$ such that $f|_U = g|_U$. The relation of having the same germ at $p$ is an equivalence relation, and the equivalence classes are called *germs.* A representative of a germ is a map $f : U \to N$ where $U$ is a neighbourhood of $p$, the set of all the germs has an obvious $\mathbb{R}-$algebra structure. We denote $\delta_p$ the set of germs at $p$ when $N = \mathbb{R}$.

**Definition 3.9** (tangent vector, tangent space, differential map)**.** A *tangent vector at* $p \in M$ is a map $X : \delta_p \to \mathbb{R}$ satisfying :

$$\forall f, g \in \delta_p, \quad X(fg) = X(f)g(p) + f(p)X(g).$$

The set of tangent vectors at $p$ forms a vector space $T_pM$ called the *tangent space at* $p$. Any germ of a map $\phi : (M,p) \to (N,q)$ (here $q = f(p)$) induces a homomorphism of $\mathbb{R}$-algebras $\phi^* : \delta_p \to \delta_q$ given by

$$\forall f \in \delta_p,\ \phi^*(f) = f \circ \phi.$$

Then we have the *differential of the map* induced by $\phi$,

$$T_p\phi : \begin{array}{ccc} T_pM & \to & T_qN \\ X & \mapsto & X \circ \phi^*. \end{array}$$

We easily see that $\phi \mapsto T_p\phi$ is functorial.

**Proposition 3.10** ([10, Ch. I Proposition 2.2])**.** *If $V$ is a finite-dimensional real vector space, then for all $p$ in $V$, $T_pV$ is canonically isomorphic to $V$.*

*Remark* 3.11. We note that with the functoriality property, each coordinate map $\phi : U \to V$ induces an isomorphism $T_p\phi : T_pM = T_pU \to T_{h(p)}V = T_{h(p)}\mathbb{R}^n \cong \mathbb{R}^n$.

**Definition 3.12** (tangent bundle)**.** The *tangent bundle* of a manifold $M$ is the disjoint union

$$TM := \bigcup_{p \in M} T_pM.$$

There is a projection $\pi : TM \to M$ given by $v \in T_pM \subset TM \mapsto p$. For any open subset $U \subset M$, we have $TU := \bigcup_{p\in U} T_pU = \bigcup_{p \in U} T_pM$. One can associate to any coordinate map $\phi$ of $M$ a coordinate map $T\phi : TM \supset TU \to TV = V \times R^n$, called the *bundle coordinate map* given by $T\phi | T_pM = T_p\phi$. This makes $TM$ into a $2n$-dimensional manifold.

**Definition 3.13** (vector field)**.** A smooth *vector field* on a manifold $M$ is a smooth map $X : M \to TM$ which verifies $X(p) \in T_pM,\ \forall p \in M$, that is to say $\pi \circ X = id_M$. For a group $G$, we denote the left action of $x \in G$ on $G$ by $l_x : G \to G,\ g \mapsto xg$. A vector field on a Lie group $G$ is said to be *left-invariant* if the following diagram commutes :

$$\begin{tikzcd} TM \arrow[r, "Tl_x"] & TM \\ M \arrow[u, "X"] \arrow[r, "l_x"'] & M. \arrow[u, "X"'] \end{tikzcd}$$

3.2. **Lie algebra associated to a Lie group.**

**Definition 3.14.** For a Lie group $G$, the *Lie algebra associated to* $G$ is $LG := T_eG$ where $e$ is the unit of $G$. Every Lie groups homomorphism $f : G \to H$ rises to a map $Lf := T_ef : LG \to LH$.

*Remark* 3.15. For $v$ in $LG$, define the vector field $X_v : G \to LG,\ x \mapsto T_e l_x(v)$, which is left-invariant. The map $v \mapsto X_v$ gives a canonical isomorphism between $LG$ and the set of left-invariant vector fields, so we will identify $LG$ with this space in the following.

We now describe the Lie algebra structure of $LG$.

**Definition 3.16** ([11, Definition 1.44])**.** The linear structure is clear, and for two left-invariant vector fields $X, Y \in LG$, define $[X, Y]$ as the vector field defined on each $g \in G$ and $\phi \in \delta_g$ by

$$[X, Y](g)(\phi) = X(g)(Y(.)(\phi)) - Y(g)(X(.)(\phi))$$

where $Y(.)(\phi) : G \to \mathbb{R},\ g \mapsto Y(g)(\phi)$ and $X(.)(\phi) : G \to \mathbb{R},\ g \mapsto X(g)(\phi)$. This is correct because we have $Y(.)(\phi) \in \delta_g$ and $X(.)(\phi) \in \delta_g$. One can verify that $[X, Y]$ is still left invariant and that the product $[\ ,\ ]$ satisfy the hypothesis of a Lie algebra, making $LG$ into a Lie algebra.

**Proposition 3.17** ([11])**.** *Let $H$ be a Lie subgroup of $G$ as the image of $H'$ under $\phi$. Then, $T_e\phi : LH' \to LG$ is a Lie algebras isomorphism onto its image, which is a subalgebra of $LG$.*

*Remark* 3.18. The previous proposition essentially asserts that the Lie algebra of a Lie subgroup is a Lie subalgebra of the corresponding Lie algebra.

**Definition 3.19** (adjoint representation)**.** Let $c(g)$ denotes the conjugation by $g \in G$ group morphism in $G$ : $c(g) : x \mapsto gxg^{-1}$. For $g \in G$, we have $Lc(g) \in \mathrm{Aut}(LG)$ the algebra of linear automorphisms of $LG$. We denote $\mathrm{Ad} : G \to \mathrm{Aut}(LG),\ g \mapsto Lc(g)$, which is called the *adjoint representation* of $G$. Then we apply $L$ and get $\mathrm{ad} := L\,\mathrm{Ad} : LG \to L\,\mathrm{Aut}(LG) = \mathrm{End}(LG)$ the *adjoint representation* of $LG$, where $\mathrm{End}(LG)$ is the Lie algebra deduced from the algebra $\mathrm{Aut}(LG)$.

**Proposition 3.20** ([10, Ch. I, Equation (2.11)])**.** *We have, for all $X, Y$ in $LG$,*

$$\mathrm{ad}(X)(Y) = [X, Y].$$

**Definition 3.21** (integral curve)**.** An integral curve $\alpha :]a, b[\to M$ of a vector field $X$ on a manifold $M$ is a differentiable curve which verifies for all $t \in ]a, b[$,

$$\dot{\alpha}(t) = X(\alpha(t)).$$

**Proposition 3.22** (existence and uniqueness, [10])**.** *For every vector field $X$, there exists an open set $A \subset \mathbb{R} \times M$ such that for every $p \in M$, $A \cap (\mathbb{R} \times \{p\})$ is an open interval containing the origin, and a differentiable map $\Psi : A \to M,\ (t, p) \mapsto \alpha_p(t)$, the* flow *of the vector field verifying $\Psi(0, t) = \alpha_p(0) = p$ for all $p$ in $M$. The curve $t \mapsto \alpha_p(t)$ is the unique maximal integral curve of $X$ with the property $\alpha_p(0) = p$.*

**Proposition 3.23** ([10]). *If $G$ is a Lie group, and $X$ a left-invariant vector field, then the flow is global, that is to say $A \cap (\mathbb{R} \times \{g\}) = \mathbb{R}$ for all $g$ in $G$. One should denote $\alpha^X$ the curve $\alpha_e$ defined on all $\mathbb{R}$.*

**Definition 3.24** (exponential map). The map

$$\exp : \begin{array}{ccc} LG & \to & G \\ X & \mapsto & \alpha^X(1) \end{array}$$

is called the exponential map.

**Proposition 3.25** ([10, Ch. I, Proposition 3.1]). *The exponential map is differentiable and its differential at the origin is the identity. Furthermore, we have $\alpha^X(t) = \exp(tX)$ for all $t \in \mathbb{R}$.*

**Proposition 3.26** ([10, Ch. I, Proposition 3.2]). *For a Lie group homomorphism $f : G \to H$, the following diagram commutes :*

$$\begin{tikzcd} LG \arrow[r, "Lf"] \arrow[d, "\exp"'] & LH \arrow[d, "\exp"] \\ G \arrow[r, "f"'] & H \end{tikzcd}$$

*Remark* 3.27. In particular, $\mathrm{Ad}(\exp) = \exp(L\,\mathrm{Ad}) = \exp(\mathrm{ad})$, with $H = \mathrm{Aut}(LG)$. In this case exp is the classic exponential map, as with all the matrix cases.

