Brownian Motion and Partial Differential Equations

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Abstract

This document is the report of an internship done during the second year in the *Ecole Normale Supérieure de Rennes*. It was done at the University of Mannheim, during 6 weeks, supervised by Andreas Neueunkirch. We study the interplay between probability, in particular Brownian motion, and partial differential equation. We see methods for solving PDEs by simulating random paths of a Brownian motion, more precisely to solve the Dirichlet Problem, the Heat Equation and other parabolic equations. We also give simulation examples for the corresponding Monte-Carlo methods.

Contents

Thanks

Thanks to the *University of Mannheim* for its welcome.

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Chapter 1

Introduction

Brownian motion is the name given to the irregular movement of pollen suspended in water, observed by the botanist Robert Brown in 1828. At the beginning of the 20^{th} century, Louis Bachelier and Albert Einstein used the Brownian motion for modeling stocks of prices and diffusion processes. But the Brownian motion was mathematicaly rigurous defined only in 1923 by Norbert Wiener. Brownian motion can be seen as the distribution limit of a normalized sequence of random walks.
Besides we find partial differential equation in many fields. They can describe many phenomena such that heat, stocks of prices and quantum mecanics. Sometimes we can solve it easily but often it is hard to find an exact solution.
However Brownian motion can help us to find solution to such equation. Therefore, we can find an approximation by the Monte-Carlo methods.

Chapter 2

Brownian Motion

In this chapter, let d be a positive integer and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

2.1 Continuous-time process

Let I be an interval. A process $(X_t,\ t \in I)$ defined on a measurable space $(E, \mathcal{E})$ is a collection of random variables, which take values in $(E, \mathcal{E})$. It can be seen as a map from Ω to E^I,

$$\begin{array}{rccc} X & : & \Omega & \to & E^I \\ & & \omega & \mapsto & (X_t(\omega))_{t\in I}. \end{array}$$

As for discrete-time random variables, we have:

Proposition 2.1.1. *Let $(X_t,\ t \in I)$ and $(Y_t,\ t \in J)$ be two processes defined on $(E, \mathcal{E})$.*

- *They have the same distribution if, $I = J$, and for all $n \in \mathbb{N}^*$, $A_1, \dots, A_n \in \mathcal{E}$ and $t_1, \dots, t_n \in I$,*

$$\mathbb{P}(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n) = \mathbb{P}(Y_{t_1} \in A_1, \dots, Y_{t_n} \in A_n).$$

- *They are independant if, for all $n \in \mathbb{N}^*$, $A_1, \dots, A_n,\ B_1, \dots, B_n \in \mathcal{E}$, $t_1, \dots, t_n \in I$ and $s_1, \dots, s_n \in J$,*

$$\mathbb{P}(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n, Y_{s_1} \in B_1, \dots, Y_{s_n} \in B_n) = \mathbb{P}(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n)\mathbb{P}(Y_{s_1} \in B_1, \dots, Y_{s_n} \in B_n).$$

Note. As for discrete random variables, we have the equivalents with the expectations of bounded measurable functions.

We must also define the notion of filtrations for continuous-time processes.

Definition 2.1.1. Let $I = [0, T]$ or $\mathbb{R}^+$. We call *filtration* a nondecreasing family $(\mathcal{F}_t)_{t\in I}$ of sub-σ-fields of $\mathcal{F}$:

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}, \quad 0 \le s < t < \infty.$$

In the following, we will write $(\mathcal{F}_t)$ for $(\mathcal{F}_t)_{t\in I}$.

We set $\mathcal{F}_\infty = \sigma\left(\cup_{t\ge 0}\mathcal{F}_t\right)$.
Given a stochastic process, the filtration generated by the process is $\mathcal{F}_t^X = \sigma(X_s; 0 \le s \le t)$. It is the smallest σ-field with respect to which X_s is measurable for every $s \in [0, t]$.

Definition 2.1.2. The stochastic process X is *adapted* to the filtration $(\mathcal{F}_t)$ if, for each $t \ge 0$, X_t is an $\mathcal{F}_t$-measurable random variable.

Obviously, every process X is adapted to $\left(\mathcal{F}_t^X\right)$.

Definition 2.1.3. The stochastic process X is called *progressively measurable* with respect to the filtration $\left(\mathcal{F}_t^X\right)$ if, for all $t \in I$,

$$\begin{array}{rcccc} X & : & ([0,t]\times\Omega,\ \mathcal{B}([0,t])\otimes\mathcal{F}_t) & \to & (\mathbb{R},\ \mathcal{B}(\mathbb{R})) \\ & & (s,\omega) & \mapsto & (X_s(\omega)) \end{array}$$

is measurable.

In this case, $\int_0^t X_s \,\mathrm{d}s$ is well defined on $\left\{\int_0^t |X_s| \,\mathrm{d}s < \infty\right\}$.

2.2 First results

We introduce here the Brownian motion and some properties we will need later.

Definition 2.2.1. Let $I = \mathbb{R}^+$ or $[0, T]$. A *standard, one dimensional, Brownian motion* on I is a collection of random variables $(W_t,\ t \in I)$ such that:

- $W_0 = 0$.
- For all $s \leq t \in I$, $W_t - W_s$ is normally distributed with mean zero and variance $t - s$.
- For all $s \leq t \in I$, $W_t - W_s$ is independant of $\mathcal{F}_s$.
- Almost surely, the map $t \in I \mapsto W_t \in \mathbb{R}$ is continuous.

Note that, based on the definition, W_t is a real random variable with law $\mathcal{N}(0, t)$. So its probability density function is:

$$p(t; 0, x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right), \quad x \in \mathbb{R}.$$

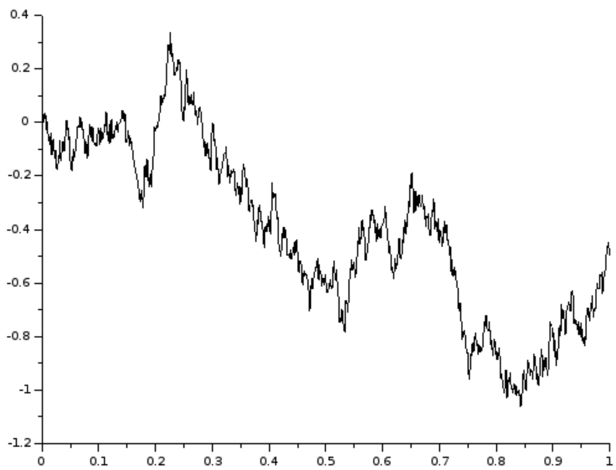

Figure 2.1: A one-dimensional Brownian sample path on $[0, 1]$.

Definition 2.2.2. Let μ be a probability measure on $\big(\mathbb{R}^d,\ \mathcal{B}(\mathbb{R}^d)\big)$. Let $W = (W_t,\ t \geq 0)$ be a continuous adapted process with values in $\mathbb{R}^d$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. This process is called a *d-dimensional Brownian motion with initial distribution* μ, if:

- $\forall A \in \mathcal{B}(\mathbb{R}^d)\ \mathbb{P}(W_0 \in A) = \mu(A)$.
- For $0 \leq s < t$, the increment $W_t - W_s$ is independant of $\mathcal{F}_s$ and is normally distributed with mean zero and covariance matrix equal to $(t - s)I_d$, where I_d is the $(d \times d)$ identity matrix.
- Almost surely, the map $t \in I \mapsto W_t \in \mathbb{R}$ is continuous.

If μ assigns measure one to some singleton $\{x\}$, we say that W is a *d-dimensional Brownian motion starting at x*. The probability density function of a Brownian motion starting at x is

$$p(t; x, y) \triangleq \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}.$$

Note. For W a Brownian motion and $x \in \mathbb{R}^d$, $x + W$ is a Brownian motion starting at x.

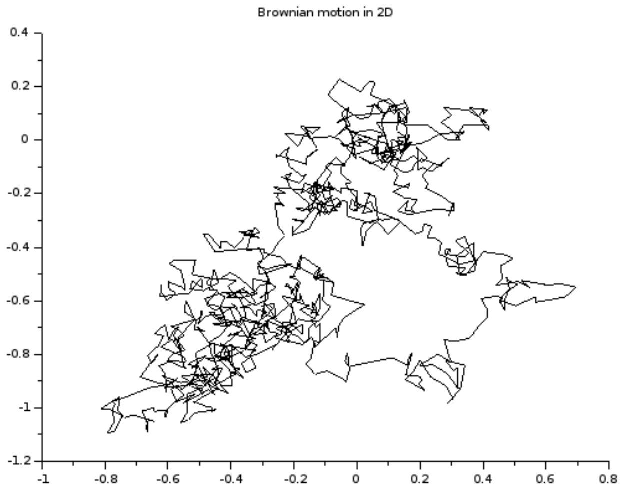

Figure 2.2: A two-dimensional Brownian motion.

Note. $(W_t,\ t \geq 0)$ is a d-dimensional Brownian motion on $\mathbb{R}^d$ if and only if its coordinates $(W_t^{(i)},\ t \geq 0)$, $i = 1, \dots, d$ are one-dimensional independant Brownian motions.

Proposition 2.2.1. *The Brownian motion has the following properties*

- $\forall t \in I,\ \mathbb{E}[W_t] = 0,$
- $\forall t \in I,\ \mathbb{E}[W_t^2] = t,$
- $\forall t, s \in I,\ \mathbb{E}[W_t W_s] = t \wedge s.$

Proof. The first two one are obvious since $W_t \sim \mathcal{N}(0, t)$. Then let's prove the last one.
Assume that $t < s$, we have $\mathbb{E}[W_t W_s] = \mathbb{E}[(W_s - W_t)W_t] + \mathbb{E}[W_t^2] = t$ because of independance of $W_s - W_t$ from $\mathcal{F}_t$. □

Proposition 2.2.2. *Let $(W_t,\ t \geq 0)$ be a Brownian motion on $\mathbb{R}$. So are the processes obtained by the following transformations:*

- *(Symmetry)* $-W = \{-W_t,\ \mathcal{F}_t;\ 0 \leq t < \infty\}$.
- *(Translation)* $Y = \{Y_t,\ \mathcal{F}_t^Y;\ 0 \leq t < \infty\}$ *defined for* $T > 0$ *by*

$$Y_t = W_{t+T} - W_T,\ \textit{for } 0 \leq t < \infty.$$

- *(Time-reversal)* $Z = \{Z_t,\ \mathcal{F}_t^Z;\ t \in [0, T]\}$ *defined for* $T > 0$ *by*

$$Z_t = W_T - W_{T-t},\ \textit{for } 0 \leq t \leq T.$$

- *(Scaling)* $X = \{X_t,\ \mathcal{F}_{ct};\ 0 \leq t < \infty\}$ *defined for* $c > 0$ *by*

$$X_t = \frac{W_{ct}}{\sqrt{c}},\ \textit{for } 0 \leq t < \infty.$$

Proof. See [1, Chapter 2 p.104] □

We can visualize these properties on Figures 2.3 and 2.4: if we erase time and space graduations, the both sample paths are alike. And we can not know which of the preceding transformations we have drawn.

Proposition 2.2.3. *Let W be a one-dimensional Brownian motion and $T > 0$. Then, $(W_t,\ t \in [0, T])$ and $(W_{T+t} - W_T,\ t \geq 0)$ are two independant Brownian motions.*

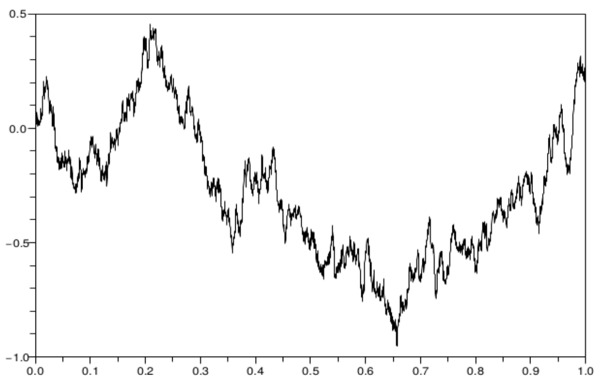

Figure 2.3: Brownian sample path on $[0,1]$.

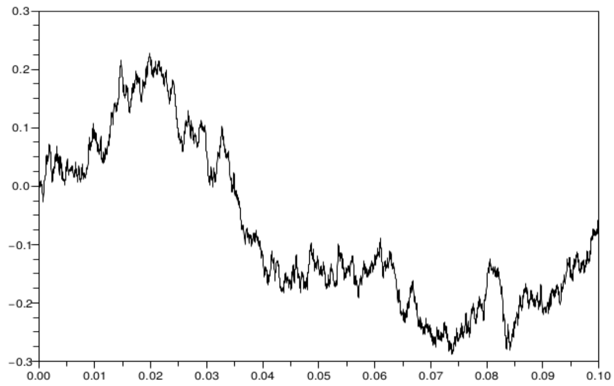

Figure 2.4: Brownian sample path on $[0,1/10]$.

Proof. We already know that the processes are Brownian motions. Let's prove that they are independant.

Let $n \in \mathbb{N}^*$, $t_1 < \ldots < t_n$, $s_1 < \ldots < s_n$ and $f, g : \mathbb{R}^n \to \mathbb{R}$ be two bounded measurable functions. Define

$$A = \mathbb{E}\left[f(W_{t_1}, \ldots, W_{t_n})g(W_{T+s_1} - W_T, \ldots, W_{T+s_n} - W_T)\right].$$

Let's consider α: $(x_1, \ldots, x_n) \mapsto (x_1, x_1 + x_2, \ldots, x_1 + \ldots + x_n)$.
We have

$$A = \mathbb{E}\left[f \circ \alpha(W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}})g \circ \alpha(W_{T+s_1} - W_T, W_{T+s_2} - W_{T+s_1}, \ldots, W_{T+s_n} - W_{T+s_{n-1}})\right].$$

Using the fact that increments are independants, it follows

$$\begin{aligned} A &= \mathbb{E}\left[f \circ \alpha(W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}})\right] \mathbb{E}\left[g \circ \alpha(W_{T+s_1} - W_T, W_{T+s_2} - W_{T+s_1}, \ldots, W_{T+s_n} - W_{T+s_{n-1}})\right] \\ &= \mathbb{E}\left[f(W_{t_1}, \ldots, W_{t_n})\right] \mathbb{E}\left[g(W_{T+s_1} - W_T, \ldots, W_{T+s_n} - W_T)\right]. \end{aligned}$$

□

Theorem 2.2.1. *For almost every $\omega \in \Omega$, we have:*

1. $\displaystyle\overline{\lim_{t\downarrow 0}} \frac{W_t(\omega)}{\sqrt{2t \log\log(1/t)}} = 1,$
2. $\displaystyle\underline{\lim_{t\downarrow 0}} \frac{W_t(\omega)}{\sqrt{2t \log\log(1/t)}} = -1,$
3. $\displaystyle\overline{\lim_{t\to\infty}} \frac{W_t(\omega)}{\sqrt{2t \log\log(t)}} = 1,$
4. $\displaystyle\underline{\lim_{t\to\infty}} \frac{W_t(\omega)}{\sqrt{2t \log\log(t)}} = -1.$

Proof. By symmetry, property 2 follows from 1, and by time-inversion, properties 3 and 4 follow from 1 and 2, respectively. It suffices to show 1. See [1, Section 2.9 p.112]. □

We introduce the stopping time:

$$T_b = \inf\{t \geq 0;\ W_t = b\}, \quad \text{for } b \in \mathbb{R}^d.$$

We will now analyse the law of $M_t \triangleq \max_{0\leq s\leq t} W_s$, when $d = 1$.
Note that we have $\{T_b \leq t\} = \{M_t \geq b\}$, for $t > 0$ and $b \in \mathbb{R}$.

