Dispersive equations and randomized initial data

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August 24, 2017

Contents

1 Introduction

The aim of this report is the study of dispersive equations, with low regularity hypothesis on the initial data. This work is based on the article of Nicolas Burq and Nikolay Tzvetkov "Probabilistic well-posedness for the cubic wave equation" published in December 2011. I have done my internship with Mr. Nicola Visciglia, professor of mathematics at the University of Pisa.

For the redaction of this report, we assume that the reader is aware of the definitions of the Sobolev spaces and the basic theory of PDE.

2 Preliminaries

This section will gather some results that we will need in some proofs. These results are very classics and their proofs can be found almost everywhere in the mathematics literature, so I chose not to prove them and just enounce them. One can read the Brezis' book for more results [2]

2.1 Sobolev embeddings

2.1.1 Continuous embeddings

Theorem 2.1.1. *(Morrey) Let $p > N$, then*

$$\mathcal{W}^{1,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$$

with continuous embedding.

Corollary 2.1.1. *Let $m \geqslant 1$ an integer and $1 \leqslant p < \infty$, then*

- *if* $\frac{1}{p} - \frac{m}{N} > 0$ *then* $\mathcal{W}^{m,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ *where* $\frac{1}{q} = \frac{1}{p} - \frac{m}{N}$,
- *if* $\frac{1}{p} - \frac{m}{N} = 0$ *then* $\mathcal{W}^{m,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N) \quad \forall q \in [p, +\infty[$,
- *if* $\frac{1}{p} - \frac{m}{N} < 0$ *then* $\mathcal{W}^{m,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$,

with continuous embeddings.

These results can also be applied to a compact manifold, like for example the torus $\mathbb{T}^3$

2.1.2 Compact embeddings

Theorem 2.1.2. *(Rellich Kondrachov)*

- *if* $p < 3$ *then* $\mathcal{W}^{1,p}(\mathbb{T}^3) \subset L^q(\mathbb{T}^3), \quad \forall q \in [p, p^*[$ *where* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$
- *if* $p = 3$ *then* $\mathcal{W}^{1,p}(\mathbb{T}^3) \subset L^q(\mathbb{T}^3), \quad \forall q \in [p, +\infty[$
- *if* $p > 3$ *then* $\mathcal{W}^{1,p}(\mathbb{T}^3) \subset \mathcal{C}(\overline{\mathbb{T}^3})$

with compact embeddings.

2.2 Notations

Notation 2.2.1.

- $\langle x \rangle := (1+|x|^2)^{\frac{1}{2}}$,
- *We will denote by* $S(t)(v_0, v_1)$ *the free evolution of the system at the time* t *for the initial data* (v_0, v_1),
- *We will denote by* $\Pi_N(u)$ *the troncature of the Fourier serie of* u *:*

$$u = a + \sum_{n \in \mathbb{Z}^3_*} (b_n cos(n \cdot x) + c_n sin(n \cdot x))$$

$$\Pi_N(u) = a + \sum_{|n| \leqslant N} (b_n cos(n \cdot x) + c_n sin(n \cdot x))$$

- $\Pi^N := (Id - \Pi_N)$

3 Randomization

Let $\mathbb{T}^3$ be the torus in dimension 3. During all this report we will study the cubic defocusing wave equation

$$\begin{aligned} \partial_t^2 u - \Delta u + u^3 = 0, \quad u : \mathbb{R} \times \mathbb{T} \to \mathbb{R}, \\ u|_{t=0} = u_0, \partial_t u|_{t=0} = u_1, \quad (u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3) := H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3) \end{aligned} \tag{3.1}$$

where $H^s(\mathbb{T}^3)$ denotes the classical Sobolev space on $\mathbb{T}^3$ and $s \in [0, 1]$.

The issue raised in the studied article is that the Cauchy problem (3.1) is not locally well-posed in $\mathcal{H}^s$ for $s < \frac{1}{2}$. The case $s \geqslant \frac{1}{2}$ can be proved by invoking the Strichartz estimates for the wave equation, but this is not the topic of the report.

This report shows that a sort of well-posedness for (3.1) still exists, if we randomize the initial data.