**Proposition 3.28** ([10, Ch. I, Theorem 5.12]). *If $G$ is a compact Lie group, there exists a positive linear form $\int_G : \mathcal{C}^0(G, \mathbb{R}) \to \mathbb{R}$ that verifies :*

*i.* $\forall f \in \mathcal{C}^0(G, \mathbb{R}),\ \forall h \in G,\ \int_G f(g)dg = \int_G f(gh)dg = \int_G f(hg)dg$

*ii.* $\int_G dg = 1.$

**Proposition 3.29.** *If $G$ is a compact Lie group, there exists a scalar product $< . , . >$ on $\mathfrak{g}$ which is $\mathrm{Ad}(G)$-invariant and $\mathrm{ad}(\mathfrak{g})$-skew-symmetric.*

*Proof.* Let $( . , . )$ be a scalar product on $\mathfrak{g}$. Set

$$< u, v > := \int_G (\mathrm{Ad}(g)u, \mathrm{Ad}(g)v)dg.$$

This is easy to verify that it is a scalar product, and thanks to the previous proposition, it is $\mathrm{Ad}(G)$-invariant. Now because of this invariance, we have for all $X \in \mathfrak{g}$,

$$< \mathrm{Ad}(\exp(tX))u, \mathrm{Ad}(\exp(tX))v > = < u, v >,$$

differentiating this equation, we get

$$< \mathrm{ad}(X)u, v > + < u, \mathrm{ad}(X)v > = 0,$$

i.e., $< . , . >$ is $\mathrm{ad}(G)$-skew-symmetric. ∎

## 4. Maximal tori

4.1. **Maximal torus of a Lie group.** All missing proofs of this section can be found in [10]. In all this section, $G$ is a compact connected Lie group.

**Definition 4.1.** A *torus* of $G$ is a compact connected abelian subgroup. A *maximal torus* $T \subset G$ is a torus which is maximal in the sense of inclusion.

*Remark* 4.2. One can prove that a torus is simply a group isomorphic to $T^k = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. If two tori are such that $T \subsetneq T'$ then $\dim T < \dim T'$ because of the connectedness and the compactness of a torus. Maximal tori then always exist.

**Definition 4.3** (Weyl group)**.** If $N := N_G(T) := \{g \in G,\ gTg^{-1} \subseteq T\}$ is the normalizer of $T$ in $G$, we define $W = N/T$ the *Weyl group* of $T$.

**Definition 4.4.** We define the action of $N$ on $T$ as the action by conjugation : $n \cdot t := ntn^{-1}$. This action is invariant modulo $T$ since $T$ is abelian, so we can define the induced action of $W$ on $T$ : $nT \cdot t := ntn^{-1}$.

**Proposition 4.5** ([10, Ch. IV, Theorem 1.5])**.** *The Weyl group $W$ is finite.*

**Theorem 4.6** (Conjugation theorem, [10, Ch. IV, Theorem 1.6])**.** *In $G$, two maximal tori $T$ and $T'$ are conjugate. Moreover, any element $g$ in $G$ is contained in a maximal torus.*

*Remark* 4.7. With this result, we get that the Weyl groups associated to $T$ and $T'$ are isomorphic, we will then say *the* maximal torus and *the* Weyl group of a Lie group.

For a subgroup $H$ of $G$, the center $Z(H)$ is the subgroup :

$$Z(H) = \{g \in G,\ gh = hg\ \forall h \in H\}.$$

**Corollary 4.8** ([10, Ch. IV, Theorem 2.3])**.** *If $S$ is a connected abelian subgroup of $G$, then $Z(S)$ is contained in the union of all the tori containing $S$. Consequently, for a torus $T$, we have $Z(T) = T$.*

**Proposition 4.9** ([10, Ch. IV, Theorem 2.11])**.** *For any torus $T$, there exists an element $t_0$, called* generic *which generates $T$, in the sense that $\{t_0^n, n \in \mathbb{N}\}$ is dense in $T$.*

*Remark* 4.10. In fact, almost all elements of $T$ are generic.

**Proposition 4.11** ([10, Ch. II, Proposition 8.4])**.** *The irreducible real representations of a torus $T$ are the trivial representation and the two dimensional given by*

$$[x_1, \ldots, x_n] \mapsto \begin{pmatrix} \cos(2\pi\alpha(x)) & \sin(2\pi\alpha(x)) \\ -\sin(2\pi\alpha(x)) & \cos(2\pi\alpha(x)) \end{pmatrix},\ \alpha \in (\mathbb{R}^n)^*,$$

*where we see $T$ as $\mathbb{R}^n/\mathbb{Z}^n$.*

4.2. **Lie algebra of a maximal torus.** Consider a maximal torus $T$ of a Lie group $G$, and $\mathfrak{t}$ and $\mathfrak{g}$ their respective Lie algebras.

*Remark* 4.12. The exponential map restricted to $\mathfrak{t}$, $\exp : \mathfrak{t} \to T$ is surjective since we can see $T$ as $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.

**Proposition 4.13.** *The Lie algebra $\mathfrak{t}$ is commutative ($[X,Y] = 0, \forall X, Y \in \mathfrak{t}$) and is a maximal commutative subalgebra of $\mathfrak{g}$.*

*Proof.* The commutativity comes from the commutativity of $T$ : the morphism $c(h)|_T$ is the identity for all $h \in T$, and therefore $\mathrm{Ad}\,|_T$ is trivial and so is $\mathrm{ad}\,|_{\mathfrak{t}}$. The maximality is due to the surjectivity of exp. □

**Definition 4.14.** A *regular element* of $\mathfrak{t}$ is an element $H_0$ whose $\mathrm{Ad}(G)$-centralizer $\{g \in G |\, \mathrm{Ad}(g)H_0 = H_0\}$ is $T$.

**Proposition 4.15.** *Any element $H_0$ such that $\exp(H_0)$ is generic is a regular element. Hence, by the surjectivity of* exp*, a regular element always exists.*

## 5. De Rham cohomology

All this part comes from [8]. In this section, let $x_1, \ldots, x_n$ be the canonical coordinates of $\mathbb{R}^n$

### 5.1. De Rham cohomology on $\mathbb{R}^n$.

**Definition 5.1** (differential form)**.** We define $\Omega^*$ to be the exterior algebra generated by the elements $dx_1, \ldots, dx_n$. The *differential forms* on $\mathbb{R}^n$ are the elements of the graded algebra

$$\Omega^*(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \otimes \Omega^*.$$

Then a differential form is a sum of elements of the form $f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ where $f$ is in $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, that we may write $f\, dx_I$, we will also forget the sign $\wedge$ in the following. We denote the grading by $\Omega^*(R^n) = \bigoplus_{k=0}^n \Omega^k(\mathbb{R}^n)$.

**Definition 5.2** (differential operator)**.** We extend $d$ to a linear map, the *differential operator* $d : \Omega^*(\mathbb{R}^n) \to \Omega^*(\mathbb{R}^n)$ by :

i. $d(f) = \displaystyle\sum_k \frac{\partial f}{\partial x_k} dx_k$

ii. $d(f dx_I) = df\, dx_I$.

One can check that it does extend $d$ and that $d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$. Therefore we have the *De Rham complex* :

$$\{0\} \xrightarrow{d=0} \Omega^0(\mathbb{R}^n) \xrightarrow{d} \Omega^1(\mathbb{R}^n) \xrightarrow{d} \cdots\cdots \xrightarrow{d} \Omega^{n-1}(\mathbb{R}^n) \xrightarrow{d} \Omega^n(\mathbb{R}^n) \xrightarrow{d} \{0\}.$$

**Proposition 5.3** ([8, Proposition 1.4])**.** *We have* $d^2 = 0$.

**Definition 5.4** (De Rham cohomology)**.** By the former proposition, we have $\operatorname{Im}(d : \Omega^{k-1}(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^n)) \subset \operatorname{Ker}(d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n))$ as vector subspaces in $\Omega^k(\mathbb{R}^n)$, so we may consider the vector space

$$H^k(\mathbb{R}^n) = \operatorname{Ker}(d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)) / \operatorname{Im}(d : \Omega^{k-1}(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^n)),$$

called the *k-th De Rham cohomology of* $\mathbb{R}^n$. The *De Rham cohomology of* $\mathbb{R}^n$ is the graded algebra $H(\mathbb{R}^n) := \bigoplus_{k=0}^n H^k(\mathbb{R}^n)$. For $\omega \in \operatorname{Ker}(d)$, we denote $[\omega] \in \operatorname{Ker}(d)/\operatorname{Im}(d)$ the cohomology class of $\omega$.

**Definition 5.5.** For an open subset $U \subset \mathbb{R}^n$, defining $\Omega^*(U) = \mathcal{C}^\infty(U, \mathbb{R}) \otimes \Omega^*$, we have in the same way $H^k(U)$, the *k-th De Rham cohomology of* $U$ and the *De Rham cohomology of* $U$, the graded algebra $H(U) := \bigoplus_{k=0}^n H^k(U)$.