Proposition 2.2.4. *M_t et $|W_t|$ are identically distributed.*

Proof. We consider the distribution function. We have, for $x \in \mathbb{R}^d$,

$$\begin{aligned}\mathbb{P}^0(M_t \geq x) &= \mathbb{P}^0(W_t \leq x, M_t \geq x) + \mathbb{P}^0(W_t > x, M_t \geq x)\\ &= \mathbb{P}^0(W_t \leq x, M_t \geq x) + \mathbb{P}^0(W_t > x)\ , \text{ since } W_t > x \Rightarrow M_t \geq x.\end{aligned}$$

Now we have to find $\mathbb{P}^0(W_t \leq x, M_t \geq x)$.

$$\begin{aligned}\mathbb{P}^0(W_t \leq x, M_t \geq x) &= \mathbb{P}^0(W_t \leq x, T_x \leq t)\\ &= \mathbb{P}^0(W_t \geq x), \text{as we will see later by the reflection principle.}\end{aligned}$$

So it follows

$$\mathbb{P}^0(M_t \geq x) = 2\mathbb{P}^0(W_t > x) = \mathbb{P}^0(|W_t| > x),$$

since $-W_t$ is also a Brownian motion. □

Thanks to this Proposition, we get

$$\mathbb{P}^0(T_b \geq T) = \mathbb{P}^0(M_T \geq b) = \mathbb{P}^0(|W_T| \geq b), \quad \forall b \in \mathbb{R},\ T > 0.$$

2.3 Continuous-time Martingales

In this section we will extend the discrete-time notions to continuous-time martingales.
We consider a real-valued process $X = (X_t,\ 0 \leq t < \infty)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to a given filtration $(\mathcal{F}_t)$ and such that $\mathbb{E}|X_t| < \infty$, for every $t \geq 0$.

Definition 2.3.1. The process $\{X_t,\ \mathcal{F}_t;\ 0 \leq t < \infty\}$ is said to be a *submartingale* (respectively, a *supermartingale*) if, for every $0 \leq s < t < \infty$, we have, a.s. $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$ (respectively, $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$).
We shall say that $\{X_t,\ \mathcal{F}_t;\ 0 \leq t < \infty\}$ is a *martingale* if, it is both a submartingale and a supermartingale.

We can now see that the Brownian motion is a martingale in respect to the filtration it generates.
Let $0 \leq s \leq t$. We know that $W_t - W_s$ is independant of $\mathcal{F}_s$, then

$$\mathbb{E}\left[W_t|\mathcal{F}_s\right] = \mathbb{E}[W_t - W_s|\mathcal{F}_s] + W_s = \mathbb{E}[W_t - W_s] + W_s = W_s.$$

2.4 Markov property of the Brownian motion

As in the discrete-time case, a process $(X_t,\ t \geq 0)$ satisfies the Markov property if, for all $T > 0$, the law of $(X_{T+t},\ t \geq 0)$ conditionally to $(X_t,\ 0 \leq t \leq T)$ is the same as the one of $(X_{T+t},\ t \geq 0)$ conditionally to X_T.
Let $0 \leq s < t$. Let us suppose that we observe a Brownian motion with initial distribution μ up to time s. In particular, we know the value of W_s^μ, we call it x. Then, $W_t^\mu = (W_t^\mu - W_s^\mu) + W_s^\mu$ and we know that $W_t^\mu - W_s^\mu$ is independant of $\mathcal{F}_s$ (the observations up to time s) and is distributed as W_{t-s} is. So $(W_t^\mu - W_s^\mu) + W_s^\mu$ is distributed as W_{t-s}^x is. So, we get

Proposition 2.4.1. *Brownian motion satisfies the Markov property. In other words:*

$$\mathbb{P}^\mu(W_t \in A|\mathcal{F}_s) = \mathbb{P}^\mu(W_{t-s} \in A|W_s),\ 0 \leq s < t,\ A \in \mathcal{B}(\mathbb{R}^d).$$

And

$$\mathbb{P}^\mu(W_t \in A|W_s = x) = \mathbb{P}^x(W_{t-s} \in A),\ 0 \leq s < t,\ A \in \mathcal{B}(\mathbb{R}^d),\ x \in \mathbb{R}^d.$$

Proposition 2.4.2 (The Strong Markov Property)**.** *Let T be a stopping time and W a Brownian motion. Then,*

$$\mathbb{E}^0[f(W_{T+s})|\mathcal{F}_T] = \mathbb{E}^0[f(W_{T+s})|W_T].$$

In particular, on $\{T < \infty\}$, $(W_{t+T} - W_T)$ is a Brownian motion independant of $\mathcal{F}_T$.

Proof. See [1, Section 2.6]. □

2.5 Reflection principle

Let $\{W_t,\ \mathcal{F}_t;\ 0 \leq t < \infty\}$ be a one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$.
For $b > 0$ and $t > 0$, we have

$$\begin{aligned}\mathbb{P}^0(T_b < t) &= \mathbb{P}^0(T_b < t, W_t > b) + \mathbb{P}^0(T_b < t, W_t < b)\\ &= \mathbb{P}^0(W_t > b) + \mathbb{P}^0(T_b < t, W_t < b).\end{aligned}$$

Let's have a look on the second term. If $T_b < t$ and $W_t < b$, then sometime before time t, the Brownian path reached level b and then traveled from b to another point $c < b$. For every path which crosses the level b and is at time t at a point below b, call it c, there is a "shadow path" which is the reflect about the level b. It exceeds this level at time t. See Figure 2.5. These two paths have the same probability, as a consequence of the strong Markov property. So, heuristically,

$$\mathbb{P}^0(T_b < t, W_t < b) = \mathbb{P}^0(T_b < t, W_t > b) = \mathbb{P}^0(W_t > b).$$

Thus,

$$\mathbb{P}^0(T_b < t) = 2\mathbb{P}^0(W_t > b) = \sqrt{\frac{2}{\pi}} \int_{b/\sqrt{t}}^{+\infty} e^{-x^2/2} \mathrm{d}x.$$

By differentiating with respect to t, we get the density of the passage time T_b:

$$\mathbb{P}^0(T_b \in \mathrm{d}t) = \frac{|b|}{\sqrt{2\pi t^3}} e^{-b^2/2t}, \quad t > 0.$$

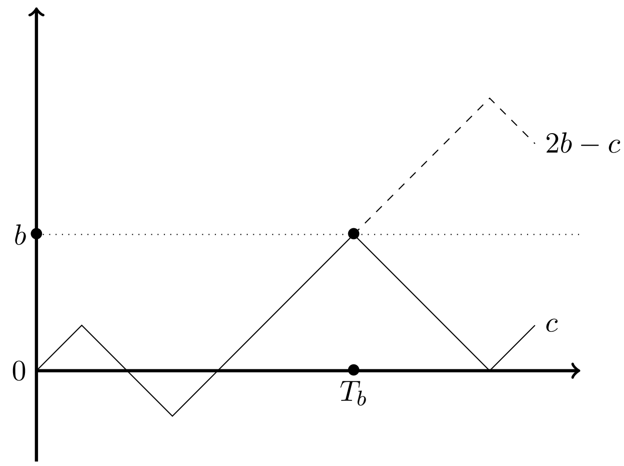

Figure 2.5: The reflection principle.

Chapter 3

Interplay with Partial Differential Equations

The solutions to many problems of elliptic and parabolic partial differential equations can be represented as expectations of stochastic functionals. It gives us a way to solve PDEs and get properties of these solutions and, conversely to determine the distributions of various functionals of stochastic processes by solving related partial differential equation problems. We see three main partial differential equations: the Dirichlet Problem, the Heat Equation and more general parabolic equations.

3.1 The Dirichlet Problem

In this section, $\{W_t,\ \mathcal{F}_t;\ 0 \le t < \infty\}$, $(\Omega, \mathcal{F})$, $\{\mathbb{P}^x\}_{x\in\mathbb{R}^d}$ is a d-dimensional Brownian family and D is an open set in $\mathbb{R}^d$.
We introduce the time of first exit from D

$$\tau_D \triangleq \inf\{t \ge 0;\ W_t \in D^c\}.$$

As each component of W is almost surely unbounded (Theorem 2.2.1), we have

$$\mathbb{P}^x(\tau_D < \infty) = 1, \quad \forall x \in D \subset \mathbb{R}^d, \quad D \text{ bounded.}$$

Let B_r be the open ball of radius r centered at the origin. Then its volume is:

$$V_r \triangleq \frac{2r^d \pi d/2}{d\Gamma(d/2)},$$

and its surface area is

$$S_r \triangleq \frac{2r^{d-1}\pi d/2}{\Gamma(d/2)} = \frac{d}{r}V_r.$$

We define a probability measure μ_r on ∂B_r by

$$\mu_r(\mathrm{d}x) = \mathbb{P}^0(W_{\tau_{B_r}} \in \mathrm{d}x), \quad r > 0.$$

3.1.1 Harmonic function and mean-value property

Let's recall some properties.

Definition 3.1.1. The function $u : D \to \mathbb{R}$ has the *mean-value property* if, for every $a \in D$ and $0 < r < \infty$ such that $a + \overline{B}_r \subset D$, we have

$$u(a) = \int_{\partial B_r} u(a+x)\mu_r(\mathrm{d}x).$$

Then,

$$\begin{aligned}\frac{1}{V_r}\int_{B_r} u(a+x)\mathrm{d}x &= \frac{1}{V_r}\int_0^r S_\rho \int_{\partial B_\rho} u(a+x)\mu_\rho(\mathrm{d}x)\mathrm{d}\rho \\ &= \frac{1}{V_r}\int_0^r S_\rho u(a)\mathrm{d}\rho, \quad \text{by definition} \\ &= u(a)\end{aligned}$$

So,

$$u(a) = \frac{1}{V_r} \int_{B_r} u(a+x)\mathrm{d}x.$$

Proposition 3.1.1. *If u is harmonic in D, then it has the mean-value property.*
If u maps D into $\mathbb{R}$ and has the mean-value property, then u is of class C^∞ and harmonic.

Proof. See [1, Section 4.2. p.241] for a proof using stochastic calculus. □

3.1.2 The Dirichlet Problem

Let $f : \partial D \to \mathbb{R}$ be a given continuous function. The *Dirichlet problem* (D, f) is finding a continuous function $u : \overline{D} \to \mathbb{R}$ such that u is harmonic in D and is equal to f on ∂D.
In other words, the last condition is that, u is of class $C^2(D)$ and

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = f & \text{on } \partial D. \end{cases} \tag{3.1}$$

We call such a function, a *solution to the Dirichlet problem* (D, f). We can see that, using Brownian motion, we can write down a solution to (D, f):

$$u(x) \triangleq \mathbb{E}^x f(W_{\tau_D}), \quad x \in \overline{D}, \tag{3.2}$$

provided the expectation exists.
Let's verify it:

- Boundary values: If $a \in \partial D$, $u(a) = \mathbb{E}^a f(W_{\tau_D}) = \mathbb{E}^a f(a) = f(a)$, by the definition of τ_D.
- if $a \in D$, choose r so that $a + \overline{B}_r \subset D$,

$$\begin{aligned} u(a) &= \mathbb{E}^a f(W_{\tau_D}) = \mathbb{E}^a \left[\mathbb{E}^a \left[f(W_{\tau_D})|\mathcal{F}_{\tau_{a+B_r}}\right]\right] = \mathbb{E}^a u\left(W_{\tau_{a+B_r}}\right), \text{ by strong Markov property} \\ &= \int_{\partial B_r} u(a+x)\mu_r(\mathrm{d}x). \end{aligned}$$

  Thus u has the mean-value property, and therefore is harmonic.

So u satisfies 3.1. The only problem is whether u is continuous up to and including ∂D.

Proposition 3.1.2. *If f is bounded and $\mathbb{P}^x(\tau_D < \infty) = 1$, for all $x \in D$, then there exists an unique bounded solution to (D, f) and it is 3.2.*

Proof. Let u be a bounded solution to (D, f) and $D_n = \{x \in D; \inf_{y \in \partial D} \| x - y \| > 1/n\}$. With Itô's rule, see B.0.1, and 3.1 we have a.s.:

$$u(W_{t\wedge\tau_{B_n}\wedge\tau_{D_n}}) = u(W_0) + \sum_{i=1}^{d} \int_0^{t\wedge\tau_{B_n}\wedge\tau_{D_n}} \frac{\partial u}{\partial x_i}(W_s)\mathrm{d}W_s^{(i)}.$$

We take the expectations and by the property of stochastic integral, we get, for $0 \le t < \infty$, $n \ge 1$ and $a \in D_n$,

$$u(a) = \mathbb{E}^a[u(W_0)] = \mathbb{E}^a[u(W_{t\wedge\tau_{B_n}\wedge\tau_{D_n}})].$$

When $t \to \infty$ and $n \to \infty$, $u(W_{t\wedge\tau_{B_n}\wedge\tau_{D_n}})$ converges to $u(W_{\tau_D}) = f(W_{\tau_D})$ a.s. And therefore, with the bounded convergence theorem,

$$u(a) = \mathbb{E}^a[f(W_{\tau_D})].$$

□

3.1.3 Regularity of the solution

We could have found a solution to the Dirichlet problem but it remains the problem of the continuity of u at the boundary of D. We want to characterize the points $a \in \partial D$ where

$$\lim_{\substack{x \to a \\ x \in D}} \mathbb{E}^x f(W_{\tau_D}) = f(a), \tag{3.3}$$

for every bounded, measurable function $f : \partial D \to \mathbb{R}$ which is continuous at a.
Let's introduce the stopping time

$$\sigma_D \triangleq \inf\{t > 0;\ W_t \in D^c\}.$$

Definition 3.1.2. A point $a \in \partial D$ is *regular* for D if, $\mathbb{P}^a(\sigma_D = 0) = 1$, i.e. a Brownian path started at a does not return to D and remains in D^c for a nonempty time interval.

Note. Regularity is local: $a \in \partial D$ is regular for D if and only if a is regular for $(a + B_r) \cap D$, for some $r > 0$.

Note. a is irregular if $\mathbb{P}^a(\sigma_D = 0) < 1$. By Blumenthal zero-one law, $\mathbb{P}^a(\sigma_D = 0) = 0$.