3.1 Principle of randomization

Let us now explain how to randomize initial data. Let $(u_0, u_1) \in \mathcal{H}^s$ be the initial data, they are given by their Fourier series

$$u_j = a_j + \sum_{n \in \mathbb{Z}^3_*} (b_{n,j} \cos(n \cdot x) + c_{n,j} \sin(n \cdot x)), \qquad j \in \{0, 1\}, \mathbb{Z}^3_* = \mathbb{Z}^3 \backslash \{(0, 0, 0)\}$$

Let now $(\alpha_j(\omega), \beta_{n,j}(\omega), \gamma_{n,j}(\omega)), n \in \mathbb{Z}^3_*, j = 0, 1$ be a sequence of real random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We can define

$$u_j^\omega = \alpha_j(\omega) a_j + \sum_{n \in \mathbb{Z}^3_*} (\beta_{n,j}(\omega) b_{n,j} \cos(n \cdot x) + \gamma_{n,j}(\omega) c_{n,j} \sin(n \cdot x)) \tag{3.2}$$

We however assume that the random variables $(\alpha_j, \beta_{n,j}, \gamma_{n,j})_{n \in \mathbb{Z}^3_*, j=0,1}$ are independent identically distributed real random Gaussian variables with a joint distribution θ.

Thanks to the Gaussian randomization, we obtain a dense set in $\mathcal{H}^s$ via the map

$$\omega \in \Omega \mapsto (u_0^\omega, u_1^\omega) \in \mathcal{H}^s \tag{3.3}$$

for many $(u_0, u_1) \in \mathcal{H}^s$.

Definition 3.1.1. *For fixed $(u_0, u_1) \in \mathcal{H}^s$, we define the probability measure $\mu_(u_0, u_1)$ by*

$$\forall A \subset \mathcal{H}^0, \mu_{(u_0,u_1)}(A) = \mathbb{P}(\{w \in \Omega : (u_0^\omega, u_1^\omega) \in A\}).$$

We can notice that this definition makes sense because the map (3.3) endows the space $\mathcal{H}^0(\mathbb{T}^3)$ with a probability measure which is the direct image of $\mathbb{P}$

Notation 3.1.1.

$$\mathcal{M}^s := \bigcup_{(u_0,u_1)\in\mathcal{H}^s} \{\mu_{(u_0,u_1)}\}.$$

Now that we have define the measures which will be used, let us see some results linked to them and some properties of them.

3.2 Random series

3.2.1 Basic large deviation estimates

In this subsection, we will only give an idea of the proof of the first result, because it is a quite simple proof and because it can be found quite easily in the literature. The results which follow it are quite direct once one knows the first result.

Now we can enounce our first proposition.

Proposition 3.2.1. *Fix* $\mu \in \mathcal{M}^s = \bigcup_{(u_0,u_1)\in\mathcal{H}^s}\{\mu_(u_0,u_1)\}, s \in [0,1)$*, and suppose that* μ *is induced via the map* (3.3) *from the couple* (u_0,u_1)*. Then there exists* $C>0$ *such that* $\forall 2 \leqslant p_1, p_2 \leqslant q < +\infty, \forall \delta > \frac{1}{p_1}$

$$\mu\big(\{(v_0,v_1)\in\mathcal{H}^s : \|\langle t\rangle^{-\delta}\Pi^0 S(t)(v_0,v_1)\|_{L^{p_1}(\mathbb{R}_t;L^{p_2}(\mathbb{T}^3))} > \lambda\}\big) \leqslant \left(C\frac{\sqrt{q}\|(u_0,u_1)\|_{\mathcal{H}^0(\mathbb{T}^3)}(\delta p_1 - 1)^{-\frac{1}{p_1}}}{\lambda}\right)^q \tag{3.4}$$

Proof. To prove this proposition, we use the definition of μ and we then decompose $\Pi^0 S(t)(u_0^\omega, u_1^\omega)$ in Fourier serie. Then we use some triangle inequalities, to rewrite the problem, which allow us to use a lemma from an article written by Kakutani [3]. We conclude with the generalized Tchebychev inequality. □

Then, for fixed p_1,p_2 we can optimize the estimate by taking

$$C\frac{\sqrt{q}\|(u_0,u_1)\|_{\mathcal{H}^0(\mathbb{T}^3)}}{\lambda} = \frac{1}{2} \iff q = \frac{\lambda^2\|(u_0,u_1)\|^{-2}_{\mathcal{H}^0(\mathbb{T}^3)}}{4C^2} \tag{3.5}$$

and we obtain

Corollary 3.2.1. *There exist* $C, c > 0$ *such that under the assumptions of the proposition 3.2.1,* $\forall\lambda > 0$

$$\mu(\{(v_0,v_1)\in\mathcal{H}^s : \|\langle t\rangle^{-\delta}\Pi^0 S(t)(v_0,v_1)\|_{L^{p_1}(\mathbb{R}_t;L^{p_2}(\mathbb{T}^3))} > \lambda\}\big) \leqslant C\exp\left(-\frac{c\lambda^2}{\|(u_0,u_1)\|^2_{\mathcal{H}^0(\mathbb{T}^3)}}\right)$$