**Definition 5.6** (pullback map)**.** A smooth map $f : R^m \to \mathbb{R}^n$ induces a *pullback map*

$$f^* : \begin{array}{ccc} C^\infty(R^n, \mathbb{R}) & \to & \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R}) \\ g & \mapsto & g \circ f. \end{array}$$

The following proposition extends it to $\Omega^*(\mathbb{R}^n)$ :

**Proposition 5.7** ([8, Proposition 2.1])**.** *For any $f$ in $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$, there exists a unique map $f^* : \Omega^*(\mathbb{R}^n) \to \Omega^*(\mathbb{R}^m)$ which extends $f^*$ and commutes with d. This map is called the* pullback map *and is defined by :*

$$f^*(g\, dx_{i_1} \dots dx_{i_k}) = (g \circ f)\, d(x_{i_1} \circ f) \dots d(x_{i_k} \circ f) = (g \circ f)\, df_{i_1} \dots df_{i_k}$$

*where $f_j$ is the $j$-th component of $f$. The following diagram commutes :*

$$\begin{tikzcd}
\{0\} \arrow[r, "d"] \arrow[d, "f^*"'] & \Omega^0(\mathbb{R}^n) \arrow[r, "d"] \arrow[d, "f^*"'] & \Omega^1(\mathbb{R}^n) \arrow[r, "d"] \arrow[d, "f^*"'] & \cdots\cdots \arrow[r, "d"] & \Omega^{n-1}(\mathbb{R}^n) \arrow[r, "d"] \arrow[d, "f^*"] & \Omega^n(\mathbb{R}^n) \arrow[r, "d"] \arrow[d, "f^*"] & \{0\} \arrow[d, "f^*"] \\
\{0\} \arrow[r, "d"'] & \Omega^0(\mathbb{R}^n) \arrow[r, "d"'] & \Omega^1(\mathbb{R}^n) \arrow[r, "d"'] & \cdots\cdots \arrow[r, "d"'] & \Omega^{n-1}(\mathbb{R}^n) \arrow[r, "d"'] & \Omega^n(\mathbb{R}^n) \arrow[r, "d"'] & \{0\}.
\end{tikzcd}$$

**Proposition 5.8** ([8, Corollary 2.1])**.** *The De Rham complex is independent of the coordinate system, i.e. if $u_1, \dots, u_n$ is a new coordinate system, then $d(g\, du_{i_1} \dots du_{i_k}) = dg\, du_{i_1} \dots du_{i_k}$.*

5.2. **De Rham cohomology on a manifold.**

**Definition 5.9** (differential form)**.** Let $M$ be a $n$-dimensional manifold with an atlas $(U_\alpha, \phi_\alpha)_{\alpha \in A}$. A *differential form* $\omega$ on $M$ is a collection of differential forms $(\omega_\alpha)_\alpha$ (where $U_\alpha$ is seen as an open subset of $\mathbb{R}^n$, and we may choose a coordinate system $u_1 = x_1 \circ \phi_\alpha, \dots, u_n = x_n \circ \phi_\alpha$) which agree on $U_\alpha \cap U_\beta$ for all $\alpha, \beta \in A$ in the sense that if $i_\alpha : U_\alpha \cap U_\beta \to U_\alpha$ and $i_\beta : U_\alpha \cap U_\beta \to U_\beta$ are the canonical inclusions, then $i_\alpha^* \omega_\alpha = i_\beta^* \omega_\beta$ in $\Omega^*(U_\alpha \cap U_\beta)$. The graded algebra of differential forms on $M$ is denoted $\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M)$.

*Remark* 5.10. Given $\omega \in \Omega * (M)$ and $x \in M$, we can consider $\omega_x$ the differential form at point $x$ since all differential forms $\omega_\alpha$ defined on a neighbourhood of $x$ have to agree on a smaller neighbourhood of $x$.

**Definition 5.11.** We can define in the same way than before $H(M) = \bigoplus_{k=0}^n H^k(M)$, the *De Rham cohomology of* $M$, and a pullback map $f^* : \Omega^*(N) \to \Omega^*(M)$ from a map $f : M \to N$.

**Proposition 5.12** ([8, Example 2.6])**.** *The De Rham cohomology of the circle is*

$$H^k(\mathbb{S}^1) = \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R} & k = 1 \end{cases}$$

**Theorem 5.13** (Künneth Formula, [8, Equation (5.9)])**.** *For any pair of manifolds* $M, N$, *we have*

$$H(M \times N) = H(M) \otimes_G H(N)$$

.

**Corollary 5.14.** *Let* $T = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ *be the* $n$*-torus. Then* $H(T) = \mathbb{R}[X_1, \dots, X_n]/I$ *where* $I$ *is the ideal generated by* $X_i^2$ *and* $X_iX_j + X_jX_i$ *for* $1 \le i, j \le n$.

*Proof.* According to 5.12, $H(\mathbb{S}^1)$ can be seen as $\mathbb{R}[X]/ < X^2 >$. Therefore, the previous theorem and the definition 2.6 gives us the result. □

## 6. Proof of the theorem

We now give the proof of theorem 1.1 following the ideas of [9].

6.1. **Root theory.** Let $G$ be a compact connected Lie group, $\mathfrak{g}$ its Lie algebra, $T$ a maximal torus of $G$ and $\mathfrak{t}$ its Lie algebra. With respect to the scalar product of 3.29 let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{t}$ : $\mathfrak{g} = \mathfrak{t} \oplus^\perp \mathfrak{m}$. We denote $l := \dim(\mathfrak{t})$ and $2v := \dim(\mathfrak{m})$ since we will see that $\mathfrak{m}$ is even dimensional.

**Proposition 6.1.** *There exists an orthogonal decomposition* $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_v$ *of* $\mathfrak{m}$ *into irreducible under the action of* $\mathrm{Ad}(T)$ *two dimensional subspaces.*

*Proof.* Since the scalar product is $\mathrm{Ad}(G)$-invariant, $\mathrm{Ad}(G)$ and therefore $\mathrm{Ad}(T)$ acts on $\mathfrak{m}$, then $\mathfrak{m}$ is a representation of the torus $T$. Since $\mathfrak{t}$ is a maximal commutative subalgebra of $\mathfrak{g}$, there is no nonzero vector in $\mathfrak{m}$ that verifies $[H, X] = 0,\ \forall X \in \mathfrak{t}$. Hence, $\mathrm{Ad}(T)$ has no nonzero invariant vectors in $\mathfrak{m}$, and there cannot be an irreducible trivial representation of $T$ in $\mathfrak{m}$. Therefore, by 4.11, all irreducible subspaces of $\mathfrak{m}$ are two dimensional. By induction, we get the decomposition. □

According to 4.11, for each $1 \leq j \leq v$, there exists $\alpha_j$ in $\mathfrak{t}^*$ such that for $H \in \mathfrak{t}$, the eigenvalues of $\mathrm{Ad}\exp(H)$ on $\mathfrak{m}_j$ are $\exp(\pm i\ \alpha_j(H))$. We choose the sign of the $\alpha_j$ as follows : take $H_0$ a regular element, as a consequence of the definition, no $\alpha_j$ is zero valued on $H_0$. Then the positive root is the one such that $\alpha_j(H_0) > 0$.

**Definition 6.2** (positive roots)**.** The linear forms defined in the preceding discussion are called *positive roots* and their set is denoted $\Delta^+$. We also denote $\Delta$ the set $\{\pm\alpha,\ \alpha \in \Delta^+\}$, whose elements are called *roots*.

**Proposition 6.3** ([9, (2.3)])**.** *The action of* $W$ *on* $\mathfrak{t}$ *is generated by orthogonal reflections about the kernels of the positive roots, denoted* $s_\alpha,\ \alpha \in \Delta^+$.

*Remark* 6.4. All $\mathfrak{m}_j$ are also preserved by $\mathrm{ad}(\mathfrak{t})$, then we can choose an orthogonal basis $\{X_j, X_{j+v}\}$ for $\mathfrak{m}_j$ such that the matrix of $\mathrm{ad}(H)|_{\mathfrak{m}_j}$ in this basis is

$$\begin{pmatrix} 0 & \alpha(H) \\ -\alpha(H) & 0 \end{pmatrix}.$$

We have, by skew-symmetricity, for all $1 \leq i \leq v$, all $1 \leq j \leq 2v$, all $H \in \mathfrak{t}$,

$$< H, [X_i, X_j] >=< [H, X_i], X_j >= -\alpha_i(H) < X_{i+v}, X_j > .$$

The last term is nonzero if and only if $j = i + v$ so if $j \neq i + v$, $[X_i, X_j] \in \mathfrak{m}$. We have the same result if $i > v$ and $j \neq i - v$.