Theorem 3.1.1. *For $d \geq 2$ and $a \in \partial D$, the following are equivalent:*

1. $\lim_{\substack{x \to a \\ x \in D}} \mathbb{E}^x f(W_{\tau_D}) = f(a)$ *for every bounded, measurable function $f : \partial D \to \mathbb{R}$ which is continuous at a.*
2. *a is regular for D.*
3. $\forall \varepsilon > 0, \quad \lim_{\substack{x \to a \\ x \in D}} \mathbb{P}^x(\tau_D > \varepsilon) = 0.$

Proof. Without loss of generality, we can assume that $a = 0$.
$1 \Rightarrow 2$: Let's suppose that 0 is irregular, then thanks to the previous Note, $\mathbb{P}^a(\sigma_D = 0) = 0$. Since a Brownian motion never returns to its starting point (See [1, Section 3.3 p.161] for more details), when $d \geq 2$,

$$\lim_{r \to 0} \mathbb{P}^0(W_{\sigma_D} \in B_r) = \mathbb{P}^0(W_{\sigma_D} = 0) = 0.$$

Let r be such that $\mathbb{P}^0(W_{\sigma_D} \in B_r) < 1/4$. We choose δ_n such that $0 < \delta_n < r$ and $\delta_n \downarrow 0$ for all n. Let's define $\tau_n = \inf\{t \geq 0;\ \| W_t \| \geq n\}$, for $n \in \mathbb{N}$. We have $\mathbb{P}^0(\tau_n \downarrow 0) = 1$ and so $\lim_{n \to \infty} \mathbb{P}^0(\tau_n < \sigma_D) = 1$. Let n be large enough so that $\mathbb{P}^0(\tau_n < \sigma_D) \geq 1/2$, we have

$$\begin{aligned}
\frac{1}{4} &\geq \mathbb{P}^0(W_{\sigma_D} \in B_r) \\
&\geq \mathbb{P}^0(W_{\sigma_D} \in B_r, \tau_n < \sigma_D) \\
&\geq \int_{\mathbb{R}} \mathbb{P}^0(W_{\sigma_D} \in B_r, \tau_n < \sigma_D, W_{\tau_n} \in \mathrm{d}x) \\
&\geq \int_{\mathbb{R}} \mathbb{P}^0(W_{\sigma_D} \in B_r | \tau_n < \sigma_D, W_{\tau_n} \in \mathrm{d}x) \mathbb{P}^0(\tau_n < \sigma_D, W_{\tau_n} \in \mathrm{d}x) \\
&\geq \int_{D \cap B_{\delta_n}} \mathbb{P}^x(W_{\sigma_D} \in B_r) \mathbb{P}^0(\tau_n < \sigma_D, W_{\tau_n} \in \mathrm{d}x) \\
&\geq \int_{D \cap B_{\delta_n}} \mathbb{P}^x(W_{\tau_D} \in B_r) \mathbb{P}^0(\tau_n < \sigma_D, W_{\tau_n} \in \mathrm{d}x) \\
&\geq \inf_{x \in D \cap B_{\delta_n}} \mathbb{P}^x(W_{\tau_D} \in B_r) \int_{D \cap B_{\delta_n}} \mathbb{P}^0(\tau_n < \sigma_D, W_{\tau_n} \in \mathrm{d}x) \\
&\geq \inf_{x \in D \cap B_{\delta_n}} \mathbb{P}^x(W_{\tau_D} \in B_r) \mathbb{P}^0(\tau_n < \sigma_D) \\
&\geq \frac{1}{2} \inf_{x \in D \cap B_{\delta_n}} \mathbb{P}^x(W_{\tau_D} \in B_r)
\end{aligned}$$

So, $\inf_{x \in D \cap B_{\delta_n}} \mathbb{P}^x(W_{\tau_D} \in B_r) \leq 1/2$. Thus, there exist $x_n \in D \cap B_{\delta_n}$ such that $\mathbb{P}^{x_n}(W_{\tau_D} \in B_r) \leq 1/2$. Now, we choose a bounded continuous function $f : \partial D \to \mathbb{R}$, such that,

- $f(0) = 1$,
- $f \leq 1$ inside B_r,
- $f = 0$ outside B_r.

With such a function, we obtain,

$$\overline{\lim_{n\to\infty}} \mathbb{P}^{x_n}(W_{\tau_D} \in B_r) \leq 1/2 < 1 = f(0).$$

But,

$$\begin{aligned}
\overline{\lim_{n\to\infty}} \mathbb{E}^{x_n} f(W_{\tau_D}) &= \overline{\lim_{n\to\infty}} \int_D f(x)\mathbb{P}^{x_n}(W_{\tau_D} \in \mathrm{d}x) \\
&= \overline{\lim_{n\to\infty}} \int_{B_r} f(x)\mathbb{P}^{x_n}(W_{\tau_D} \in \mathrm{d}x) \\
&\leq \overline{\lim_{n\to\infty}} \int_{B_r} \mathbb{P}^{x_n}(W_{\tau_D} \in \mathrm{d}x) \\
&\leq \overline{\lim_{n\to\infty}} \mathbb{P}^{x_n}(W_{\tau_D} \in B_r).
\end{aligned}$$

So 1 fails.
2 $\Rightarrow$ 3: See [1, Section 4.2 p.247].
3 $\Rightarrow$ 1: See [1, Section 4.2 p.247]. □

Definition 3.1.3. Let $a \in \partial D$. A *barrier at a* is a continuous function $v : \overline{D} \to \mathbb{R}$ which is harmonic in D, positive on $\overline{D}\backslash\{a\}$, and equal to zero at a.

Proposition 3.1.3. *Let D be bounded and $a \in \partial D$. If there exists a barrier at a, then a is regular.*

Proof. Let's suppose that v is a barrier at a, let $f : \partial D \to \mathbb{R}$ be a bounded, continuous at a, function. We define $M = \sup_{x\in\partial D} |f(x)|$. We want to show 1 of Theorem 3.2. Let $\epsilon > 0$, and δ such that

$$x \in \partial D, \ \| x - a \| < \delta \Rightarrow |f(x) - f(a)| < \epsilon.$$

Then let k be such that $kv(x) \geq 2M$ for $x \in \overline{D}$ and $\| x - a \| \geq \delta$.
Then for $x \in D$, $|f(x) - f(a)| < \epsilon + kv(x)$.
Thus for $x \in D$, $|\mathbb{E}^x f(W_{\tau_D}) - f(a)| < \epsilon + k\mathbb{E}^x v(W_{\tau_D}) = \epsilon + kv(x)$, because of Proposition 3.1.2.
But v is continuous and $v(a) = 0$ so $\lim_{\substack{x\to a\\x\in D}} |\mathbb{E}^x f(W_{\tau_D}) - f(a)| < \epsilon$. And so a is regular. □

Example 3.1.1. Let $D \subset B_r \subset \mathbb{R}^2$ be an open set, where $0 < r < 1$ and assume that $(0,0) \in \partial D$. If log is well-defined on $\overline{D}\backslash(0,0)$, we let

$$v(x_1,x_2) \triangleq \begin{cases} -Re\dfrac{1}{\log(x_1+ix_2)} = -\dfrac{\log(\sqrt{x_1^2+x_2^2})}{|\log(x_1+ix_2)|^2}, & (x_1,x_2) \in D\backslash(0,0), \\ 0, & (x_1,x_2) = (0,0). \end{cases}$$

Indeed, v is the real part of an analytic function, so v is harmonic, and as $0 < \sqrt{x_1^2+x_2^2} < r < 1$ in $\overline{D}\backslash(0,0)$, v is positive on this set. So v is a barrier at $(0,0)$.

Proposition 3.1.4. *Let $D \subset \mathbb{R}^2$ be open, and suppose that $a \in \partial D$ has the property that there exist a point $b \neq a$ in $\mathbb{R}^2\backslash D$, and a simple arc in $\mathbb{R}^2\backslash D$ connecting a to b. Then a is regular.*

Note. We call a *simple arc* an arc which never crosses itself.

Proof. We can assume without loss of generality that $a = (0,0)$. We choose r such that $0 < r <\| b \| \wedge 1$. We see before that it suffices to show that b is regular for $D \cap B_r$. We know that C is a simple arc in $\mathbb{R}^2\backslash D$ connecting a to b. On $B_r\backslash C$, log is well-defined because we can not turn around the origin. Then, the Example 3.1.1 gives us a barrier at a and so by Proposition 3.1.3, a is regular. □

Example 3.1.2 (Lebesgue's Thorn)**.** Let us see an example of an irregular point. With $d = 3$ and $\{\epsilon_n\}_{n\geq 1}$ a sequence of positive numbers decreasing to zero, we define

$$E \triangleq \{(x_1, x_2, x_3);\ -1 < x_1 < 1,\ x_2^2 + x_3^2 < 1\},$$

$$F_n \triangleq \{(x_1, x_2, x_3);\ 2^{-n} \leq x_1 \leq 2^{-n+1},\ x_2^2 + x_3^2 \leq \epsilon_n\}$$

$$\text{and } D \triangleq E\backslash(\bigcup_{n\geq 1} F_n).$$

We know that $\mathbb{P}^0((W_t^{(2)}, W_t^{(3)}) = (0, 0),\ \text{for some } t > 0) = 0$, so the $\mathbb{P}^0$-probability that $W = (W^{(1)}, W^{(2)}, W^{(3)})$ ever hits the compact set $K_n \triangleq \{(x_1, x_2, x_3);\ 2^{-n} \leq x_1 \leq 2^{-n+1},\ x_2 = x_3 = 0\}$ equals to zero. Moreover, $\lim_{t\to\infty} \parallel W_t \parallel = \infty$ a.s., thus, if ϵ_n is enough small, we can assume that $\mathbb{P}^0(W_t \in F_n,\ \text{for some } t \geq 0) \leq 3^{-n}$. If W, starting at the origin, does not return to D immediatly, it must avoid D by entering $\bigcup_{n\geq 1} F_n$. That is to say,

$$\mathbb{P}^0(\sigma_D = 0) \leq \mathbb{P}^0(W_t \in F_n,\ \text{for some } t \geq 0 \text{ and } n \geq 1) \leq \sum_{n=1}^{+\infty} 3^{-n} < 1.$$

And thus, 0 is an irregular point.

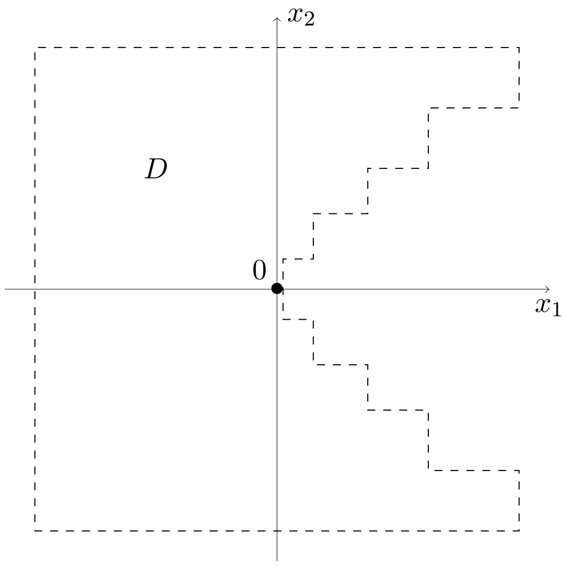

Figure 3.1: Lebesgue's Thorn.

Definition 3.1.4. For $y \in \mathbb{R}^d\backslash\{0\}$ and $0 \leq \theta \leq \pi$, we define the *cone* $C(y, \theta)$ *with direction* y *and aperture* θ by

$$C(y, \theta) = \{x \in \mathbb{R}^d;\ (x, y) \geq\parallel x \parallel \cdot \parallel y \parallel \cdot \cos\theta\}.$$

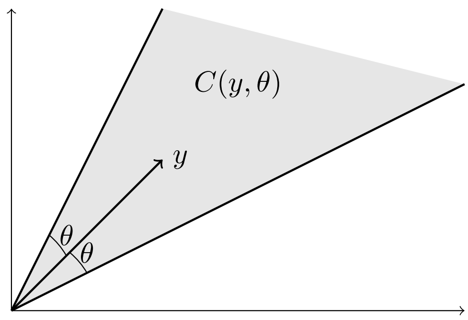

Figure 3.2: The cone $C(y, \theta)$ with direction y and aperture θ.

Definition 3.1.5. The point $a \in \partial D$ satisfies *Zaremba's cone condition* if, there exist $y \neq 0$ and $0 < \theta < \pi$ such that the translated cone $a + C(y, \theta)$ is contained in $\mathbb{R}^2 \backslash D$.

Theorem 3.1.2. *If a point $a \in \partial D$ satisfies Zaremba's cone condition, then a is regular.*

Proof. See [1, Section 4.2 p.250]. $\square$

Thus we have a complete solution to the Dirichlet problem for many open sets D. If every bounded points of D is regular and D satisfies $\mathbb{P}^a(\tau_D < \infty) = 1$ for all $a \in D$, then the unique bounded solution to (D, f) is given by 3.2.

3.1.4 Integral formulas of Poisson

If D is a half-space or a d-dimensional sphere, we can compute the solution to get Poisson integral formulas.

Theorem 3.1.3. *For $d \geq 2$, $D = \{(x_1, \ldots, x_d); x_d > 0\}$ and $f : \partial D \to \mathbb{R}$ a bounded and continuous function, the unique bounded solution to the Dirichlet problem (D, f) is given by*

$$u(x) = \frac{\Gamma(d/2)}{\pi^{d/2}} \int_{\partial D} \frac{x_d f(y)}{\| y - x \|^d} \mathrm{d}y, \quad x \in D.$$

Proof. See [1, Section 4.2.D]. $\square$

Theorem 3.1.4. *For $d \geq 2$, $B_r = \{x \in \mathbb{R}^d; \| x \| < r\}$, and $f : \partial B_r \to \mathbb{R}$ continuous, the unique solution to the Dirichlet problem (B_r, f) is given by*

$$u(x) = r^{d-2}(r - \| x \|^2) \int_{\partial B_r} \frac{f(y)\mu_r(\mathrm{d}y)}{\| y - x \|^d}, \quad x \in B_r.$$

Proof. See [1, Section 4.2.D]. $\square$

Example 3.1.3. Let u be the stationnary temperature on the unit-disk, then u verifies the Laplace equation $\Delta u = 0$. Let assume we have the Dirichlet condition $u(1, \theta) = \phi(\theta)$ for all $\theta \in [0, 2\pi)$, on the unit circle. Then, thanks to the integral formula of Poisson on the 2-dimensional sphere, u is given by

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) \frac{1 - r^2}{1 - r\cos(t + \theta) + r^2} \mathrm{d}t, \quad 0 \leq r \leq 1, \ \theta \in [0, 2\pi).$$

3.2 The one-dimensional Heat Equation

This equation describes the temperatures in infinite, semi-infinite, and finite rods. We consider an infinite rod, insulated and extended along the x-axis of the (t, x) plane, $f(x)$ denotes the temperature of the rod at $t = 0$ and location x. Let $u(t, x)$ be the temperature of the rod at time $t \geq 0$ and position $x \in \mathbb{R}$. Then u satisfies the *heat equation*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 u}{\partial x^2} \\ u(0, x) = f(x), \quad x \in \mathbb{R}. \end{cases} \tag{3.4}$$

Note. We observe, it will be useful later, that the transition density of the one-dimensional Brownian motion

$$p(t; x, y) = \frac{1}{\mathrm{d}y} \mathbb{P}^x(W_t \in \mathrm{d}y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}, \quad t > 0, \ x, y \in \mathbb{R},$$

satisfies the partial differential equation

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial x^2}.$$

We call $p(t; x, y)$ a *fundamental solution* to our problem.

Theorem 3.2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel-measurable continuous function satisfying*

$$\int_{-\infty}^{+\infty} e^{-ax^2}|f(x)|\mathrm{d}x < \infty,$$

for some $a > 0$. Then

$$u(t,x) \triangleq \mathbb{E}^x f(W_t) = \int_{-\infty}^{+\infty} f(y)p(t;x,y)\mathrm{d}y \tag{3.5}$$

is defined for $0 < t < 1/(2a)$, $x \in \mathbb{R}$, has derivatives of all orders, and satisfies the heat equation 3.4.

Proof. u is well-defined thanks to the condition on the integral. Then, by derivating under the integral, u has derivatives of all orders. Since p satisfies the heat equation 3.4, u satisfies the partial differential equation. And by dominated convergence theorem,

$$\lim_{t\to 0} \mathbb{E}^x f(W_t) = \lim_{t\to 0} \mathbb{E}^0 f(x + W_t) = f(x).$$

□

3.2.1 The Tychonoff uniqueness theorem

Definition 3.2.1. We say that a function $u : \mathbb{R}^m \to \mathbb{R}$ has *continuous derivatives up to a certain order on a set G*, if these derivatives exist and are continuous in the interior of G, and have continuous extensions to the boundary ∂G.

With this convention, we can now give the following theorem, about uniqueness.