Remark 3.2.1. *We can notice that the measure μ can be seen as a tensor product of the two probability measures μ_N and μ^N defined on the images of the projectors Π_N and Π^N respectively.*

Thus we can apply the previous corollary to μ^N, and we get $\forall\lambda > 0$

$$\begin{aligned}
\mu\big(\{(v_0,v_1) \in& \mathcal{H}^s : \|\langle t\rangle^{-\delta}\Pi^N S(t)(v_0,v_1)\|_{L^{p_1}(\mathbb{R}_t;L^{p_2}(\mathbb{T}^3))} > \lambda\}\big) \\
&= \mu^N\big(\{(v_0,v_1) \in \Pi^N(\mathcal{H}^s) : \|\langle t\rangle^{-\delta} S(t)(v_0^N,v_1^N)\|_{L^{p_1}(\mathbb{R}_t;L^{p_2}(\mathbb{T}^3))} > \lambda\}\big) \\
&\leqslant C\exp\left(-\frac{c\lambda^2}{\|\Pi^N(u_0,u_1)\|^2_{\mathcal{H}^0(\mathbb{T}^3)}}\right) \\
&\leqslant C\exp\left(-\frac{c\lambda^2}{N^{-2s}\|\Pi^N(u_0,u_1)\|^2_{\mathcal{H}^s(\mathbb{T}^3)}}\right).
\end{aligned} \tag{3.6}$$

Corollary 3.2.2. *Fix $\mu \in \mathcal{M}^s = \bigcup_{(u_0,u_1)\in\mathcal{H}^s}\{\mu_(u_0,u_1)\}$, $s \in [0,1)$, and suppose that μ is induced via the map (3.3) from $(u_0,u_1) \in \mathcal{H}^s$. Fix also $2 \leqslant p_1$, $p_2 < +\infty$ and $\delta > 1 + \frac{1}{p_1}$.*

Then there exists a positive constant C such that $\forall\lambda > 0$

$$\mu\big(\{(v_0,v_1) \in \mathcal{H}^s : \|\langle t\rangle^{-\delta} S(t)(v_0,v_1)\|_{L^{p_1}(\mathbb{R}_t;L^{p_2}(\mathbb{T}^3))} > \lambda\}\big) \leqslant C\exp\left(-\frac{c\lambda^2}{\|(u_0,u_1)\|^2_{\mathcal{H}^0(\mathbb{T}^3)}}\right).$$

Finally, if we use the Sobolev embeddings $\mathcal{W}^{\sigma,p}(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$, $\sigma > \frac{3}{p}$ we can obtain the following statement.

Corollary 3.2.3. *Fix $s > 0$ and $\mu \in \mathcal{M}^s$. Let $0 < \sigma \leqslant s$ and suppose that μ is induced via (3.3) from $(u_0,u_1) \in \mathcal{H}^s$. Fix also $2 \leqslant p_1 < +\infty$, $2 \leqslant p_2 \leqslant +\infty$ and $\delta > 1 + \frac{1}{p_1}$.*

Then there exists a positive constant C such that $\forall\lambda > 0$

$$\mu\big(\{(v_0,v_1) \in \mathcal{H}^s : \|\langle t\rangle^{-\delta} S(t)(v_0,v_1)\|_{L^{p_1}(\mathbb{R}_t;L^{p_2}(\mathbb{T}^3))} > \lambda\}\big) \leqslant C\exp\left(-\frac{c\lambda^2}{\|(u_0,u_1)\|^2_{\mathcal{H}^\sigma(\mathbb{T}^3)}}\right)$$

Remark 3.2.2. *We can also show (same reasonment as in 3.2.1) that for every $2 \leqslant p < +\infty$ and $s \geqslant 0$ there exists $C, c > 0$ such that under the assumption of the proposition 3.2.1 defining μ, for every $\lambda > 0$ and an integer $N \geqslant 0$*

$$\mu\big(\{(v_0,v_1) \in \mathcal{H}^s : \|\Pi_N v_0\|_{L^p(\mathbb{T}^3)} > \lambda\}\big) \leqslant C\exp\left(-\frac{c\lambda^2}{\|(u_0,u_1)\|^2_{\mathcal{H}^0(\mathbb{T}^3)}}\right) \tag{3.7}$$

$$\mu\big(\{(v_0,v_1) \in \mathcal{H}^s : \|(v_0,v_1)\|_{\mathcal{H}^s(\mathbb{T}^3)} > \lambda\}\big) \leqslant C\exp\left(-\frac{c\lambda^2}{\|(u_0,u_1)\|^2_{\mathcal{H}^s(\mathbb{T}^3)}}\right) \tag{3.8}$$

All these results will be useful in several proofs of the article.