For each $1 \leq i \leq l$, define $H_i := [X_i, X_{i+v}]$. This is $\mathrm{Ad}(T)$-invariant, indeed, we want to prove that $\mathrm{Ad}(t)H_i = H_i$ for $t$ in $T$, this is equivalent to prove that $\mathrm{ad}(Y)H_i = 0$ for $Y$ in $\mathfrak{t}$, and the later holds because $[Y, H_i] = [Y, [X_i, X_{i+v}]] = [X_{i+v}, [X_i, Y]] + [[X_{i+v}, Y], X_i] = -\alpha(Y)[X_{i+v}, X_{i+v}] + \alpha(Y)[X_i, X_i] = 0$. Then we have that $H_i$ belongs in $\mathfrak{t}$, hence $\mathrm{ad}(H_i)m_i \subseteq m_i$. Therefore, the span of $X_i, X_{i+v}$ and $H_i$ is a Lie subalgebra $\mathfrak{g}_i$ of $\mathfrak{g}$.

6.2. **Invariant theory.** See [5].

**Notation.** We denote $\mathscr{S} = \bigoplus_{k\geq 0} \mathscr{S}^k := S(\mathfrak{t}^*)$ the symmetric algebra of $\mathfrak{t}^*$ and $\Lambda = \bigoplus_{k=0}^{l} \Lambda^k := \Lambda(\mathfrak{t}^*)$ the exterior algebra of $\mathfrak{t}^*$.

*Remark* 6.5. If $x_1, \dots, x_l$ is a basis of $\mathfrak{t}$, $\mathscr{S}$ can be seen as the polynomial ring $\mathbb{R}[x_1, \dots, x_l]$, and $\Lambda$ as the set of differential forms $\omega = \sum_{i_1 \leq \dots \leq i_k} a_{i_1,\dots,i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$ where $dx_i$ is the canonical form associated to $x_i$.

**Definition 6.6** (action of $W$)**.** $W$ acts on $\mathfrak{t}^*$ by $w \cdot \alpha : x \mapsto \alpha(w^{-1} \cdot x)$. It naturally induces an action of $W$ on $\mathscr{S}$ and $\Lambda$.

**Notation.** If $W$ acts on a space $\mathscr{A}$, then we denote $\mathscr{A}^W$ the space of all elements of $\mathscr{A}$ that are invariant under the action of $W$.

We now proceed the proof of Chevalley's theorem 6.14 as made in [6].

**Notation.** We denote $(\mathscr{S}^W)_+$ the ideal of $S^W$ consisting of elements of constant term (whose in zero degree) equals to zero.

**Proposition 6.7** ([6, Proposition (3.1)])**.** *The field of fractions of $\mathscr{S}^W$ has transcendence degree $l$ over $\mathbb{R}$.*

**Definition 6.8.** For any polynomial $f$ in $\mathscr{S}$, we define $\widehat{f}$ with :

$$\widehat{f} = \frac{1}{|W|} \sum_{w \in W} w \cdot f.$$

*Remark* 6.9. We note that the map $f \mapsto \widehat{f}$ is a linear map from $\mathscr{S}$ to $\mathscr{S}^W$, which is the identity on $\mathscr{S}^W$, and that preserves degrees. We also see that if $p \in \mathscr{S}$ and $q \in \mathscr{S}^W$,

$$\widehat{pq} = \widehat{p}q. \tag{2}$$

**Proposition 6.10.** *Let $\mathscr{K}$ be the ideal in $\mathscr{S}$ generated by $(\mathscr{S}^W)_+$. If $F_1, \dots, F_n$ are homogeneous elements of $(\mathscr{S}^W)_+$ which generate the ideal $\mathscr{K}$, then $\mathscr{S}^W$ is generated as an algebra by $1, F_1, \dots, F_n$.*

*Proof.* We have to show that every element $f$ of $\mathscr{S}^W$ is a polynomial in $F_1, \dots, F_n$. It suffices to prove it for homogeneous elements, and we proceed by induction on $\deg(f)$. The case $\deg(f) = 0$ is trivial, so we take $f \in \mathscr{S}^W$ with $\deg(f) > 0$. We have since $f$ is homogeneous that $f \in \mathscr{K}$ so there exists $g_i \in \mathscr{S}$ such that

$$f = g_1 F_1 + \dots + g_n F_n. \tag{3}$$

We may assume (after simplifications) that all $g_i$ are homogeneous of degree $\deg(f) - \deg(F_i)$. Now we apply $\widehat{\ }$ and use (2) to get

$$f = \widehat{g_1} F_1 + \dots + \widehat{g_n} F_n. \tag{4}$$

But $\widehat{g_i}$ is an homogeneous element of $\mathscr{S}^W$ of degree lower than $f$, so we may conclude by induction. $\square$

**Lemma 6.11.** *Let $g$ be a homogeneous polynomial of $\mathbb{R}[x_1, \ldots, x_n]$ of degree one. If the polynomial $f$ vanishes at all zeros of $g$, then $g$ divides $f$ in $\mathbb{R}[x_1, \ldots, x_n]$.*

*Proof.* We can assume, without loss of generality, that $g$ is of degree one in $x_n$. Now, with euclidean division in $\mathbb{R}[x_1, \ldots, x_{n-1}][x_n]$, we get $r \in \mathbb{R}[x_1, \ldots, x_{n-1}]$ such that

$$f = gq + r.$$

But if $r$ is nonzero, take $(a_1, \ldots, a_{n-1})$ such that $r(a_1, \ldots, a_{n-1}) \neq 0$. Resolving a linear equation, we can find $a_n$ such that $g(a_1, \ldots, a_{n-1}, a_n) = 0$, therefore, by hypothesis, we get $f(a_1, \ldots, a_n) = 0$, and a contradiction. □

**Lemma 6.12.** *Suppose that $f_1, \ldots, f_r$ are elements of $\mathscr{S}^W$ and that $f_1$ is not in the ideal of $\mathscr{S}^W$ generated by $f_2, \ldots, f_r$. If $g_1, \ldots, g_r$ are homogeneous elements of $\mathscr{S}$ verifying*

$$g_1 f_1 + \cdots + g_r f_r = 0 \tag{5}$$

*then $g_1 \in \mathscr{K}$.*

*Proof.* We first note that $f_1$ is not in the ideal of $\mathscr{S}$ generated by $f_2, \ldots, f_r$. Indeed, otherwise we would have

$$f_1 = q_2 f_2 + \cdots + q_r f_r$$

for $q_i$ in $\mathscr{S}$ and we would apply $\widehat{\ }$ to obtain

$$f_1 = \widehat{q_2} f_2 + \cdots + \widehat{q_r} f_r$$

which would contradict the hypothesis that $f_1$ is not in the ideal of $\mathscr{S}^W$ generated by $f_2, \ldots, f_r$.

We now proceed by induction on $\deg(g_1)$. If $g_1$ is of degree zero, it must be zero, because otherwise $f_1$ would be in the ideal of $\mathscr{S}$ generated by $f_2, \ldots, f_r$.

Now, if $\deg(g_1) > 0$, take $s := s_\alpha \in W$ a simple reflection, and denote $H := H_\alpha$ the corresponding hyperplane. We consider $g$ a linear polynomial whose zero set is exactly $H$. Now $s \cdot g_i - g_i$ is zero on all $H$ since $s$ is the identity on $H$, hence, by 6.11, we get $h_i$ such that

$$s \cdot g_i - g_i = g h_i. \tag{6}$$

Since $g$, $g_i$ and therefore $s \cdot g_i$ are homogeneous, $h_i$ is also homogeneous, of degree $\deg(h_i) = \deg(g_i) - 1$. Now we apply $s$ to equation (5), subtract it to (5), and finally use (6) to obtain :

$$g \cdot (f_1 h_1 + \cdots + f_r h_r) = 0 \tag{7}$$

and then, since $g$ is nonzero,

$$f_1 h_1 + \cdots + f_r h_r = 0. \tag{8}$$

By induction, we hence have $h_1 \in \mathscr{K}$, so $s \cdot g_1 - g_1 \in \mathscr{K}$, that is to say, $s \cdot g_1 = g_1 \mod \mathscr{K}$. By its action on $\mathscr{S}$ and because $W$ stabilizes $(\mathscr{S}^W)_+$ and so $\mathscr{K}$, $W$ acts naturally on $\mathscr{S}/\mathscr{K}$. Now, since we have just shown that $s \cdot g_1 = g_1 \mod \mathscr{K}$ for every simple reflection $s$ in $W$, we get that for all elements $w$ of $W$, $w \cdot g_1 = g_1 \mod \mathscr{K}$, and therefore, $\widehat{g_1} = g_1 \mod \mathscr{K}$. But $\widehat{g_1}$ is in $(\mathscr{S}^W)_+$ because $\deg(g_1) > 0$, $\widehat{g_1} \in \mathscr{S}^W$ and $\widehat{\ }$ preserves degrees. Therefore $\widehat{g_1}$ belongs in $\mathscr{K}$, and so does $g_1$, concluding the proof. ☐

**Proposition 6.13** (Euler formula, [6, Chap. 3, Equation (10)]). *Given a homogeneous polynomial $f(x_1, \dots, x_n)$, we have the formula*

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = \deg(f) f \tag{9}$$

**Theorem 6.14** (Chevalley). *The algebra $\mathscr{S}^W$ is generated as an algebra by $l$ homogeneous algebraically independent elements $F_1, \dots, F_l$ of positive degree :*

$$\mathscr{S}^W = \mathbb{R}[F_1, \dots, F_l].$$

*The degrees $d_1, \dots, d_l$, unique up to order, are supposed to be numbered so that $d_1 \leq \dots \leq d_l$.*

*Proof.* Hilbert's basis theorem gives $n$ homogeneous invariants of positive degree $F_1, \dots, F_n$ which generate $\mathscr{K}$. It follows from 6.10 that $1, F_1, \dots, F_n$ generate $\mathscr{S}^W$ as an algebra. It remains to show that they are algebraically independent, and we will deduce from 6.7 that $n = l$.