Theorem 3.2.2. *If u is $C^{1,2}$ on the strip $(0, T] \times \mathbb{R}$, satisfies the heat equation 3.4 there and if there exist K and a such that*

$$\lim_{\substack{t\downarrow 0\\ y\to x}} u(t,y) = 0, \quad x \in \mathbb{R}, \tag{3.6}$$

$$\sup_{0<t\leq T} |u(t,x)| \leq K e^{ax^2}, \quad x \in \mathbb{R}^2. \tag{3.7}$$

Then $u = 0$ on $(0, T] \times \mathbb{R}$.

Note. If u_1 and u_2 satisfy the heat equation 3.4, the condition 3.7 and

$$\lim_{\substack{t\downarrow 0\\ y\to x}} u_1(t,y) = \lim_{\substack{t\downarrow 0\\ y\to x}} u_2(t,y),$$

then we can apply Theorem 3.2.2 to $u_1 - u_2$ and get the uniqueness.

Proof. Recall that we note T_y the first passage time of W to y. Let $x \in \mathbb{R}$, $t \in [0, T)$ and choose $n > |x|$. We define $R_n = T_n \wedge T_{-n}$ and $v(\theta, x) \triangleq u(T - t - \theta, x)$, for $0 \leq \theta < T - t$. By Itô's rule, and since u satisfies the heat equation 3.4, we have, a.s., for $0 \leq s < T - t$,

$$\begin{aligned} v(s\wedge R_n, W_{s\wedge R_n}) &= v(0, W_0) + \int_0^{s\wedge R_n} \partial_\theta v(s, W_s)\mathrm{d}s + \int_0^{s\wedge R_n} \partial_x v(s, W_s)\mathrm{d}W_s + \frac{1}{2}\int_0^{s\wedge R_n} \partial_x^2 v(s, W_s)\mathrm{d}s \\ &= v(0, W_0) + \int_0^{s\wedge R_n} \partial_x v(s, W_s)\mathrm{d}W_s. \end{aligned}$$

Then we take the expectations, and because of properties of the stochastic integral, we obtain

$$\begin{aligned} u(T-t, x) = v(0,x) &= \mathbb{E}^x v(s\wedge R_n, W_{s\wedge R_n}) \\ &= \mathbb{E}^x[v(s, W_s)1_{s<R_n}] + \mathbb{E}^x[v(R_n, W_{R_n})1_{s\geq R_n}]. \end{aligned}$$

But

- $|v(s, W_s)1_{s<R_n}| \leq \max_{\substack{0\leq s<T-t\\ |y|\leq n}} |u(T-t-s, y)| \leq K e^{ay^2}$ a.s., because of 3.7,

- $\lim_{s\to T-t} v(s, W_s) = 0$ a.s. because of 3.6,

- $|v(R_n, W_{R_n})1_{s\geq R_n}| \leq Ke^{an^2}$ a.s., as before, since $|W_{R_n}| = n$ a.s.,

- $\lim_{s\to T-t} v(R_n, W_{R_n})1_{s\geq R_n} = v(R_n, W_{R_n})1_{T-t>R_n}$ a.s. .

So from the bounded convergence theorem, we get, for $s \to T-t$,

$$u(T-t, x) = \mathbb{E}^x[v(R_n, W_{R_n})1_{T-t>R_n}].$$

And,

$$\begin{aligned}
|u(T-t,x)| &\leq Ke^{an^2}\mathbb{P}^x(R_n < T-t) \\
&\leq Ke^{an^2}\left[\mathbb{P}^x(T_n < T-t, T_n < T_{-n}) + \mathbb{P}^x(T_{-n} < T-t, T_n \geq T_{-n})\right] \\
&\leq Ke^{an^2}\left[\mathbb{P}^0(T_{n-x} < T-t, T_{n-x} < T_{-n-x}) + \mathbb{P}^0(T_{-n-x} < T-t, T_{n-x} \geq T_{-n-x})\right] \\
&\leq Ke^{an^2}\left[\mathbb{P}^0(T_{n-x} < T) + \mathbb{P}^0(T_{n+x} < T)\right] \text{ by symetry of the Brownian motion,} \\
&\leq Ke^{an^2}\left[\mathbb{P}^0(M_T < n-x) + \mathbb{P}^0(M_T < n+x)\right] \\
&\leq Ke^{an^2}\left[\sqrt{\frac{2}{\pi}}\int_{(n-x)/\sqrt{T}}^{+\infty} e^{-z^2/2}\mathrm{d}z + \sqrt{\frac{2}{\pi}}\int_{(n+x)/\sqrt{Tt}}^{+\infty} e^{-z^2/2}\mathrm{d}z\right].
\end{aligned}$$

Thanks to Lemma A.0.1, by letting $n \to \infty$, we get $u(T-t, x) = 0$ because $a < 1/(2T)$. We can extend it to the case where $a < 1/(2T)$ does not hold, by choosing $T_0 = 0 < T_1 < \ldots < T_k = T$ such that $a < 1/(2(T_i - T_{i-1}))$, $i \in \{1, \ldots, k\}$ and then showing that $u = 0$ on each strip.

□

Note. The function

$$h(t,x) \triangleq \frac{x}{t}p(t;x,0) = -\frac{\partial}{\partial x}p(t;x,0), \quad t > 0,\ x \in \mathbb{R}, \tag{3.8}$$

satisfies the heat equation 3.4 on every strip of the form $(0, T] \times \mathbb{R}$, the condition 3.7 for every $0 < a < (1/2T)$, as well as 3.6 for every $x \neq 0$.

3.2.2 Nonnegative solutions of the Heat Equation

Thanks to the representation 3.5, if the initial temperature f is nonnegative, the temperature u should remain nonnegative for all $t > 0$. We now want to characterize the nonnegative solutions of the heat equation.

Theorem 3.2.3. *Let $v(t, x)$ be a nonnegative function defined on a strip $(0, T) \times \mathbb{R}$, where $0 < T < \infty$. The following four conditions are equivalent.*

1. *For some nondecreasing function $F : \mathbb{R} \to \mathbb{R}$,*

$$v(t,x) = \int_{-\infty}^{+\infty} p(T-t;x,y)\mathrm{d}F(y), \quad 0 < t < T,\ x \in \mathbb{R}. \tag{3.9}$$

2. *v is of class $C^{1,2}$ on $(0,T) \times \mathbb{R}$ and satisfies the backward heat equation*

$$\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2} = 0 \tag{3.10}$$

   on this strip.

3. *For a Brownian family $\{W_s,\ \mathcal{F}_s;\ 0 \leq s < \infty\}$, $(\Omega, \mathcal{F})$, $\{\mathbb{P}^x\}_{x\in\mathbb{R}}$ and each fixed $t \in (0,T)$, $x \in \mathbb{R}$, the process $\{v(t+s, W_s),\ \mathcal{F}_s;\ 0 \leq s < T-t\}$ is a martingale on $(\Omega, \mathcal{F}, \mathbb{P}^x)$.*

4. *For a Brownian family $\{W_s,\ \mathcal{F}_s;\ 0 \leq s < \infty\}$, $(\Omega, \mathcal{F})$, $\{\mathbb{P}^x\}_{x\in\mathbb{R}}$ we have*

$$v(t,x) = \mathbb{E}^x v(t+s, W_s), \quad 0 < t \leq t+s < T,\ x \in \mathbb{R}. \tag{3.11}$$

Proof. $1 \Rightarrow 2$: We have $\frac{\partial}{\partial t}p(T-t;x,y) + \frac{1}{2}\frac{\partial^2}{\partial x^2}p(T-t;x,y) = 0$, then by differentiating under the integral, v satisfies the backward heat equation 3.10. We have for $a > 1/(2T)$,

$$\int_{\mathbb{R}} e^{-ay^2}\mathrm{d}F(y) = \sqrt{\frac{\pi}{a}}v(T-1/2a,0) < \infty,$$

it is the same condition as in the Theorem 3.2.1. We conclude likewise.

$2 \Rightarrow 3$: For $0 \leq s < T-t$ and $a < x < b$, we apply the Itô's rule and use the backward heat equation, to get

$$v(t+s\wedge T_a \wedge T_b, W_{s\wedge T_a\wedge T_b}) = v(t,W_0) + \int_0^{s\wedge T_a\wedge T_b} \frac{\partial}{\partial x}v(t+\sigma,W_\sigma)\mathrm{d}W_\sigma \text{ a.s. } .$$

Then, by taking the expectations, we obtain

$$v(t,x) = \mathbb{E}^x[v(t+s\wedge T_a\wedge T_b, W_{s\wedge T_a\wedge T_b})]. \tag{3.12}$$

Therefore, by Fatou's Lemma, by letting $a \downarrow -\infty$ and $b \to +\infty$, we get

$$v(t,x) \geq \mathbb{E}^x v(t+s,W_s), \quad 0 < t \leq t+s < T,\ x\in\mathbb{R}. \tag{3.13}$$

Let $0 \leq s_1 \leq s_2 < T-t$, by the Markov property, we have

$$\mathbb{E}^x[v(t+s_2,W_{s_2})|\mathcal{F}_{s_1}] = f(W_{s_1}), \tag{3.14}$$

where

$$f(y) = \mathbb{E}^y v(t+s_2, W_{s_2-s_1}).$$

From 3.13, we obtain

$$v(t+s_1) \geq \mathbb{E}^x[v(t+s_2,W_{s_2})|\mathcal{F}_{s_1}].$$

Thus $\{v(t+s,W_s),\ 0\leq s < T-t\}$ is a supermartingale on $(\Omega,\mathcal{F},\mathbb{P}^x)$. Let us now prove the reverse inequality. We use 3.12,

$$\begin{aligned}
v(t,x) &= \mathbb{E}^x[v(t+s\wedge T_a\wedge T_b, W_{s\wedge T_a\wedge T_b})]\\
&= \mathbb{E}^x[v(t+s,W_s)1_{s\leq T_a\wedge T_b}] + \mathbb{E}^x[v(t+T_a,a)1_{T_a<s\wedge T_b}] + \mathbb{E}^x[v(t+T_b,b)1_{T_b<s+T_a}]\\
&\leq \mathbb{E}^x[v(t+s,W_s)] + \mathbb{E}^x[v(t+T_a,a)1_{T_a<s}] + \mathbb{E}^x[v(t+T_b,b)1_{T_b<s}].
\end{aligned}$$

Now we want to show that the two last terms converge to zero when a (respectively b) converges to $-\infty$ (respectively $+\infty$). Both are the same, we do it only for b. Let us show that for B large enough,

$$\int_B^{+\infty} \mathbb{E}^x[v(t+T_b,b)1_{T_b<s}]\mathrm{d}b < +\infty.$$

We choose $x\in\mathbb{R}$, $0<t<T$, $0\leq s<t$ so that $s+t<T$. We have

$$\mathbb{P}^x(T_b\in\mathrm{d}\sigma) = \frac{b-x}{\sqrt{2\pi\sigma^3}}e^{-(b-x)^2/2\sigma}\mathrm{d}\sigma = h(\sigma,b-x)\mathrm{d}\sigma, \quad b>x, \sigma>0.$$

For $B \geq x$ enough large and $b\geq B$, $h(\sigma,b-x)$ is an increasing function of $\sigma\in(0,s)$. Therefore for $r\in(s,t)$ and B perhaps larger, we have

$$h(s,b-x) \leq \sqrt{\frac{r}{s^3}}p(r;x,b).$$

It follows that

$$\begin{aligned}
\int_B^{+\infty} \mathbb{E}^x[v(t+T_b,b)1_{T_b<s}]\mathrm{d}b &= \int_B^{+\infty}\int_0^s v(t+\sigma,b)h(\sigma,b-x)\mathrm{d}\sigma\mathrm{d}b\\
&\leq \sqrt{\frac{r}{s^3}}\int_0^s\int_B^{+\infty} v(t+\sigma,b)p(r,x,b)\mathrm{d}b\mathrm{d}\sigma\\
&\leq \sqrt{\frac{r}{s^3}}\int_0^s \mathbb{E}^x v(t+\sigma,W_r)\mathrm{d}\sigma\\
&\leq \sqrt{\frac{r}{s^3}}\int_0^s v(t+\sigma-r,x)\mathrm{d}\sigma < \infty \text{ with 3.13.}
\end{aligned}$$

It proves 3.11 for $x \in \mathbb{R}$, $0 < t \leq t+s < T$ with $s < t$. Now we want to remove the restriction $s < t$. We prove by induction on k that $0 < t \leq t+s < T$ and $s < kt$ implies $v(t,x) = \mathbb{E}^x v(t+s, W_s)$. We have etablished $k = 1$. Assume it for $k \geq 1$, then as we have done, $\{v(t+s, W_s),\ 0 \leq s < kt\}$ is a martingale. Let $s_2 \in [kt, (k+1)t]$, $s_1 \in [0, kt)$ with $0 < s_2 - s_1 < t$. Then,

$$\begin{aligned}
\mathbb{E}^x v(t+s_2, W_{s_2}) &= \mathbb{E}^x \left[\mathbb{E}^x \left[v(t+s_2, W_{s_2}) | \mathcal{F}_{s_1}\right]\right] \\
&= \mathbb{E}^x v(t+s_1, W_{s_1}) \text{ thanks to 3.14} \\
&= v(t,x) \text{ by induction.}
\end{aligned}$$

$3 \Rightarrow 4$: With the previous proof, we have the result.
$4 \Rightarrow 1$: See [1, Section 4.3. p.260].

□

Corollary 3.2.4. *Let $u(t,x)$ be a nonnegative function defined on a strip $(0,T) \times \mathbb{R}$, where $0 < T \leq \infty$. The following four conditions are equivalent.*

1. *For some nondecreasing function $F : \mathbb{R} \to \mathbb{R}$,*

$$u(t,x) = \int_{-\infty}^{+\infty} p(t;x,y) \mathrm{d}F(y), \quad 0 < t < T,\ x \in \mathbb{R}. \tag{3.15}$$

2. *u is of class $C^{1,2}$ on $(0,T) \times \mathbb{R}$ and satisfies the heat equation 3.4 there.*

3. *For a Brownian family $\{W_s,\ \mathcal{F}_s;\ 0 \leq s < \infty\}$, $(\Omega, \mathcal{F})$, $\{\mathbb{P}^x\}_{x \in \mathbb{R}}$ and each fixed $t \in (0,T)$, $x \in \mathbb{R}$, the process $\{u(t-s, W_s),\ \mathcal{F}_s;\ 0 \leq s < t\}$ is a martingale on $(\Omega, \mathcal{F}, \mathbb{P}^x)$.*

4. *For a Brownian family $\{W_s,\ \mathcal{F}_s;\ 0 \leq s < \infty\}$, $(\Omega, \mathcal{F})$, $\{\mathbb{P}^x\}_{x \in \mathbb{R}}$ we have*

$$u(t,x) = \mathbb{E}^x u(t-s, W_s), \quad 0 < s \leq t < T,\ x \in \mathbb{R}. \tag{3.16}$$

Proof. When $T < \infty$, we use the previous Theorem with $v(t,x) = u(T-t,x)$.
When $T = \infty$, let us define for all $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $0 < t < n$, $v_n(t,x) \triangleq u(n-t,x)$. Then we apply Theorem 3.2.3 to each v_n with $T = n$ and get that 2, 3 and 4 are equivalent. They are implied by 1 and they imply that there exists for all $n \in \mathbb{N}^*$, a nondecreasing function $F : \mathbb{R} \to \mathbb{R}$ such that we have 3.15 on $(0,n) \times \mathbb{R}$. Then, for $t \geq n$, we have from 3.16,

$$\begin{aligned}
u(t,x) = \mathbb{E}^x \left(\frac{n}{2}, W_{t-\frac{n}{2}}\right) &= \int_{\mathbb{R}} u\left(\frac{n}{2}, z\right) p\left(t - \frac{n}{2}; x, z\right) \mathrm{d}z \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} p\left(\frac{n}{2}; z, y\right) p\left(t - \frac{n}{2}; x, z\right) \mathrm{d}z \mathrm{d}F(y) \\
&= \int_{\mathbb{R}} p(t;x,y) \mathrm{d}F(y).
\end{aligned}$$

Thus 1 is proved. □

Proposition 3.2.1. *Let $v(t,x)$ be a nonnegative function defined on the half-plane $(0,\infty) \times \mathbb{R}$. With $T = \infty$, conditions 2, 3, and 4 of Theorem 3.2.3 are equivalent to one another and to :*

1. *for some nondecreasing function $F :\ \mathbb{R} \to \mathbb{R}$,*

$$v(t,x) = \int_{-\infty}^{+\infty} \exp(yx - \frac{1}{2}y^2 t) \mathrm{d}F(y), \quad 0 < t < \infty,\ x \in \mathbb{R}. \tag{3.17}$$

Proof. The equivalence of 2, 3 and 4 follow from Corollary 3.2.4. If v satisfies 3.17, then by differentiating under the integral, justified as in Theorem 3.2.3, we get the backward heat equation, and thus 2. Now, if v satisfies 2 then u defined by

$$v(t,x) = \sqrt{\frac{2\pi}{t}} \exp\left(\frac{x^2}{2t}\right) u\left(\frac{1}{t}, \frac{x}{t}\right), \quad 0 < t < \infty,\ x \in \mathbb{R},$$

satisfies condition 2 of Corollary 3.2.4 and then condition 1 of the same Corollary. Thus follows 3.17. □

3.2.3 Boundary crossing probabilities for Brownian motion

We have established stochastic representations for the temperatures on different rods and we can now use it to compute boundary-crossing probabilities for Brownian motion.
The representation 3.17 has some consequences. Let us consider a positive function $v(t,x)$ which is defined and of class $C^{1,2}$ on $(0,\infty)\times\mathbb{R}$, and satisfies the backward heat equation 3.10. Then v admits the representation 3.17 for some F, thanks to 3.2.1.
We differentiate under the integral and get

$$\frac{\partial}{\partial t}v(t,x) = -\frac{1}{2}\int_{\mathbb{R}} y^2 \exp\left(yx - \frac{1}{2}y^2 t\right) \mathrm{d}F(y) < 0, \quad 0<t<\infty,\ x\in\mathbb{R}.$$

And because of the backward heat equation $v(t,\cdot)$ is convex for each $t>0$. In particular, $\lim_{t\downarrow 0} v(t,0)$ exists. Now we assume that this limit is finite, and, without loss of generality (by scaling), that

$$\lim_{t\downarrow 0} v(t,0) = 1. \tag{3.18}$$

We also assume that

$$\lim_{t\to\infty} v(t,0) = 0, \tag{3.19}$$

$$\lim_{x\to+\infty} v(t,x) = +\infty, \quad 0<t<\infty, \tag{3.20}$$

$$\text{and} \lim_{x\to-\infty} v(t,x) = 0, \quad 0<t<\infty. \tag{3.21}$$

3.18-3.21 are satisfied if and only if F is a probability distribution function with $F(0+)=0$. By imposing this condition, 3.17 becomes

$$v(t,x) = \int_{0+}^{+\infty} \exp\left(yx - \frac{1}{2}y^2 t\right) \mathrm{d}F(y), 0<t<\infty,\ x\in\mathbb{R}, \tag{3.22}$$

where $F(+\infty)=1$ and $F(0+)=0$. This shows that $v(t,\cdot)$ is strictly increasing. Thus by the implicit function theorem, for each $t>0$ and $b>0$, there is an unique number $A(t,b)$ such that $v(t,A(t,b))=b$ and $A(\cdot,b)$ is continuous. We can verify that $A(\cdot,b)$ is strictly increasing. We define $A(0,b)=\lim_{t\downarrow 0} A(t,b)$.
We want to know how one can compute the probability that a Brownian path W will eventually cross the curve $A(\cdot,b)$. Computing the probability that a Brownian motion crosses a given, time-dependant continuous boundary $\{\psi(t);\ 0\le t<\infty\}$ is thereby reduced to find a solution v to the backward heat equation which also satisfied 3.18- 3.21 and $v(t,\psi(t))=b,\ 0\le t<\infty$, for some $b>0$.

Let $\{W_t,\ \mathcal{F}_t;\ 0\le t<\infty\}$,$(\Omega,\mathcal{F})$,$\{\mathbb{P}^x\}_{x\in\mathbb{R}}$ be a Brownian family, and define

$$Z_t \triangleq v(t,W_t), \quad 0<t<\infty.$$

For $0<s<t$, we have from the Markov property and condition 4 of Proposition 3.2.1:

$$\mathbb{E}^0[Z_t|\mathcal{F}_s] = f(W_s) = v(s,W_s) = Z_s \quad \text{a.s.},$$

where $f(y)\triangleq \mathbb{E}^y v(t,W_{t-s})$. That is to say that $\{Z_t,\ \mathcal{F}_t;\ 0<t<\infty\}$ is a continuous, nonnegative martingale. Let $\{t_n\}$ be a sequence of positive numbers with $t_n\downarrow 0$, and we can define $Z_0=\lim_{n\to\infty} Z_{t_n}$. We have that, thanks to the Blumenthal Law, Z_0 is a.s. constant.

Lemma 3.2.1. $Z\triangleq\{Z_t,\ \mathcal{F}_t;\ 0\le t<\infty\}$ *is a continuous, nonnegative martingale under* $\mathbb{P}^0$ *and satisfies* $Z_0=1,\ Z_\infty=0,\ \mathbb{P}^0$*-a.s.*

Proof. See [1, Section 4.3 p.263]. □

We can now obtain the probability that a Brownian path ever crosses the boundary $\{A(t,b);\ 0 \le t < \infty\}$, for $b \in \mathbb{R}$. Define the stopping time $T \triangleq \inf\{t \ge 0;\ Z_t = b\}$. Then, $\{Z_{T\wedge t},\ \mathcal{F}_t,\ 0 \le t < \infty\}$ is a martingale. It follows that for every $A \in \mathcal{F}_0$ and $t \ge 0$,

$$\begin{aligned}\mathbb{E}^0\left[Z_0 1_{A\cap\{Z_0<b\}}\right] &= \mathbb{E}^0\left[1_{A\cap\{Z_0<b\}}\mathbb{E}^0\left[Z_{T\wedge t}|\mathcal{F}_0\right]\right]\\ &= \mathbb{E}^0\left[Z_{T\wedge t}1_{A\cap\{Z_0<b\}}\right]\\ &= \mathbb{E}^0\left[b1_{A\cap\{Z_0<b\}\cap\{T\le t\}}\right] + \mathbb{E}^0\left[Z_t 1_{A\cap\{Z_0<b\}\cap\{T>t\}}\right].\end{aligned}$$

But $Z_t 1_{T<t} \le b$ and $\lim\limits_{t\to\infty} Z_t 1_{T<t} = 0$. So, by the dominated convergence theorem, we obtain

$$\begin{aligned}\mathbb{E}^0\left[Z_0 1_{A\cap\{Z_0<b\}}\right] &= b\mathbb{E}^0\left[1_{A\cap\{Z_0<b\}\cap\{T<\infty\}}\right]\\ &= b\mathbb{E}^0\left[\mathbb{E}^0\left[1_{A\cap\{Z_0<b\}\cap\{T\le t\}}|\mathcal{F}_0\right]\right]\\ &= b\mathbb{E}^0\left[1_{A\cap\{Z_0<b\}}\mathbb{P}^0(T<\infty|\mathcal{F}_0)\right].\end{aligned}$$

It follows that $\dfrac{Z_0}{b} = \mathbb{P}^0(T<\infty|\mathcal{F}_0)$ a.s. on $\{Z_0 < b\}$. And then, using that $Z_0 = 1$ a.s.,

$$\mathbb{P}^0(T<\infty) = \frac{1}{b}\mathbb{P}^0(1<b) + \mathbb{P}^0(1 \ge b).$$

So the probability that a Brownian path ever crosses the boundary $\{A(t,b);\ 0 \le t < \infty\}$, for $b \in \mathbb{R}$, is

$$\mathbb{P}^0(T<\infty) = \begin{cases} 1 & \text{if } b \le 1\\ \dfrac{1}{b} & \text{if } 1 < b.\end{cases}$$

3.2.4 Mixed initial-boundary value problems

Let's discuss about the temperature in a semi-infinite rod and the relation of it to Brownian motion absorbed at the origin. We suppose that $f : (0,\infty) \to \mathbb{R}$ is a Borel-measurable function and that there exists $a > 0$ such that

$$\int_0^{+\infty} e^{-ax^2}|f(x)|\mathrm{d}x < \infty. \tag{3.23}$$

We define

$$u_1(t,x) \triangleq \mathbb{E}^x[f(W_t)1_{\{T_0>t\}}], \quad 0 < t < \frac{1}{2a},\ x > 0. \tag{3.24}$$

Moreover, we have

$$\begin{aligned}\mathbb{P}^x(W_t > y, T_0 > t) &= \mathbb{P}^x(W_t > y) - \mathbb{P}^x(W_t > y, T_0 \le t)\\ &= \mathbb{P}^x(W_t > y) - \mathbb{P}^x(W_t < -y),\end{aligned}$$

where the last equality comes from the reflection principle. Indeed, the Brownian path has to go to zero before t, so as we have done previously, we can draw a "shadow path" which is the reflect about the level zero. See on Figure 3.3.

Then by derivating, we obtain

$$\mathbb{P}^x(W_t \in \mathrm{d}y, T_0 > t) = (p(t;x,y) - p(t;x,-y))\mathrm{d}y,$$

for $t > 0$, $x > 0$, $y > 0$ and thus

$$u_1(t,x) = \int_0^{+\infty} f(y)p(t;x,y)\mathrm{d}y - \int_{-\infty}^0 f(-y)p(t;x,y)\mathrm{d}y. \tag{3.25}$$

As seen before, u_1 has derivative of all orders, and if $f(y) = -f(-y)$ for all $y > 0$, satisfies the heat equation 3.4,

$$f(x) = \lim_{\substack{t\downarrow 0\\ y\to x}} u_1(t,y)$$

at all continuity points of f and

$$\lim_{y\downarrow 0} u_1(t,y) = 0, \quad 0 < t < \frac{1}{2a}.$$

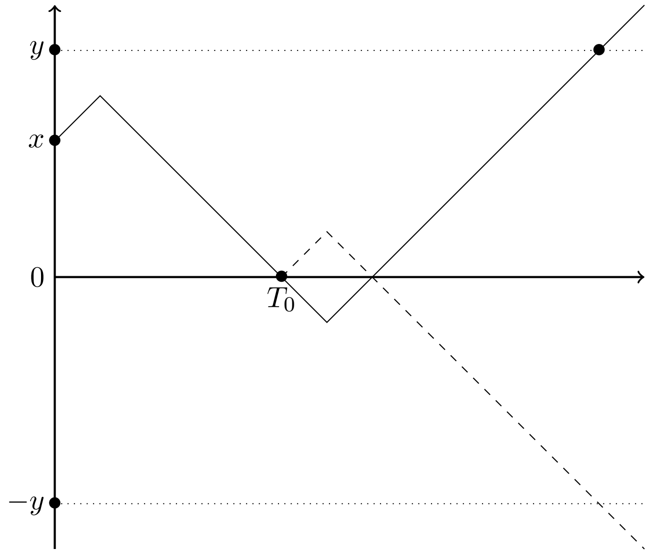

Figure 3.3: The reflection principle.

We can see $u_1(t,x)$ as the temperature in a semi-infinite rod along the nonnegative x-axis, when the temperature at $x=0$ equals to zero and the initial temperature at $y>0$ is $f(y)$. Now we suppose that the initial temperature in a semi-infinite rod is identically zero, but the temperature at the endpoint $x=0$ at time t is $g(t)$, where $g:(0,1/2a)\to\mathbb{R}$ is bounded and continuous. We write

$$\begin{aligned} u_2(t,x) &\triangleq \mathbb{E}^x[g(t-T_0)1_{\{T_0\le t\}}] \\ &= \int_0^t g(t-s)h(s,x)\mathrm{d}s \\ &= \int_0^t g(s)h(t-s,x)\mathrm{d}s, \quad 0<t<\frac{1}{2a},\ x>0, \end{aligned} \tag{3.26}$$

with h given by 3.8. g is a solution to the heat equation 3.4, because h is (Note 3.2.1) and $h(0,x)=0$, for $x>0$. We can rewrite it to

$$u_2(t,x) = \mathbb{E}^0[g(t-T_x)1_{\{T_x\le t\}}], \quad 0<t<\frac{1}{2a},\ x>0.$$

By bounded convergence theorem, we get that

$$\lim_{\substack{s\to t\\ x\downarrow 0}} u_2(s,x) = g(t), \quad 0<t<\frac{1}{2a},$$

$$\lim_{\substack{t\downarrow 0\\ y\to x}} u_2(t,y) = 0, \quad 0<x<\infty.$$

So now we can add u_1 and u_2 to get a solution to the problem with initial datum f and time-dependant boundary condition $g(t)$ at $x=0$.

3.3 The parabolic equation and the formulas of Feynman and Kac

We will now see a representation for the solution of the parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + ku = \dfrac{1}{2}\Delta u + g, & (t,x)\in(0,\infty)\times\mathbb{R}^d \\ u(0,x) = f(x), & x\in\mathbb{R}^d, \end{cases} \tag{3.27}$$

for $k:\mathbb{R}^d\to[0,\infty)$, $g:(0,\infty)\times\mathbb{R}^d\to\mathbb{R}$ and $f:\mathbb{R}^d\to\mathbb{R}$.
In the case $g=0$, we define the Laplace transform

$$z_\alpha(x) \triangleq \int_0^{+\infty} e^{-\alpha t}u(t,x)\mathrm{d}t, \quad x\in\mathbb{R}^d. \tag{3.28}$$

We have

$$\begin{aligned}\frac{1}{2}\Delta z_\alpha &= \frac{1}{2}\int_0^{+\infty} e^{-\alpha t}\Delta u \mathrm{d}t\\ &= \int_0^{+\infty} e^{-\alpha t}\left(\frac{\partial u}{\partial t}+ku\right)\mathrm{d}t \quad \text{by using 3.27,}\\ &= (k+\alpha)z_\alpha - f \quad \text{ by using } \lim_{t\to+\infty} e^{-\alpha t}u(t,x)=0.\end{aligned}$$

Let $\{W_t,\ \mathcal{F}_t;\ 0\le t<\infty\}, (\Omega,\mathcal{F}), \{\mathbb{P}^x\}_{x\in\mathbb{R}^d}$ be a d-dimensional Brownian family.

3.3.1 The multidimensional formula

Definition 3.3.1. Let $f:\mathbb{R}^d\to\mathbb{R}$, $k:\mathbb{R}^d\to[0,\infty)$, and $g:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ be continuous functions. We suppose that v is a continuous, real-valued function on $[0,T]\times\mathbb{R}^d$, of class $C^{1,2}$ on $[0,T)\times\mathbb{R}^d$ and satisfies

$$-\frac{\partial v}{\partial t}+kv=\frac{1}{2}\Delta v+g, \text{ on } [0,T)\times\mathbb{R}^d, \tag{3.29}$$

$$v(T,x)=f(x), \quad x\in\mathbb{R}^d. \tag{3.30}$$

We call such function a *solution of the Cauchy problem for the backward heat equation 3.29 with potential k and Lagrangian g subject to the terminal condition 3.30.*

Theorem 3.3.1. *Let v be as in the previous definition and assume that there exist some constants $K>0$ and $0<a<1/(2Td)$ such that*

$$\max_{0\le t\le T}|v(t,x)|+\max_{0\le t\le T}|g(t,x)|\le Ke^{a\|x\|^2}, \ \forall x\in\mathbb{R}^d. \tag{3.31}$$

Then v admits the stochastic representation

$$v(t,x)=\mathbb{E}^x\left[f(W_{T-t})\exp\left(-\int_0^{T-t}k(W_s)\mathrm{d}s\right)+\int_0^{T-t}g(t+\theta,W_\theta)\exp\left(-\int_0^\theta k(W_s)\mathrm{d}s\right)\mathrm{d}\theta\right],\ 0\le t\le T,\ x\in\mathbb{R}^d. \tag{3.32}$$

And thus, v is unique.