Suppose that the elements are dependent, thus there exists a nonzero polynomial $h(x_1, \dots, x_n)$ such that

$$h(F_1, \dots, F_n) = 0. \tag{10}$$

We want to analyse the expression of $h$ and simplify it : take $a\ x_1^{e_1} \cdots x_n^{e_n}$ a monomial in $h$. Then, if $d_i := \deg(F_i)$, $a\ F_1^{e_1} \cdots F_n^{e_n}$ has degree $d := \sum_i d_i e_i$. It is clear that all monomials of degree $d$ after being evaluate in $F_1, \dots, F_n$ sum up to zero. We will then keep only this monomials in $h$.

We now differentiate (10) with respect to $x_k$ for each $k$ :

$$\sum_{i=1}^{n} \frac{\partial h}{\partial x_i}(F_1, \dots, F_n) \frac{\partial F_i}{\partial x_k} = 0. \tag{11}$$

We denote $h_i := \frac{\partial h}{\partial x_i}(F_1, \dots, F_n)$. We have that $h_i$ is an homogeneous element of $\mathscr{S}^W$ and $\frac{\partial F_i}{\partial x_k}$ is an homogeneous element of $\mathscr{S}$, so we would like to apply 6.12, but $h_i$ may not verify the hypothesis. We renumbered $h_i$ so that $h_i,\ 1 \le i \le m$ is a minimal set of generators for the ideal of $\mathscr{S}^W$ generated by $h_i,\ 1 \le i \le n$, so we may write

$$h_i = \sum_{j=1}^{m} g_j^i h_j \tag{12}$$

for $i > m$, with $g_j^i \in \mathscr{S}^W$. Since $h_i$ is of degree $d - d_i$, if we discard redundant terms, we can assume that $g_j^i$ is of degree $\deg(h_i) - \deg(h_j) = d_j - d_i$. Now, for each $k$, we put (12) into (11) to obtain

$$\sum_{i=1}^{m} h_i \left( \frac{\partial F_i}{\partial x_k} + \sum_{j=m+1}^{n} g_j^i \frac{\partial F_j}{\partial x_k} \right) = 0. \tag{13}$$

We write $p_i$ for the term in parenthesis, which is an homogeneous polynomial of degree $d_i - 1$, and get by lemma 6.12 that $p_1 \in \mathscr{K}$. Therefore, we have,

$$\frac{\partial F_1}{\partial x_k} + \sum_{j=m+1}^{n} g_j^1 \frac{\partial F_j}{\partial x_k} = \sum_{i=1}^{n} q_i F_i \tag{14}$$

with $q_i \in \mathscr{S}$. Now, we multiply (13) by $x_k$ and sum over $k$ :

$$\sum_{k=1}^{n} x_k \frac{\partial F_1}{\partial x_k} + \sum_{k=1}^{n} x_k \sum_{j=m+1}^{n} g_j^1 \frac{\partial F_j}{\partial x_k} = \sum_{k=1}^{n} x_k \sum_{i=1}^{n} q_i F_i$$

thus,

$$\sum_{k=1}^{n} x_k \frac{\partial F_1}{\partial x_k} + \sum_{j=m+1}^{n} g_j^1 \sum_{k=1}^{n} x_k \frac{\partial F_j}{\partial x_k} = \sum_{i=1}^{n} \left( \sum_{k=1}^{n} x_k q_i \right) F_i$$

therefore we have, using (9) :

$$d_1 F_1 + \sum_{j=m+1}^{n} d_j g_j^1 F_j = \sum_{i=1}^{n} r_i F_i. \tag{15}$$

But now $\deg(r_i) > 0$, and we see that the left side of (15) is of degree $d_1$, so the term $r_1 F_1$ on the right side must simplify with the other terms of degree different from $d_1$. With this simplification, we see that (15) expresses $F_1$ as an element of the ideal in $\mathscr{S}$ generated by $F_2, \ldots, F_n$, contrary to the hypothesis made about $F_1, \ldots, F_n$. $\square$

**Definition 6.15** (exponents)**.** The *exponents* of $W$ acting on $\mathscr{S}$ are defined as $m_i = d_i - 1$ for $1 \le i \le l$.

**Proposition 6.16** ([6, Theorem 3.9])**.** *We have* $m_1 + \cdots + m_l = v$ *and* $(m_1 + 1) \cdots (m_l + 1) = d_1 \cdots d_l = |W|$.

**Definition 6.17.** We set $\mathscr{D}$ to be the set of *constant coefficient differential operators on* $S$. Such an operator is a functional on $S$ of the form $\sum_I s_I \frac{\partial^{j_1}}{\partial x_1^{j_1}} \cdots \frac{\partial^{j_l}}{\partial x_l^{j_l}}$ where $s_I \in \mathscr{S}^0$ and $\frac{\partial^{j_1}}{\partial x_i^{j_i}}$ is the classical partial derivative on a polynomial.

**Definition 6.18.** For each $H \in \mathfrak{t}$, define a functional $D_H : \mathscr{S} \to \mathscr{S}$ as the derivation extending the evaluation map $H : \mathfrak{t}^* \to \mathbb{R}$. With this identification, we consider $S(\mathfrak{t})$ to be the symmetric algebra of the vector space of maps $D_H$, $H \in \mathfrak{t}$. We should denote $D_a$ the map $\mathscr{S} \to \mathscr{S}$ corresponding to $a \in S(\mathfrak{t})$.

*Remark* 6.19. We should explicit the map $D_{H_1 \cdots H_k} = D_{H_1} \cdots D_{H_k}$ : for $k = 1$, we have for example $D_H(\alpha_1\alpha_2) = \alpha_1(H)\alpha_2 + \alpha_1\alpha_2(H)$, and $D_{H_1} \cdots D_{H_k}(\alpha_1\alpha_2) = D_{H_1} \cdots D_{H_k - 1}(D_{H_k}(\alpha_1\alpha_2))$. Then we easily get :

$$D_{H_1} \cdots D_{H_k}(\alpha_1 \cdots \alpha_k) = \sum_{\sigma \in S_k} \alpha_1(H_{\sigma(1)}) \cdots \alpha_k(H_{\sigma(k)}). \tag{16}$$

*Remark* 6.20. We can identify $\mathscr{D}$ to be $S(\mathfrak{t})$ as defined, and we shall use the most appropriate definition when needed.

**Definition 6.21.** The action of $W$ on $\mathscr{D} = S(\mathfrak{t})$ is natural, and we define the *harmonic polynomials*, the polynomials $f$ in $\mathscr{S}$ that are annihilated by all the functionals in $\mathscr{D}$ that are invariant under the action of $W$. The algebra of all harmonic polynomials is denoted $\mathscr{H}$ so we have

$$\mathscr{H} = \{f \in \mathscr{S} | \mathscr{D}^W f = 0\}.$$

**Proposition 6.22.** *We have* $\mathscr{H} = \bigoplus_k \mathscr{H}^k$ *where* $\mathscr{H}^k := \mathscr{H} \cap \mathscr{S}^k$. *Furthermore,* $\mathscr{H}$ *is invariant under the action of* $W$ *on* $\mathscr{S}$.

*Proof.* To prove that $\mathscr{H} = \bigoplus_k \mathscr{H}^k$ it remains to show that $\mathscr{H} \subset \sum_k \mathscr{H}^k$. Take $f = \sum_i f_i$ in $\mathscr{H}$. We want to show that $f_i \in \mathscr{H}$, so choose $p \in \mathscr{D}^W$. An element in $\mathscr{D} = S(\mathfrak{t})$ is $W$-invariant if and only if each of its homogeneous components is $W$-invariant (since the degree is invariant under the action of $W$), so we can suppose that $p$ is homogeneous. Then $p$ reduce the degree of each homogeneous component of $f$ by the same integer, so if $p(f) = 0$, then $p(f_i) = 0$ for all $i$.