Note. If $g\ge 0$ on $[0,T]\times\mathbb{R}^d$, then we can replace the condition 3.31 by

$$\max_{0\le t\le T}|v(t,x)|\le Ke^{a\|x\|^2}, \quad \forall x\in\mathbb{R}^d, \tag{3.33}$$

since the second integral is well-defined.

Note. If v satisfies 3.33 and the differential inequality

$$-\frac{\partial v}{\partial t}+kv\ge\frac{1}{2}\Delta v \text{ on } [0,T)\times\mathbb{R}^d,$$

with a continuous potential $k:\mathbb{R}^d\to[0,\infty)$, then by applying Theorem 3.3.1, $v\ge 0$ on $[0,T]\times\mathbb{R}^d$.

Proof. Define $S_n=\inf\{t\ge 0;\ \| W_t\|\ge n\sqrt{d}\}$, for $n\ge 1$. Let $0<r<T-t$. By Itô's rule, we have

$$\begin{aligned}\mathrm{d}\left[v(t+\theta,W_\theta)\exp\left(-\int_0^\theta k(W_s)\mathrm{d}s\right)\right] &= \partial_\tau\left(v(t+\theta,W_\theta)\exp\left(-\int_0^\theta k(W_s)\mathrm{d}s\right)\right)\\ &+\partial_x\left(v(t+\theta,W_\theta)\exp\left(-\int_0^\theta k(W_s)\mathrm{d}s\right)\right)\\ &+\frac{1}{2}\partial_x^2\left(v(t+\theta,W_\theta)\exp\left(-\int_0^\theta k(W_s)\mathrm{d}s\right)\right)\\ &=\exp\left(-\int_0^\theta k(W_s)\mathrm{d}s\right)[\partial_\tau v(t+\theta,W_\theta)-k(W_\theta)v(t+\theta,W_\theta)]\\ &+\exp\left(-\int_0^\theta k(W_s)\mathrm{d}s\right)\left[\partial_x v(t+\theta,W_\theta)+\frac{1}{2}\partial_x^2 v(t+\theta,W_\theta)\right]\\ &=\exp\left(-\int_0^\theta k(W_s)\mathrm{d}s\right)[\partial_x v(t+\theta,W_\theta)-g(t+\theta,W_\theta)] \text{ with 3.29.}\end{aligned}$$

Now, let us integrate on $[0, r \wedge S_n]$, we get a.s.,

$$v(t+r\wedge S_n, W_{r\wedge S_n}) \exp\left(-\int_0^{r\wedge S_n} k(W_s)\mathrm{d}s\right) = v(t, W_0) + \int_0^{r\wedge S_n} \exp\left(-\int_0^{\theta} k(W_s)\mathrm{d}s\right)[\partial_x v(t+\theta, W_\theta) - g(t+\theta, W_\theta)]$$

Then we take the expectations,

$$\begin{aligned}
v(t,x) &= \mathbb{E}^x\left[v(t+r\wedge S_n, W_{r\wedge S_n}) \exp\left(-\int_0^{r\wedge S_n} k(W_s)\mathrm{d}s\right)\right] + \mathbb{E}^x\left[\int_0^{r\wedge S_n} \exp\left(-\int_0^{\theta} k(W_s)\mathrm{d}s\right) g(t+\theta, W_\theta)\mathrm{d}\theta\right] \\
&= \mathbb{E}^x\left[v(t+r, W_r) \exp\left(-\int_0^{r} k(W_s)\mathrm{d}s\right) 1_{r\le S_n}\right] + \mathbb{E}^x\left[v(t+S_n, W_{S_n}) \exp\left(-\int_0^{S_n} k(W_s)\mathrm{d}s\right) 1_{r<S_n}\right] \\
&\quad + \mathbb{E}^x\left[\int_0^{r\wedge S_n} \exp\left(-\int_0^{\theta} k(W_s)\mathrm{d}s\right) g(t+\theta, W_\theta)\mathrm{d}\theta\right].
\end{aligned}$$

Therefore, we have

$$\left|\int_0^{r\wedge S_n} \exp\left(-\int_0^{\theta} k(W_s)\mathrm{d}s\right) g(t+\theta, W_\theta)\mathrm{d}\theta\right| \le \int_0^{T-t} |g(t+\theta, W_\theta)|\,\mathrm{d}\theta,$$

which has a finite expectation because of 3.31. By dominated convergence the last term of the right-hand side converges to

$$\mathbb{E}^x\left[\int_0^{T-t} \exp\left(-\int_0^{\theta} k(W_s)\mathrm{d}s\right) g(t+\theta, W_\theta)\mathrm{d}\theta\right],$$

as $n \to \infty$ and $r \uparrow T-t$. Then, the second term is dominated by

$$\begin{aligned}
\mathbb{E}^x\left[|v(t+S_n, W_{S_n})|\, 1_{S_n \le T-t}\right] &\le K e^{adn^2} \mathbb{P}^x(S_n \le T) \\
&\le K e^{adn^2} \sum_{j=1}^{d} \mathbb{P}^x\left(\max_{0\le t\le T} \left|W_t^{(j)}\right| \ge n\right) \\
&\le 2K e^{adn^2} \sum_{j=1}^{d} \mathbb{P}^x\left(W_T^{(j)} \ge n\right) + \mathbb{P}^x\left(-W_T^{(j)} \ge n\right).
\end{aligned}$$

But, with Lemma A.0.1, and since $0 < a < 1/(2Td)$,

$$e^{adn^2}\mathbb{P}^x\left(\pm W_T^{(j)} \ge n\right) \le e^{adn^2}\sqrt{\frac{T}{2\pi}}\frac{1}{n \pm x^{(j)}} e^{-(n\pm x^{(j)})^2/2T} \underset{n\to+\infty}{\longrightarrow} 0.$$

By the dominated convergence theorem, the first term converges likewise to

$$\mathbb{E}^x\left[v(T, W_{T-t}) \exp\left(-\int_0^{T-t} k(W_s)\mathrm{d}s\right)\right],$$

as $n \to \infty$ and $r \uparrow T-t$. Thus, we get the Feynman-Kac formula 3.32. □

Corollary 3.3.2. *Assume that $f : \mathbb{R}^d \to \mathbb{R}$, $k : \mathbb{R}^d \to [0,\infty)$, and $g : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ are continuous, and that the continuous function $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ is of class $C^{1,2}$ on $(0,\infty) \times \mathbb{R}^d$ and satisfies 3.27. If for each finite $T > 0$ there exist constants $K > 0$ and $0 < a < 1/(2Td)$ such that*

$$\max_{0\le t\le T} |u(t,x)| + \max_{0\le t\le T} |g(t,x)| \le K e^{a\|x\|^2}, \quad \forall x \in \mathbb{R},$$

then u admits the stochastic representation

$$u(t,x) = \mathbb{E}^x[f(W_t)\exp\left(-\int_0^t k(W_s)\mathrm{d}s\right) + \int_0^t g(t-\theta, W_\theta)\exp\left(-\int_0^\theta k(W_s)\mathrm{d}s\right)\mathrm{d}\theta], \quad 0 \le t < \infty,\ x \in \mathbb{R}^d. \tag{3.34}$$

In the case $g = 0$, we can think of $u(t,x)$ as the temperature at time $t \ge 0$ at the point $x \in \mathbb{R}^d$ of a medium which is not a perfect heat conductor but instead dissipates heat locally at the rate k (heat flow with cooling).

Example 3.3.1. Let us consider the equation

$$-\frac{\partial u}{\partial t}(x,t) + x^2 u(x,t) = \frac{1}{2}\Delta u, \quad x \in \mathbb{R},\ t \geq 0,$$

with initial condition

$$u(0,x) = 42,\ \forall x \in \mathbb{R}.$$

Thanks to the formula of Feynman and Kac, u admits the stochastic representation for $x \in \mathbb{R}$ and $t \geq 0$,

$$u(t,x) = 42\mathbb{E}^x\left[\exp\left(-\int_0^t W_s^2 \mathrm{d}s\right)\right] = 42\mathbb{E}^x\left[\exp\left(-\lim_{n\to\infty}\frac{t}{n}\sum_{k=1}^{n} W_{k\frac{t}{n}}^2\right)\right].$$

The Figure 3.4 shows the solution.

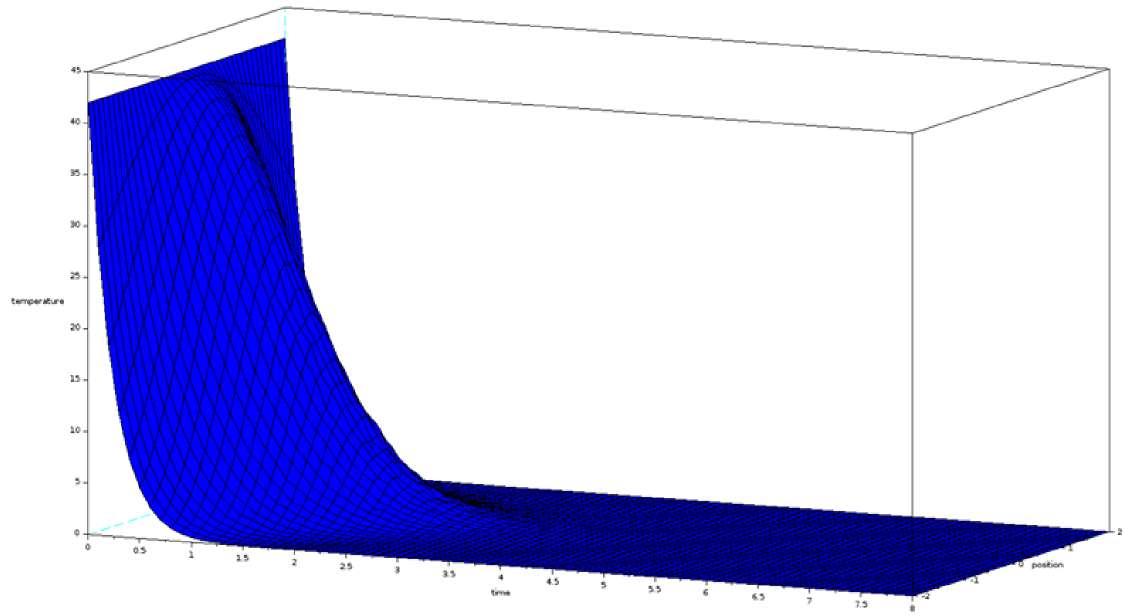

Figure 3.4: Solution to the problem.

The Feynman-Kac formula 3.34 suggests that this situation is equivalent to Brownian motion with killing of particles at the same rate k. The probability that the particles survives up to time t, conditional on the path $\{W_s,\ 0 \leq s \leq t\}$ is $\exp\{-\int_0^t k(W_s)\mathrm{d}s\}$.

3.3.2 The one-dimensional formula

Definition 3.3.2. A Borel-measurable function $f : \mathbb{R} \to \mathbb{R}$ is called *piecewise-continuous* if, it admits left- and right-hand limits everywhere on $\mathbb{R}$ and it has only finitely many points of discontinuity in every bounded interval. We note D_f the set of discontinuity points of f. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called *piecewise* C^j, $j \geq 1$ if, its derivatives $f^{(i)}, 1 \leq i \leq j-1$, are continuous, and the derivative $f^{(j)}$ is piecewise-continuous.

Theorem 3.3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R} \to [0,\infty)$ be piecewise-continuous functions with*

$$\int_{-\infty}^{+\infty} |f(x+y)| e^{-|y|\sqrt{2\alpha}} \mathrm{d}y < \infty, \quad \forall x \in \mathbb{R}, \tag{3.35}$$

for some fixed constant $\alpha > 0$. Then the function z defined by

$$z(x) = \mathbb{E}^x \int_0^{+\infty} f(W_t) \exp\left(-\alpha t - \int_0^t k(W_s)\mathrm{d}s\right) \mathrm{d}t \tag{3.36}$$

is piecewise C^2 and satisfies

$$(\alpha+k)z=\frac{1}{2}z''+f,\ \text{on } \mathbb{R}\backslash(D_f\cup D_k). \tag{3.37}$$

Note. With the Laplace transform computation,

$$\int_0^{+\infty} e^{-\alpha t}\frac{1}{\sqrt{2\pi t}}e^{-\xi^2/2t}\mathrm{d}t=\frac{1}{\sqrt{2\alpha}}e^{-|\xi|\sqrt{2\alpha}},\quad \alpha>0,\ \xi\in\mathbb{R},$$

we can replace 3.35 by the equivalent condition

$$\mathbb{E}^x\int_0^{+\infty} e^{-\alpha t}|f(W_t)|\mathrm{d}t<\infty,\quad \forall x\in\mathbb{R}. \tag{3.38}$$

Proof. We define the *resolvent operator* G_α for a piecewise-continuous function g, by

$$G_\alpha g(x)\triangleq\mathbb{E}^x\int_0^{+\infty} e^{-\alpha t}g\left(W_t\right)\mathrm{d}t=\frac{1}{\sqrt{2\alpha}}\int_{\mathbb{R}} e^{-|x-y|\sqrt{2\alpha}}g(y)\mathrm{d}y,\quad x\in\mathbb{R}.$$

We differentiate it and obtain

$$\left(G_\alpha g\right)'(x)=\int_x^{+\infty} e^{(x-y)\sqrt{2\alpha}}g(y)\mathrm{d}y-\int_{-\infty}^x e^{(y-x)\sqrt{2\alpha}}g(y)\mathrm{d}y,\quad x\in\mathbb{R}$$

and

$$\left(G_\alpha g\right)''(x)=-2g(x)+2\alpha\left(G_\alpha g\right)(x),\quad x\in\mathbb{R}\backslash D_g.$$

Let us apply this equality for $g=f$ and $g=kz$, we obtain

$$\left(G_\alpha f\right)''(x)=-2f(x)+2\alpha\left(G_\alpha f\right)(x),\quad x\in\mathbb{R}\backslash D_f$$

and

$$\left(G_\alpha kz\right)''(x)=-2kz(x)+2\alpha\left(G_\alpha kz\right)(x),\quad x\in\mathbb{R}\backslash D_{kz}.$$

Moreover, we will show later that

$$\left(G_\alpha kz\right)=\left(G_\alpha f\right)-z \tag{3.39}$$

and that

$$\left(G_\alpha|kz|\right)(x)<\infty,\quad x\in\mathbb{R}. \tag{3.40}$$

Thus,

$$\left(G_\alpha f\right)''-z''=\left(G_\alpha kz\right)''=-2kz+2\alpha\left(G_\alpha f\right)-2\alpha z$$

so,

$$-2f+2\alpha\left(G_\alpha f\right)-z''=-2kz+2\alpha\left(G_\alpha f\right)-2\alpha z.$$

Thanks to 3.39, it follows, on $\mathbb{R}\backslash(D_f\cup D_{kz})$,

$$\frac{1}{2}z''+f=z(k+\alpha).$$

With the dominated convergence theorem, we can show that z is continuous, so $D_{kz}\subseteq D_k$ and we have 3.37.
By integrated 3.37, we check that z' is continuous.
Now we have to show 3.39. We observe that

$$\begin{aligned}0\leq\int_0^t k\left(W_s\right)\exp\left(-\int_s^t k(W_u)\mathrm{d}u\right)\mathrm{d}s&=\int_0^t k\left(W_s\right)\exp\left(-\int_0^t k(W_u)\mathrm{d}u+\int_0^s k(W_u)\mathrm{d}u\right)\mathrm{d}s\\&=\exp\left(-\int_0^t k(W_u)\mathrm{d}u\right)\left[\exp\left(\int_0^s k(W_u)\mathrm{d}u\right)\right]_0^t\\&=1-\exp\left(-\int_0^t k(W_u)\mathrm{d}u\right)\leq 1.\end{aligned}$$

Then we have, for $x \in \mathbb{R}$,

$$
\begin{aligned}
\left(G_\alpha f - z\right)(x) &= \mathbb{E}^x\left[\int_0^{+\infty} e^{-\alpha t} f\left(W_t\right)\left(1-\exp\left(-\int_0^t k\left(W_s\right)\mathrm{d}s\right)\right)\mathrm{d}t\right] \\
&= \mathbb{E}^x\left[\int_0^{+\infty} e^{-\alpha t} f\left(W_t\right)\int_0^t k\left(W_s\right)\exp\left(-\int_s^t k\left(W_u\right)\mathrm{d}u\right)\mathrm{d}s\mathrm{d}t\right] \\
&= \mathbb{E}^x\left[\int_0^{+\infty} k\left(W_s\right)\int_s^{+\infty} e^{-\alpha t} f\left(W_t\right)\exp\left(-\int_s^t k\left(W_u\right)\mathrm{d}u\right)\mathrm{d}t\mathrm{d}s\right] \text{ by Fubini's theorem,} \\
&= \int_0^{+\infty}\mathbb{E}^x\left[k\left(W_s\right)\int_s^{+\infty} e^{-\alpha t} f\left(W_t\right)\exp\left(-\int_s^t k\left(W_u\right)\mathrm{d}u\right)\mathrm{d}t\right]\mathrm{d}s \text{ by Fubini's theorem,} \\
&= \int_0^{+\infty}\mathbb{E}^x\left[k\left(W_s\right)e^{-\alpha s}\int_0^{+\infty} e^{-\alpha t} f\left(W_{t+s}\right)\exp\left(-\int_0^t k\left(W_{s+u}\right)\mathrm{d}u\right)\mathrm{d}t\right]\mathrm{d}s \\
&= \int_0^{+\infty}\mathbb{E}^x\left[\mathbb{E}^x\left[k\left(W_s\right)e^{-\alpha s}\int_0^{+\infty} e^{-\alpha t} f\left(W_{t+s}\right)\exp\left(-\int_0^t k\left(W_{s+u}\right)\mathrm{d}u\right)\mathrm{d}t|\mathcal{F}_s\right]\right]\mathrm{d}s \\
&= \int_0^{+\infty}\mathbb{E}^x\left[k\left(W_s\right)e^{-\alpha s}\mathbb{E}^x\left[\int_0^{+\infty} e^{-\alpha t} f\left(W_{t+s}\right)\exp\left(-\int_0^t k\left(W_{s+u}\right)\mathrm{d}u\right)\mathrm{d}t|\mathcal{F}_s\right]\right]\mathrm{d}s \\
&= \mathbb{E}^x\left[\int_0^{+\infty} k\left(W_s\right)e^{-\alpha s}\mathbb{E}^x\left[\int_0^{+\infty} e^{-\alpha t} f\left(W_{t+s}\right)\exp\left(-\int_0^t k\left(W_{s+u}\right)\mathrm{d}u\right)\mathrm{d}t|\mathcal{F}_s\right]\mathrm{d}s\right]
\end{aligned}
$$

by Fubini's theorem,

$$
\begin{aligned}
&= \mathbb{E}^x\left[\int_0^{+\infty} e^{-\alpha s} k\left(W_s\right) z\left(W_s\right)\mathrm{d}s\right] \text{ by Markov property,} \\
&= \left(G_\alpha k z\right)(x).
\end{aligned}
$$

We have proved 3.39.
Now let's replace f by $|f|$ in 3.36, then we get a nonnegative function $\tilde{z} \geq |z|$ and thanks to 3.35, for $x \in \mathbb{R}$,

$$
\left(G_\alpha |kz|\right)(x) \leq \left(G_\alpha k\tilde{z}\right)(x) = \left(\left(G_\alpha |f|\right) - \tilde{z}\right)(x) < \infty.
$$

We have proved 3.40. ☐

Let us see some applications of Theorem 3.3.3.
As for the discrete-time random walk, we have:

Proposition 3.3.1 (Arc-Sine Law for the Occupation Time of $(0,\infty)$). *Let* $\Gamma_+(t) \triangleq \int_0^t 1_{(0,\infty)}(W_s)\mathrm{d}s$. *Then*

$$
\mathbb{P}^0(\Gamma_+(t) \leq \theta) = \int_0^{\theta/t} \frac{\mathrm{d}s}{\pi\sqrt{s(1-s)}} = \frac{2}{\pi}\arcsin\sqrt{\frac{\theta}{t}}, \quad 0 \leq \theta \leq t. \tag{3.41}
$$

Proof. For $\alpha > 0$, and $\beta > 0$, we define

$$
z(x) \triangleq \mathbb{E}^x \int_0^{+\infty} \exp\left(-\alpha t - \beta \int_0^t 1_{(0,+\infty)}\left(W_s\right)\mathrm{d}s\right)\mathrm{d}t, \ x \in \mathbb{R}.
$$

By Theorem 3.3.3, it satisfies the equation

$$
\alpha z = \begin{cases} \frac{1}{2}z'' - \beta z + 1 & x > 0, \\ \frac{1}{2}z'' + 1 & x < 0, \end{cases}
$$

and the conditions

$$
z(0+) = z(0-), \quad z'(0+) = z'(0-).
$$

So bounded solutions of this equation have the form

$$
z(x) = \begin{cases} Ae^{-x\sqrt{2(\alpha+\beta)}} + \frac{1}{\alpha+\beta} & x > 0, \\ Be^{x\sqrt{2\alpha}} + \frac{1}{\alpha} & x < 0. \end{cases}
$$

Using the continuity of z and z' at $x = 0$, we get

$$A = \frac{\sqrt{\alpha+\beta} - \sqrt{\alpha}}{(\alpha+\beta)\sqrt{\alpha}}.$$

Thus

$$\mathbb{E}^0 \left[\int_0^{+\infty} e^{-\alpha t} \exp\left(-\beta \int_0^t 1_{(0,+\infty)}(W_s)\,\mathrm{d}s \right) \mathrm{d}t \right] = z(0) = \frac{1}{\sqrt{(\alpha+\beta)\alpha}}.$$

But

$$\begin{aligned}
\int_0^{+\infty} e^{-\alpha t} \int_0^t \frac{e^{-\beta\theta}}{\pi\sqrt{\theta(t-\theta)}} \mathrm{d}\theta \mathrm{d}t &= \int_0^{+\infty} \frac{e^{-\beta\theta}}{\pi\sqrt{\theta}} \int_0^{+\infty} \frac{e^{-\alpha t}}{\sqrt{t-\theta}} \mathrm{d}t\mathrm{d}\theta \\
&= \frac{1}{\pi} \int_0^{+\infty} \frac{e^{-(\alpha+\beta)\theta}}{\sqrt{\theta}} \int_0^{+\infty} \frac{e^{-\alpha s}}{\sqrt{s}} \mathrm{d}s\mathrm{d}\theta \\
&= \frac{1}{\sqrt{\alpha(\alpha+\beta)}} \\
&= z(0),
\end{aligned}$$

since

$$\int_0^{+\infty} \frac{e^{-\gamma t}}{\sqrt{t}} = \frac{1}{\sqrt{\gamma}} \Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{\gamma}}, \quad \gamma > 0.$$

The uniqueness of Laplace transform implies

$$\mathbb{E}^0 e^{-\beta\Gamma_+(t)} = \int_0^t \frac{e^{-\beta\theta}}{\pi\sqrt{\theta(t-\theta)}} \mathrm{d}\theta.$$

And thus,

$$\mathbb{P}^0(\Gamma_+(t) \le \theta) = \int_0^{\theta/t} \frac{\mathrm{d}s}{\pi\sqrt{s(1-s)}}, \quad 0 \le \theta \le t.$$

□

Proposition 3.3.2 (Occupation Time of $(0,\infty)$ until First Hitting $b > 0$)**.** *For $\beta > 0$, $b > 0$, we have*

$$\mathbb{E}^0 \exp(-\beta\Gamma_+(T_b)) \triangleq \mathbb{E}^0 \exp\left(-\beta \int_0^{T_b} 1_{(0,\infty)}(W_s)\mathrm{d}s \right) = \frac{1}{\cosh b\sqrt{2\beta}}. \tag{3.42}$$

Proof. For b, α, β, γ positive numbers, we define

$$\Gamma_b(t) \triangleq \int_0^t 1_{(b,+\infty)}(W_s)\,\mathrm{d}s$$

and

$$z(x) \triangleq \mathbb{E}^x \int_0^{+\infty} 1_{(0,+\infty)}(W_t) \exp(-\alpha t - \beta\Gamma_+(t) - \gamma\Gamma_b(t))\,\mathrm{d}t, \quad x \in \mathbb{R}.$$

We have

$$\begin{aligned}
z(0) &= \mathbb{E}^0 \int_0^{+\infty} 1_{(0,+\infty)}(W_t) \exp(-\alpha t - \beta\Gamma_+(t) - \gamma\Gamma_b(t))\,\mathrm{d}t \\
&= \mathbb{E}^0 \int_0^{T_b} 1_{(0,+\infty)}(W_t) \exp(-\alpha t - \beta\Gamma_+(t))\,\mathrm{d}t + \mathbb{E}^0 \int_{T_b}^{+\infty} 1_{(0,+\infty)}(W_t) \exp(-\alpha t - \beta\Gamma_+(t) - \gamma\Gamma_b(t))\,\mathrm{d}t.
\end{aligned}$$

Since $\Gamma_b(t) > 0$ a.s. on $\{T_b < t\}$, we obtain

$$\lim_{\gamma\uparrow+\infty} z(0) = \mathbb{E}^0 \int_0^{T_b} 1_{(0,+\infty)}(W_t) \exp(-\alpha t - \beta\Gamma_+(t))\,\mathrm{d}t$$

and

$$\begin{aligned}
\lim_{\alpha\downarrow 0} \lim_{\gamma\uparrow+\infty} z(0) &= \mathbb{E}^0 \int_0^{T_b} 1_{(0,+\infty)}(W_t) \exp(-\beta\Gamma_+(t))\,\mathrm{d}t \\
&= \mathbb{E}^0 \int_0^{T_b} \exp(-\alpha t - \beta\Gamma_+(t))\,\mathrm{d}\Gamma_+(t) \\
&= \frac{1}{b}\left[1 - \mathbb{E}^0 \exp(-\beta\Gamma_+(T_b))\right].
\end{aligned}$$

According to Theorem 3.3.3, z is piecewise C^2 on $\mathbb{R}$ and satisfies

$$\begin{cases} \alpha z = \frac{1}{2}z'' + f & x < 0, \\ z\sigma = \frac{1}{2}z'' + 1 & 0 < x < b, \\ (\sigma+\gamma)z = \frac{1}{2}z'' + 1 & x > b, \end{cases}$$

where $\sigma = \alpha + \beta$.
The bounded solutions have the form

$$z(x) = \begin{cases} Ae^{x\sqrt{2\alpha}} & x < 0, \\ Be^{x\sqrt{2\alpha}} + Ce^{-x\sqrt{2\alpha}} + \frac{1}{\sigma} & 0 < x < b, \\ De^{-x\sqrt{2(\sigma+\gamma)}} + \frac{1}{\sigma+\gamma} & x > b. \end{cases}$$

Now, using the continuity of z and z' at $x = 0$, we get the value of A, B, C and D. In particular, using $z(0) = A$, we obtain as $\gamma \uparrow +\infty$ and then $\alpha \downarrow 0$ that

$$\mathbb{E}^0 \exp(-\beta \Gamma_+(T_b)) = \frac{1}{\cosh b\sqrt{2\beta}}.$$

□

3.4 Monte-Carlo methods and Coding

3.4.1 Theory

Let g be a measurable function such that $\mathbb{E}^x |g(W_t)|^2 < \infty$ and $x \in \mathbb{R}^d$. By the Law of Large Numbers, we have

$$u_n(x) \triangleq \frac{1}{n}\sum_{i=1}^{n} g\left(W_t^{(i),x}\right) \underset{n\to+\infty}{\longrightarrow} u(x) \triangleq \mathbb{E}^x g(W_t) \text{ a.s. } .$$

u_n is called to be a *Monte-Carlo estimator*. It is unbiaised since $\mathbb{E}u_n(x) = \mathbb{E}^x g(W_t)$. We now use the Central Limit Theorem to find a confidence interval. We have

$$\frac{u_n(x) - u(x)}{\sigma/\sqrt{n}} \underset{n\to+\infty}{\Longrightarrow} \mathcal{N}(0,1),$$

where $\sigma = \mathbb{E}^x g^2(W_t) - u^2(x)$. We approximate σ with

$$\hat{\sigma} \triangleq \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(g\left(W_t^{(i),x}\right) - u_n(x)\right)^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} g^2\left(W_t^{(i),x}\right) - u_n^2(x)}.$$

Thus, we get the confidence interval of 95% for $u(x)$

$$\left[u_n(x) - 1,96\frac{\hat{\sigma}}{\sqrt{n}};\ u_n(x) + 1,96\frac{\hat{\sigma}}{\sqrt{n}}\right].$$

3.4.2 Dirichlet Problem in the two-dimensional unit disk

We consider the Dirichlet problem on the two-dimensional unit disk B,

$$\begin{cases} \Delta u(X) = 0 & X = (x,y) \in B, \\ u(X) = \frac{1}{2}\ln\left[(x-2)^2 + y^2\right] & X = (x,y) \in \mathbb{S}^1. \end{cases} \tag{3.43}$$

u can be defined on B, and is harmonic there. So the exact solution to this problem is given by

$$u(X) = \frac{1}{2}\ln\left[(x-2)^2 + y^2\right], \quad X = (x,y) \in B.$$

Let us now use our Monte-Carlo estimator to approximate the value of our solution at points $(0,0)$, $(0.25, 0.25)$ and $(0.5, 0.5)$. See Figure 3.5.

Let M be the number of random walks we use to do the mean. We can calculate the absolute error between the approximation and the real value, the confidence interval and also the running time. See Figure 3.12.

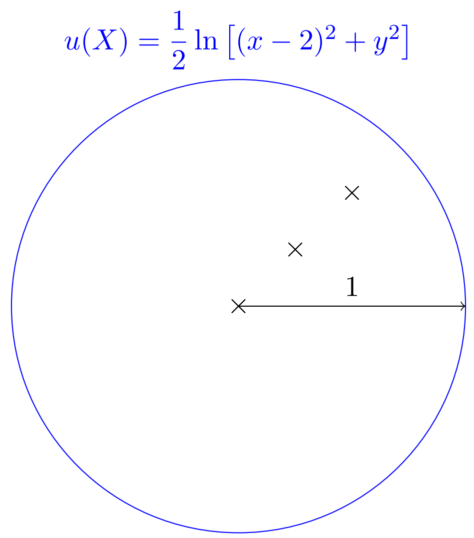

Figure 3.5: Dirichlet problem on unit disk.