To see the second statement, take $w \in W$, $f \in \mathscr{H}$, $p \in \mathscr{D}^W$, and note that $p(w \cdot f) = (w \cdot p)(f) = p(f) = 0$ since $p \in \mathscr{D}^W$. □

**Proposition 6.23** ([5, Ch. III, Theorem 3.4])**.** *We denote* $\mathscr{I}$ *the ideal of* $\mathscr{S}$ *generated by the homogeneous elements of* $\mathscr{S}^W$ *of positive degree. Then we have*

$$\mathscr{S} = \mathscr{H} \oplus \mathscr{I}. \tag{17}$$

*Hence,*

$$\mathscr{S}/\mathscr{I} \cong \mathscr{H}. \tag{18}$$

**Theorem 6.24** ([5, Ch. III, Theorem 3.4])**.** *Let $\phi : \mathscr{H} \otimes \mathscr{S}^W \to \mathscr{S}$ be the linear map extending the map $\phi : f \otimes g \mapsto fg$. Then $\phi$ is a linear isomorphism :*

$$\mathscr{S} \cong \mathscr{H} \otimes \mathscr{S}^W. \tag{19}$$

*That is to say, every element $s$ in $\mathscr{S}$ can be written uniquely as $s = \sum_i h_i g_i$ where $h_i$ is harmonic and $g_i$ is $W$-invariant.*

**Corollary 6.25.** *We have the identity*

$$\sum_{k \geq 0} (\dim \mathscr{H}^k) t^k = \prod_{j=1}^{l} (1 + t + t^2 + \cdots + t^{m_j}). \tag{20}$$

*Therefore, $\dim \mathscr{H}^v = 1$, $\mathscr{H}^k = 0$ for $k > v$ and $\dim \mathscr{H} = |W|$.*

*Proof.* We shall prove the very last statement $\dim \mathscr{H} = |W|$, since it is not made in the reference and it is not obvious. The dimension of $\mathscr{H}$ is the sum of all the coefficient of the polynomial given by (20). But by the presentation of it, this sum is exactly $(1 + m_1) \cdots (1 + m_l) = |W|$. □

**Definition 6.26.** We set the *primordial harmonic polynomial* $\Pi \in \mathscr{S}^v$ to be :

$$\Pi := \prod_{\alpha \in \Delta^+} \alpha. \tag{21}$$

**Proposition 6.27** ([6, p.69])**.** *Any polynomial $f \in \mathscr{S}$ verifying $w \cdot f = \det(w) f$ is divisible by $\Pi$.*

*Remark* 6.28. We have for example, and it is used in the proof, $w \cdot \Pi = \det(w)\Pi$ for all $w \in W$. Indeed, since $W$ is generated by simple reflections, we have to prove that $s_{\alpha_j} \cdot \Pi = -\Pi$ for $1 \leq j \leq l$. This statement holds because $s_{\alpha_j}$ is a permutation of $\Delta^+ \backslash \{\alpha_j\}$ and $s_{\alpha_j}(\alpha_j) = -\alpha_j$.

**Corollary 6.29.** *The polynomial $\Pi$ is harmonic, therefore $\Pi$ spans $\mathscr{H}^v$ because of 6.25.*

*Proof.* Take $D_a \in \mathscr{D}^W$, $a \in S(\mathfrak{t})^W$. We want to show that $D_a(\Pi) = 0$. We have, for $w \in W$ :

$$w \cdot D_a(\Pi) = D_{w \cdot a}(w \cdot \Pi) = \det(w) D_(a)(\Pi).$$

Therefore, by the preceding proposition, $D_a(\Pi)$ is divisible by $\Pi$. But $D_a(\Pi)$ has smaller degree than $\Pi$, hence is zero. □

**Definition 6.30.** For a polynomial $F \in \mathscr{S}$, define $dF \in \mathscr{S} \otimes \Lambda^1$ to be the classical differential :

$$dF = \sum_i \frac{\partial F}{\partial x_i} dx_i.$$

**Theorem 6.31** (Solomon)**.** *The space* $(\mathscr{S} \otimes \Lambda)^W$ *is a free* $\mathscr{S}^W$*-module with basis*

$$\{dF_{i_1} \wedge \cdots \wedge dF_{i_q}, 1 \leq i_1 < \cdots < i_q \leq l\}.$$

*Where the* $F_i$ *are the homogeneous generators defined in 6.14.*

**Lemma 6.32.** *We have,*

$$dF_1 \wedge \cdots \wedge dF_l = c\ \Pi\ dx_1 \wedge \cdots \wedge dx_l$$

*for a nonzero real number* $c$*.*

*Proof.* Because of the homogeneousness of the $F_i$ and the sum formula about their degrees, we have :

$$dF_1 \wedge \cdots \wedge dF_l = J\ dx_1 \wedge \cdots \wedge dx_l$$

for a polynomial $J$ of degree $m_1 + \cdots + m_l = v$. The left-hand side is $W$-invariant and $dx_1 \wedge \cdots \wedge dx_l$ verifies $w \cdot f = \det(w) f$, therefore so does $J$. Hence, because $J$ and $\Pi$ are the same degree and with 6.27 :

$$dF_1 \wedge \cdots \wedge dF_l = c\ \Pi\ dx_1 \wedge \cdots \wedge dx_l$$

for a real number $c$.

We now show that $c$ is nonzero. For $1 \leq i \leq l$, because of the independence of the $F_i$ and the dimension of $\mathbb{R}[x_1, \ldots, x_n]$, the polynomials $x_i, F_1, \ldots, F_l$ are algebraically dependent. Let $Q_i(x_i, z_1, \ldots, z_l)$ be a polynomial of minimal degree in $x_i$ such that

$$Q_i(x_i, F_1, \ldots, F_l) = 0.$$

We now take the partial derivative over $x_j$ of this expression, for $1 \leq j \leq l$ to obtain :

$$\sum_{k=1}^{l} \frac{\partial Q_i}{\partial z_k}(x_i, F_1, \ldots, F_l)\frac{\partial F_k}{\partial x_j} + \delta_i^j \frac{\partial Q_i}{\partial x_j}(x_i, F_1, \ldots, F_l) = 0.$$

We write this set of equalities with matrices : $AB = -C$ where $A = (\frac{\partial Q_i}{\partial z_j}(x_i, F_1, \ldots, F_l))_{i,j}$, $B = (\frac{\partial F_i}{\partial x_j})_{i,j}$ and $C$ is the diagonal matrix $C = (\frac{\partial Q_i}{\partial x_i}(x_i, F_1, \ldots, F_l))_i$. Now, since $\deg_{x_i}(\frac{\partial Q_i}{\partial x_i}) < \deg_{x_i}(Q_i)$, by minimality $\frac{\partial Q_i}{\partial x_i}(x_i, F_1, \ldots, F_l)$ is nonzero, hence $\det(C) \neq 0$ and then $\det(B) \neq 0$. This proves that the form $dF_1 \wedge \cdots \wedge dF_l$ is nonzero and therefore $c \neq 0$. □

*Proof.* (theorem 6.31) Let $k$ be the quotient field of $\mathscr{S}$. For a sequence $I = i_1 < \cdots < i_q$, we denote $dF_I = dF_{i_1} \wedge \cdots \wedge dF_{i_q}$ and $I'$ the increasing sequence of $\{1, \ldots, l\} \backslash \{i_1, \ldots, i_q\}$. We now show that the elements of $\{dF_{i_1} \wedge \cdots \wedge dF_{i_q}, 1 \leq i_1 < \cdots < i_q \leq l\}$ form a $k$-basis of $k \otimes \Lambda$, hence they are linearly independent over $\mathscr{S}^W$, and we will also deduce that they are generators of $(\mathscr{S} \otimes \Lambda)^W$.

Suppose that there exists a collection $f_I \in k$ such that $\sum_I f_I dF_I = 0$. For any sequence $I$, we multiply by $dF_{I'}$, which kills all the terms but the one in $I$ because of the alternating product. We now have

$$\pm c \ f_I \ \Pi \ dx_1 \wedge \cdots \wedge dx_l = 0$$

and then $f_I = 0$. We have the independence of the $dF_I$, but because there are the same number as the dimension over $k$ of $k \otimes \Lambda$, they form a $k$-basis of $k \otimes \Lambda$.

Now, take $\omega \in (\mathscr{S} \otimes \Lambda)^W \subset k \otimes \Lambda$. There exists some $g_I \in k$ such that $\omega = \sum_I g_I dF_I$. Again, we multiply by $dF_{I'}$ to obtain for each $I$,

$$\omega \wedge dF_{I'} = \pm c \ g_I \ \Pi \ dx_1 \wedge \cdots \wedge dx_l.$$

But since both $\omega \wedge dF_{I'}$ and $\Pi \ dx_1 \wedge \cdots \wedge dx_l$ are in $(\mathscr{S} \otimes \Lambda)^W$, they are $W$-invariant and polynomial, and $g_I$ has to be $W$-invariant and polynomial. This conclude the proof. $\square$

**Definition 6.33.** For $\omega \in \mathscr{S} \otimes \Lambda$, we define $\omega' \in \mathscr{S}/\mathscr{I} \otimes \Lambda$ obtained by reducing the coefficients of $\omega$ modulo $\mathscr{I}$.