3.4.3 Dirichlet Problem on the square

In this section, we consider the Dirichlet problem on the square $[0,1]\times[0,1]$,

$$\begin{cases} \Delta u(X) = 0 & X = (x,y) \in [0,1]\times[0,1] \\ u(x,0) = 0 & 0 < x < 1, \\ u(x,1) = f(x) & 0 < x < 1, \\ u(0,y) = u(1,y) = 0 & 0 < y < 1, \end{cases} \tag{3.44}$$

where

$$f(x) = \begin{cases} 75x & \text{if } 0 \leq x \leq \frac{2}{3} \\ 150(1-x) & \text{if } \frac{2}{3} < x \leq 1. \end{cases}$$

Note. The sine series for f is

$$f(x) = \frac{450}{\pi^2}\sum_{n=1}^{+\infty} \frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2}\sin(n\pi x).$$

Assume that the solution to this problem is given by

$$u(x,y) = \frac{450}{\pi^2}\sum_{n=1}^{+\infty} \frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2\sinh(n\pi)}\sin(n\pi x)\sinh(n\pi y).$$

As previously we use our Monte-Carlo estimator to approximate the value of our solution at points $(0.5, 0.5)$, $(0.25, 0.25)$, $(0.25, 0.9)$ and $(0.5, 0.9)$. See Figure 3.6. Because of the boundary conditions which are often zero, we use an ϵ border. That is to say that we define f on $[0,1]\times[1-\epsilon, 1+\epsilon]$. Here we choose $\epsilon = 0.05$. See Figure 3.13.

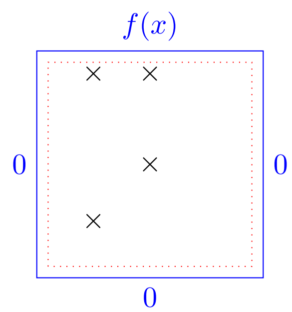

Figure 3.6: Dirichlet problem on the unit square.

As we can see, the error is not very good. It could be due to the fact that a Brownian motion goes uniformly in the four directions and that three directions on four have a zero boundary. The error is proportionnal to the value of u.

3.4.4 Dirichlet Problem using the Spherical Process

In this section we will use a property of the Brownian motion: a Brownian motion is uniformly distributed on the 2-dimensional unit circle, it goes uniformly in all directions. So, instead of doing all the path of the Brownian motion still it goes out our domain Ω, we will proceed differently. Let X be the starting point, the point where we want to calculate the value of our solution. Let ϵ be a positive number. We calculate the distance from X to the boundary of Ω. And then using the fact that $W_{\tau_{B_r}}$ is distributed as $r(\cos(U), \sin(U))$, where $r = dist(X, \partial\Omega) - \epsilon$ and U is uniformly distributed on $[0, 2\pi)$, we take $X_1 = r(\cos(U), \sin(U))$. And we repeat the same with X_1 until there exist n such that $dist(X_n, \partial\Omega) < \epsilon$. Then we keep X_n as W_{τ_Ω}. See Figure 3.7 to see such a path.

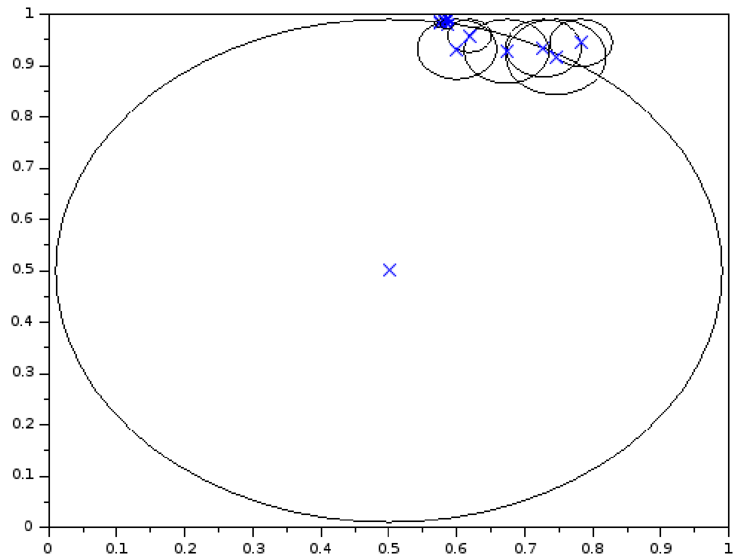

Figure 3.7: A path starting at $(0.5, 0.5)$ in the unit square, according to the spherical process.

With this method, we obtain quite the same results concerning the absolute error but the running time is much better. See 3.8.

	$X = (0.5, 0.5)$		
	$M = 100$	$M = 1000$	$M = 10000$
running time (sec)	0.02	0.22	2.08

Figure 3.8: Running time for the spherical process.

Let us now use our method to calculate the solution of

$$\begin{cases} \Delta u(x) = 0 & X = (x, y) \in D, \\ u(x, y) = f(x, y) & X = (x, y) \in \partial D, \end{cases} \tag{3.45}$$

where

$$\begin{cases} f(x, y) = -y + 2 & 1 \leq y \leq 2, \quad x \in \{0, 1\}, \\ f(0, y) = y & 0 \leq y < 1, \\ f(x, 1) = 2 - x & 1 < x \leq 2, \\ f(x, y) = 0, \text{ else}, \end{cases}$$

and D is a L-shaped domain, see Figure 3.9.

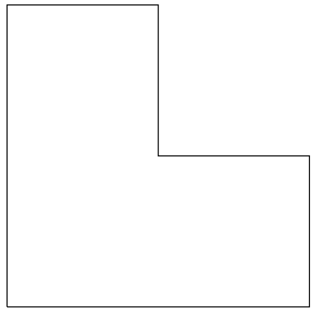

Figure 3.9: The L-shaped domain D.

Figures 3.10 and 3.11 show the solution obtained with this method. The Figure 3.10 was made with $M = 10$, the approximation in the middle of the shape is not pretty good but we can see the continuity of the function. Whereas on Figure 3.11, which was made with $M = 100$, the approximation is quite better in the middle but we can't see very well the continuity of the function on boundaries. In the second Figure, the discretisation step is not the same in the middle of the shap and on boundaries: it is chosen smaller on the boundaries in order to have a better approximation of the solution.

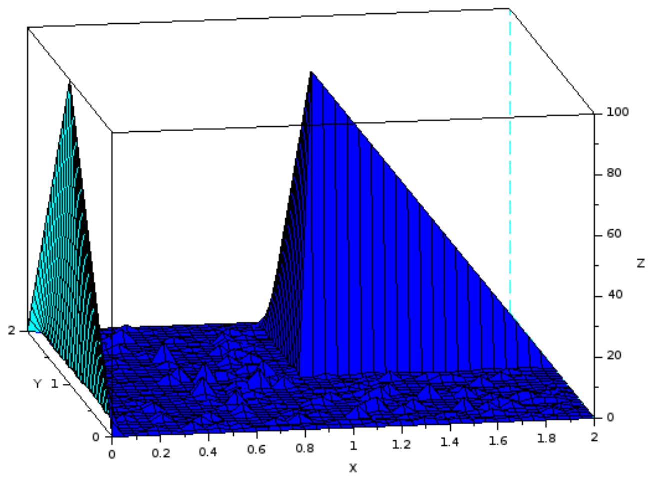

Figure 3.10: Solution with M=10.

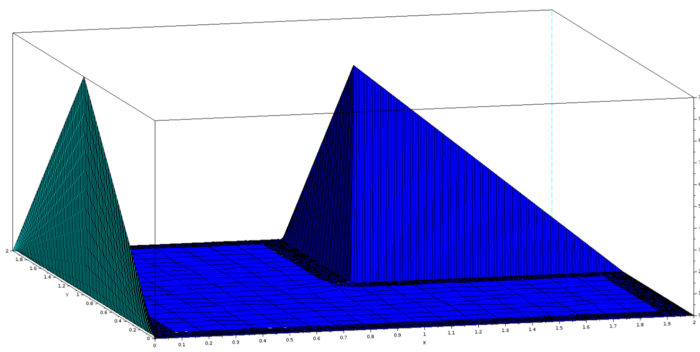

Figure 3.11: Solution with M=100.

	$X = (0, 0)$		
	$M = 100$	$M = 1000$	$M = 10000$
absolute error	0.016	0.002	0.001
confidence interval	0.073	0.023	0.007
running time (sec)	0.3	2.8	31.6
	$X = (0.25, 0.25)$		
	$M = 100$	$M = 1000$	$M = 10000$
absolute error	0.027	0.003	0.0004
confidence interval	0.073	0.023	0.007
running time (sec)	0.3	2.6	25.5
	$X = (0.5, 0.5)$		
	$M = 100$	$M = 1000$	$M = 10000$
absolute error	0.029	0.014	0.0006
confidence interval	0.006	0.019	0.006
running time (sec)	0.2	1.5	15.3

Figure 3.12: Results for the unit circle

	$X = (0.5, 0.5)$		
	$M = 100$	$M = 1000$	$M = 10000$
absolute error	7.874	7.963	7.953
confidence interval	0.049	0.04	0.012
running time (sec)	0.2	1.7	16.0
	$X = (0.25, 0.25)$		
	$M = 100$	$M = 1000$	$M = 10000$
absolute error	2.075	2.064	2.045
confidence interval	0.017	0.012	0.006
running time (sec)	0.1	1.2	12.3
	$X = (0.25, 0.9)$		
	$M = 100$	$M = 1000$	$M = 10000$
absolute error	15.429	15.483	15.455
confidence interval	0.126	0.043	0.015
running time (sec)	0.1	1.2	12.5
	$X = (0.5, 0.9)$		
	$M = 100$	$M = 1000$	$M = 10000$
absolute error	28.532	28.547	28.582
confidence interval	0.252	0.050	0.0172
running time (sec)	0.2	1.3	15.5

Figure 3.13: Results for the unit square

Appendix A

Some results

Lemma A.0.1. *For all $x > 0$,*

$$\frac{x}{1+x^2}e^{-x^2/2} \leq \int_x^{+\infty} e^{-u^2/2}\mathrm{d}u \leq \frac{1}{x}e^{-x^2/2}.$$

Proof. Let $x > 0$. Using integration by parts, we get

$$\begin{aligned}\int_x^{+\infty} e^{-u^2/2}\mathrm{d}u &= \left[\frac{e^{-u^2/2}}{u}\right]_x^{+\infty} - \int_x^{+\infty} \frac{e^{-u^2/2}}{u^2}\mathrm{d}u \\ &\leq \frac{1}{x}e^{-x^2/2}.\end{aligned}$$

Then, using integration by parts again, we obtain

$$\int_x^{+\infty} \frac{x^2}{u^2}e^{-u^2/2}\mathrm{d}u = xe^{-x^2/2} - \int_x^{+\infty} x^2 e^{-u^2/2}\mathrm{d}u.$$

And then,

$$xe^{-x^2/2} \leq \int_x^{+\infty} (x^2+1)e^{-u^2/2}\mathrm{d}u.$$

The first inequality follows. $\square$

Proposition A.0.1 (Blumenthal zero-one Law)**.** *Let $\{W_t,\ \mathcal{F}_t;\ 0 \leq t < \infty\}$, $(\Omega, \mathcal{F})$, $\{\mathbb{P}^x\}_{x\in\mathbb{R}^d}$ be a d-dimensional Brownian family. If $F \in \tilde{\mathcal{F}}_0 \triangleq \cap_{s>0}\mathcal{F}_s$, then for each $x \in \mathbb{R}^d$, we have either $\mathbb{P}^x(F) = 0$ or $\mathbb{P}^x(F) = 1$.*

Proof. See [1, Section 2.7. p.94]. $\square$

Appendix B

Stochastic calculus

Here are some properties of stochastic calculus we need.

Theorem B.0.1 (Itô's rule)**.** *If W is a Brownian motion and f is a function of class C^2 on $\mathbb{R}^d$, then*

$$f(W_t) = f(W_0) + \int_0^t f'(W_s)\mathrm{d}W_s + \frac{1}{2}\int_0^t f''(W_s)\mathrm{d}s \ \textit{a.s.} \ .$$

If f depends on time and is continuous differentiable in time, then

$$f(t, W_t) = f(0, W_0) + \int_0^t \partial_t f(s, W_s)\mathrm{d}s + \int_0^t \partial_x f(s, W_s)\mathrm{d}W_s + \frac{1}{2}\int_0^t \partial_x^2 f(s, W_s)\mathrm{d}s \ \textit{a.s.} \ .$$

Moreover if X is a stochastic process such that for almost all $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is differentiable, then

$$f(W_t)X_t = f(W_0)X_0 + \int_0^t X_s f'(W_s)\mathrm{d}W_s + \int_0^t f(W_s)X_s'\mathrm{d}s + \frac{1}{2}\int_0^t X_s f''(W_s)\mathrm{d}s \ \textit{a.s.} \ .$$

Proof. See [1, Section 3.3. p.149]. □

Proposition B.0.2. *Let X be a process, W be a Brownian motion on $\mathbb{R}^d$ and I an interval of $\mathbb{R}$, then*

$$\mathbb{E}\int_I X_t \mathrm{d}W_t = 0.$$

Proof. See [4]. □

Appendix C

Scilab Code

Here is the main code used to do the simulation.

```
function [Xt,Yt]=BMcircle(X0,Y0,N,r,a0,b0)
        h=1/N
        x=X0; y=Y0;
        while (x-a0)**2+(y-b0)**2 <r**2
                bruitX = grand(1,1,"nor",0, sqrt(h));
                bruitY = grand(1,1,"nor",0, sqrt(h));
                x=x+bruitX;
                y=y+bruitY;
        end
        Xt=x; Yt=y;
        endfunction
```

Listing C.1: To calculate W_{τ_D}, starting at $(X0, Y0)$, where D is a circle of radius r and center $(a0, b0)$.

```
function [Xt,Yt]=BMrect(X0,Y0,N,r, R, a0, b0)
        h=1/N
        x=X0; y=Y0;
        while (abs(x-a0)<r/2) and (abs(y-b0)<R/2)
                bruitX = grand(1,1,"nor",0, sqrt(h));
                bruitY = grand(1,1,"nor",0, sqrt(h));
                x=x+bruitX;
                y=y+bruitY;
        end
        Xt=x; Yt=y;
endfunction
```

Listing C.2: To calculate W_{τ_D}, starting at $(X0, Y0)$, where D is a rectangle of largor r, high R and center $(a0, b0)$.

```
function [Xt,Yt]= BMuniform(X0,Y0,D,eps)
// D is an array nx2 containing the coordinates of the domain.
        pt=[X0,Y0]
        r=orthProj(D,pt)
        while r>eps
                U= 2 * %pi * grand(1,1,'def')
                pt=[r * cos(U), r * sin(U)]
                r=orthProj(D,pt)
        end
        Xt=pt(1); Yt=pt(2);
endfunction
```

Listing C.3: To calculate W_{τ_D}, starting at $(X0, Y0)$, where D is a domain, using the spherical process.

Bibliography

[1] KARATZAS I, SHREVE S.E.,1996. *Brownian motion and Stochastic Calculus.* Springer.

[2] GOBET E, 2016. *Méthodes de Monte-Carlo et processus stochastiques.* CRC Press.

[3] GRAHAM C, TALAY D, 2011. *Simulation stochastique et méthodes de Monte-Carlo*. Ellipses.

[4] BOSSY M, 2013. *Introduction à la modélisation financière en temps continu et calcul stochastique.* [Online]. Available on `http://math.unice.fr/~binard/polycopieMathFi%281%29.pdf`

[5] DAILEDA R.C, 2014. *The Dirichlet Problem on a Rectangle.* [Online]. Available on `http://ramanujan.math.trinity.edu/rdaileda/teach/s14/m3357/lectures/lecture_3_18_slides.pdf`

[6] MULLER M.E, 1956. *Some continuous Monte-Carlo methods for the Dirichlet problem.*[Online]. Available on `http://projecteuclid.org/euclid.aoms/1177728169`

Figures done with *Scilab* and *Tikz*.