**Corollary 6.34.** *$(\mathscr{S}/\mathscr{T} \otimes \Lambda)^W$ is an exterior algebra generated by*

$$dF_i' \in [(\mathscr{S}/\mathscr{T})^{m_i} \otimes \Lambda^1]^W$$

*for $1 \leq i \leq l$.*

**Definition 6.35.** The *$q$-th elementary symmetric polynomial in $l$ variables* is the polynomial

$$s_q(x_1, \ldots, x_l) = \sum_{1 \leq i_1 < \cdots < i_q \leq l} x_{i_1} \cdots x_{i_q}.$$

**Corollary 6.36.** *We have :*

$$\sum_{k=1}^{v} \dim \mathrm{Hom}_W(\Lambda^q, \mathscr{H}^k) u^k = s_q(u^{m_1}, \ldots, u^{m_l}).$$

6.3. **Invariant differential forms.** The elements presented in this section can be found in [2].

In this part, we consider a transitive action of a compact Lie group $G$ on a manifold $M$, and we denote $\tau_g$ the diffeomorphism of $M$ corresponding to $g \in G$. It will be used later in the cases $M = G$ and $M = G/T$.

**Definition 6.37.** A differential form $\omega \in \Omega^p(M)$ is said to be *$G$-invariant* if for all $g \in G$, $\tau_g^* \omega = \omega$.

*Remark* 6.38. An invariant differential form is then determined by its value at any one point of $M$.

**Proposition 6.39** ([9, (4.1)]). *Any De Rham cohomology class is represented by a $G$-invariant differential form and the subcomplex of invariant differential forms is preserved by the differential operator.*

*Remark* 6.40. Taking $x \in M$, and considering $K := K_x := \{g \in G \mid g \cdot x = x\}$ the stabilizer of $x$, we have the identification $M = G/K$ via the map $g \mapsto g \cdot x$, surjective by transitivity and with kernel $K$. We define $\mathfrak{r} := LK$ and $\mathfrak{n}$ its orthogonal complement in $\mathfrak{g}$. Note that this decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{r}$ is preserved by $\mathrm{Ad}(K)$.

Differentiating at $x$ the map introduced before, we identify $\mathfrak{n}$ and $T_x(M)$. The two particular cases given before give $K = 1$, $\mathfrak{n} = \mathfrak{g}$ and $K = T$, $\mathfrak{n} = \mathfrak{m}$.

**Proposition 6.41.** *We may identify the $G$-invariant $p$-forms on $M$ with the space $(\Lambda^p \mathfrak{n}^*)^K$.*

*Proof.* With the identification between $\mathfrak{n}$ and $T_x(M)$ and the fact that an invariant differential form is determined by its value at $x$, any form $\widetilde{\omega} \in \Omega^p(M)$ is determined by the skew-symmetric multilinear map

$$\omega = \widetilde{\omega}_x : \mathfrak{n} \times \cdots \times \mathfrak{n} \to \mathbb{R} \in \Lambda^p \mathfrak{n}^*.$$

$\omega$ is $K$-invariant because $\widetilde{\omega}$ is $G$-invariant, hence $K$-invariant.

On the other hand, given $\omega \in (\Lambda^p \mathfrak{n}^*)^K$, we can define a $G$-invariant differential form $\widetilde{\omega}$ on $M$ with

$$\widetilde{\omega}_{g \cdot x}(T_x \tau_g(X_1), \ldots, T_x \tau_g(X_p)) = \omega(X_1, \ldots, X_p).$$

□

**Proposition 6.42** ([2, p. 97]). *In the identification of the previous proposition, the differential operator becomes the map $\delta : (\Lambda^p \mathfrak{n}^*)^K \to (\Lambda^{p+1} \mathfrak{n}^*)^K$ given by*

$$\delta\omega(X_0, \ldots, X_p) := \frac{1}{p+1} \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j]_{\mathfrak{n}}, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p)$$

*where $\hat{}$ means that the term is omitted and $[X_i, X_j]_{\mathfrak{n}}$ denote the projection of $[X_i, X_j]$ into $\mathfrak{n}$ along $\mathfrak{r}$.*

**Corollary 6.43.** *The propositions 6.39, 6.41 and 6.42 show that the De Rham cohomology of $M$ is computed by the complex $\{(\Lambda^p \mathfrak{n}^*)^K, \delta\}$.*

In the following, we consider the case $M = G$. In this case, there is also the action of right multiplication in $G$, so we may consider bi-invariant differential forms, differential forms that are invariant under the action of both left and right multiplication.

**Proposition 6.44** ([9, (4.1)])**.** *Any De Rham cohomology class is represented by a $G$-bi-invariant differential form.*

**Proposition 6.45.** *We have*

$$H(G) \cong (\Lambda\mathfrak{g}^*)^G.$$

*Proof.* The value at the identity of a bi-invariant differential form is $\mathrm{Ad}(G)$-invariant, i.e. for such a differential form $\omega$, for all $g$ in $G$, for all $X_1, \dots, X_p$ in $\mathfrak{g}$,

$$\omega(\mathrm{Ad}(g)X_1, \dots, \mathrm{Ad}(g)X_p) = \omega(X_1, \dots, X_p).$$

In particular, for all $X$ in $\mathfrak{g}$,

$$\omega(\mathrm{Ad}(\exp(tX))X_1, \dots, \mathrm{Ad}(\exp(tX))X_p) = \omega(X_1, \dots, X_p).$$

Differentiating at the identity we get,

$$\omega([X, X_1], \dots, X_p) + \cdots + \omega(X_1, \dots, [X, X_p]) = 0.$$

The previous equation shows that if $\omega$ is bi-invariant, $\delta(\omega) = 0$, and it gives the result. $\square$

## 6.4. **Main theorems.**

**Proposition 6.46.** *There is an isomorphism of graded ring*

$$H(T) \cong \Lambda. \tag{22}$$

*Proof.* The presentation of $H(T)$ given by 5.14 is exactly the presentation of $\Lambda$, hence there exists an isomorphism between the two graded rings. $\square$

**Proposition 6.47** ([9, (5.3)])**.** *We have, as modules,*

$$H(G/T) \cong \mathbb{R}[W] \tag{23}$$

*where $\mathbb{R}[W]$ is the group ring of $W$. We say that $H(G/T)$ is the regular representation of $W$.*

**Theorem 6.48** (Borel)**.** *There exists a degree doubling $W$-equivariant rings isomorphism*

$$c : \mathscr{S}/\mathscr{T} \to H(G/T).$$

*Therefore, recalling that $\mathscr{S}/\mathscr{T} \cong \mathscr{H}$, if $\mathscr{H}_{(2)}$ is $\mathscr{H}$ with the degrees doubled then $\mathscr{H}_{(2)} \cong H(G/T)$.*

*Proof.* To define the morphism $c$, we start by setting its image on $\lambda \in \mathfrak{t}^*$. For $\lambda \in \mathfrak{t}^*$, we extend it on all $\mathfrak{g}$ by making it zero on $\mathfrak{m}$, and we define $\omega_\lambda$ a $\mathrm{Ad}(T)$-invariant two-form on $\mathfrak{m}$ given by

$$\forall X, Y \in \mathfrak{m},\ \omega_\lambda(X,Y) := \lambda([X,Y]). \tag{24}$$

The Jacobi identity gives $\delta\omega_\lambda = 0$. Then we use 6.43 in the case $K = T$ and $M = G/T$ to identify $\omega_\lambda \in (\Lambda\mathfrak{m}^*)^T$ with $\tilde{\omega}_\lambda \in \Omega(G/T)$. Hence we can define $c(\lambda) := [\tilde{\omega}_\lambda] \in H^2(G/T)$ the cohomology class of $\tilde{\omega}_\lambda$. We extend this to a degree doubling rings homomorphism $c : \mathscr{S} \to H(G/T)$.

The group $W$ acts on $\mathfrak{t}^*$ naturally, and on $\Omega(G/T)$ by its action on $G/T$. Then one can note that we have for all $w$ in $W$, $w^*\omega_\lambda = \omega_{w\cdot\lambda}$. Therefore, we have for $\lambda \in \mathfrak{t}^*$, $w^*c(\lambda) = c(w\cdot\lambda)$, and it still holds in $\mathscr{S}$ : $c$ preserves the action of $W$. Since, (23), $H(G/T)$ is the regular representation of $W$, the $W$-invariants are copies of the trivial representation. Yet, we know by 2.13 that the regular representation contains each irreducible representation with multiplicity equals to its dimension. But there is already a copy of the trivial representation in $H^0(G/T)$, hence all $W$-invariants lie in $H^0(G/T)$. Now, take $\lambda$ in $\mathscr{I}$ the ideal of $\mathscr{S}$ generated by $W$-invariant polynomials of positive degree. As told before, we have for $w$ in $W$, $w^*c(\lambda) = c(w\cdot\lambda) = c(\lambda)$ so $c(\lambda)$ is $W$-invariant, and generates a $W$-invariant, therefore lie in $H^0(G/T)$. But on the other hand, by degree preserving, $c(\lambda)$ is of positive degree, and therefore $c(\lambda) = 0$. We have $\mathscr{I} \subseteq \mathrm{Ker}(c)$. In the following, we show that they are equal. Recalling (17) : $\mathscr{S} = \mathscr{H} \oplus \mathscr{I}$, we prove that $c$ is injective on $\mathscr{H}$. We use the grading structure on $\mathscr{H}$, we first prove that $c$ is injective on $\mathscr{H}^v$ and then proceed by decreasing induction.

We start by showing that $c(\Pi) \neq 0$, where we recall that $\Pi$ is the primordial harmonic polynomials given by (21) which spans $\mathscr{H}^v$. We use part 6.1 : consider $\Delta^+ = \{\alpha_1, \dots, \alpha_v\}$ the set of positive roots, the decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_v$ with basis $\{X_i, X_{i+v}\}$ for $\mathfrak{m}_i$ and relations given in 6.4. We denote $\omega_i := \omega_{\alpha_i}$, so $c(\Pi) = [\tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_v]$. We want to evaluate $c(\Pi)$ on the given basis of $\mathfrak{m}$, we have :

$$\begin{aligned}
&\omega_1 \wedge \cdots \wedge \omega_v(X_1, X_{1+v}, \dots, X_v, X_{2v}) \\
&= \frac{1}{(2v)!} \sum_{\sigma \in S_{2v}} \mathrm{sgn}(\sigma)\omega_1(X_{\sigma(1)}, X_{\sigma(1+v)}) \cdots \omega_v(X_{\sigma(v)}, X_{\sigma(2v)}) \\
&= \frac{1}{(2v)!} \sum_{\sigma \in S_{2v}} \mathrm{sgn}(\sigma)\alpha_1([X_{\sigma(1)}, X_{\sigma(1+v)}]) \cdots \alpha_v([X_{\sigma(v)}, X_{\sigma(2v)}])
\end{aligned}$$

where $S_{2v}$ is the symmetric group and $\mathrm{sgn}(\sigma)$ is the sign of the permutation $\sigma$. Recall that $\alpha_i$ is zero on $\mathfrak{m}$, so the term corresponding to $\alpha$ is nonzero only if $\sigma$ permutes the pairs $\pi_i = \{i, i+v\}$ and switches members of each pair. Taking $\sigma_0$

that only permutes pairs, and $\sigma$ that permutes pairs the same way than $\sigma_0$ but also switches some members of each pair, we have that the terms corresponding to $\sigma_0$ and $\sigma$ are equal, because of the skew-symmetricity of the bracket and the fact that sgn corresponds to the number of switches minus one. Since permutations that only permutes pairs are even we have, according to (16) :

$$\begin{aligned}
&\omega_1 \wedge \cdots \wedge \omega_v(X_1, X_{1+v}, \ldots, X_v, X_{2v}) \\
&= \frac{2^v}{(2v)!} \sum_{\sigma \in S_v} \alpha_1([X_{\sigma(1)}, X_{\sigma(1)+v}]) \cdots \alpha_v([X_{\sigma(v)}, X_{\sigma(v)+v}]) \\
&= \frac{2^v}{(2v)!} \sum_{\sigma \in S_v} \alpha_1(H_{\sigma(1)}) \cdots \alpha_v(H_{\sigma(v)}) \\
&= \frac{2^v}{(2v)!} D_1 \cdots D_v \Pi
\end{aligned}$$

where $D_i := D_{H_i} \in \mathscr{D}$ is the functional on $\mathscr{S}$ as defined in 6.18. We now use the following lemma too conclude.

**Lemma 6.49** ([9, (5.4)])**.** *We have $D_1 \cdots D_v \Pi \neq 0$.*

The form $\omega_1 \wedge \cdots \wedge \omega_v$ is nonzero, and thus neither is $c(\Pi)$.

We know proceed by induction : suppose that $c : \mathscr{H}^k \to H^{2k}(G/T)$ is injective for a given $k \leq v$, and let $V := \mathscr{H}^{k-1} \cap \mathrm{Ker}(c)$. By $W$-equivariance ($c(wf) = wc(f)$), $V$ is preserved by $W$. There exists a positive root $\alpha$ such that $s_\alpha$ does not act by $-I$ on $V$. Indeed, if this was not the case, then the action of $W$ on $V$ would be given by the determinant action : $w \cdot f = \det(w)f$, but this is impossible because of 6.27 and the degree of polynomials in $V$. Then we decompose $V = V_+ \oplus V_-$ according to the eigenspaces of $s_\alpha$. If $V \neq 0$, then $V_+ \neq 0$, so take $f$ in $V_+$ nonzero. We have $c(\alpha f) = c(\alpha)c(f) = 0$ and therefore $\alpha f \in \mathscr{I}$ by induction and because of its degree. Now, take $h_1, \ldots, h_{|W|}$ to be a basis of $\mathscr{H}$, where $h_1, dots, h_r$ are $s_\alpha$-skew and the rest is invariant. By Chevalley's theorem 6.14 and theorem 6.24, we write $\alpha f = \sum_i h_i \sigma_i$ where $\sigma_i$ are $W$-invariant of positive degree. The polynomial $\alpha$ is skew and $f$ is invariant, so $\alpha f$ is skew, and then the sum ends at $r$. By hypothesis, for $i \leq r$, $h_i$ must vanish on $\mathrm{Ker}(\alpha)$, hence be written $h_i = \alpha h_i'$ for $h_i' \in \mathscr{S}$. We then have $f = \sum_i h_i' \sigma_i \in \mathscr{I}$. Therefore, $f \in \mathscr{H} \cap \mathscr{I} = 0$ and $c : \mathscr{H}^k \to H^{2k}(G/T)$ is injective. By induction, we have proved that $c$ is injective on $H$.

The surjectivity is then clear by dimension : we have proven in 6.25 that $\dim \mathscr{H} = |W|$, and by (23) $H(G/T)$ is of the same dimension. ☐

**Theorem 6.50** ([9, Proposition (6.1)])**.** *The pullback map $\Psi^*$ associated to $\Psi$ is an isomorphism of graded rings*

$$\Psi^* : H(G) \xrightarrow{\cong} [H(G/T) \otimes H(T)]^W. \tag{25}$$

**Theorem 6.51.** *The cohomology ring $H(G)$ is an exterior graded algebra with generators of degree $(2m_i, 1)$, for $1 \leq i \leq l$.*

*Proof.* We know from (25) that

$$H(G) \cong [H(G/T) \otimes H(T)]^W.$$

Therefore, using (22) and 6.48 we obtain

$$H(G) \cong [\mathscr{H}_{(2)} \otimes \Lambda]^W.$$

We conclude with 6.34 : $H(G)$ is an exterior algebra with generators of degree $(2m_i, 1)$, for $1 \leq i \leq l$. $\square$

## References

[1] Nicolas Bourbaki. *Algèbre : Chapitres 1 à 3*. Eléments de mathématique [2.1]. Springer, 2006.

[2] Samuel Eilenberg Claude Chevalley. Cohomology theory of lie groups and lie algebras. *Trans. Amer. Math. Soc. 63 (1948), 85-124*.

[3] Brian Hall. *Lie groups, Lie algebras, and representations : an elementary introduction*. Graduate texts in mathematics 222. Springer, 2ed. edition, 2015.

[4] William Fulton; Joe Harris. *Representation theory : a first course*. Graduate texts in mathematics, 129.; Graduate texts in mathematics., Readings in mathematics. Springer-Verlag, 1991.

[5] Sigurdur Helgason. *Groups and Geometric Analysis*. Pure and Applied Mathematics Academic Pr. Academic Press, 1984.

[6] James E. Humphreys. *Reflection groups and Coxeter groups*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1992.

[7] Serge Lang. *Algebra*. Springer, 2002.

[8] Loring W. Tu Raoul Bott. *Differential forms in algebraic topology*. Graduate texts in mathematics 082. Springer, 1982.

[9] Mark Reeder. On the cohomology of compact lie groups. *L'enseignement mathématiques*, 1995.

[10] Tammo Tom Dieck Theodor Brocker. *Representations of Compact Lie Groups*. Graduate Texts in Mathematics. Springer-Verlag, corr. 2nd print edition, 1985.

[11] Frank W. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Graduate Texts in Mathematics. Springer, 1st ed. 1971. 2nd printing edition, 2010.