3.2.2 Conditionned large deviation estimates

The purpose of this paragraph is to deduce the following conditionned versions of our previous large deviation estimates.

Proposition 3.2.2. *Let $\mu \in \mathcal{M}^s = \bigcup_{(u_0,u_1)\in\mathcal{H}^s}\{\mu_(u_0,u_1)\}$, $s \in (0,1)$ and suppose that the real random variable with distribution θ, involved in the definition of μ, is symmetric. Then for $2 \leqslant p_1 < +\infty$, $2 \leqslant p_2 \leqslant +\infty$ and $\delta > 1 + \frac{1}{p_1}$ there exists positive constants c, C such that $\forall \varepsilon, \lambda, \Lambda, A > 0$*

$$\begin{aligned}\mu \otimes \mu\Big(\{((v_0,v_1),(v_0',v_1')) \in \mathcal{H}^s \times \mathcal{H}^s : \|\langle t\rangle^{-\delta} S(t)(v_0 - v_0', v_1 - v_1')\|_{L^{p_1}(\mathbb{R}_t;L^{p_2}(\mathbb{T}^3))} > \lambda \\ or \|\langle t\rangle^{-\delta} S(t)(v_0 + v_0', v_1 + v_1')\|_{L^{p_1}(\mathbb{R}_t;L^{p_2}(\mathbb{T}^3))} > \Lambda \\ \mid \|(v_0 - v_0', v_1 - v_1')\|_{\mathcal{H}^s} \leqslant \varepsilon, \quad \|(v_0 + v_0', v_1 + v_1')\|_{\mathcal{H}^s} \leqslant A\}\Big) \\ \leqslant C\left(e^{-\frac{c\lambda^2}{\varepsilon^2}} + e^{-\frac{c\Lambda^2}{A^2}}\right)\end{aligned} \tag{3.9}$$

We can deduce this result directly from the large deviation estimates of the previous section. However we will need some lemmas.

Lemma 3.2.1. *Let E_1,E_2 be two Banach spaces endowed with measure μ_1,μ_2 respectively. Let $f : (E_1, E_2) \to \mathbb{C}$, $g_1, g_2 : E_2 \to \mathbb{C}$ be three measurable functions. Then*

$$\begin{aligned}\mu_1 \otimes \mu_2\big(\{(x_0, x_1) \in E_1 \times E_2 : |f(x_1, x_2)| > \lambda \mid |g_1(x_2)| \leqslant \varepsilon, |g_2(x_2)| \leqslant A\}\big) \\ \leqslant \sup_{x_2\in E_2, |g_1(x_2)|\leqslant\varepsilon, |g_2(x_2)|\leqslant A} \mu_1\left(\{x_1 \in E_1 : |f(x_1, x_2)| > \lambda\}\right)\end{aligned}$$

Proof.

$$\begin{aligned}&\int_{E_1} \mathbb{1}\big(\{|f(x_1, x_2)| > \lambda\}\big)\mathbb{1}\big(\{|g_1(x_2)| \leqslant \varepsilon\}\big)\mathbb{1}\big(\{|g_2(x_2)| \leqslant A\}\big)d\mu_1(x_1) \\ &\leqslant \Big(\sup_{\substack{X_2\in E_2, |g_1(X_2)|\leqslant\varepsilon, \\ |g_2(X_2)|\leqslant A}} \mu_1\left(\{x_1 \in E_1 : |f(x_1, X_2)| > \lambda\}\right)\Big)\mathbb{1}\big(\{|g_1(x_2)| \leqslant \varepsilon\}\big)\mathbb{1}\big(\{|g_2(x_2)| \leqslant A\}\big)\end{aligned} \tag{3.10}$$

for almost every $x_2 \in E_2$, where by $\mathbb{1}(\cdot)$ we denote the characteristic function of the corresponding set. Now we just integrate (3.10) over $x_2 \in E_2$ with respect to μ_2 to achieve the claimed bound. □

Lemma 3.2.2. *Let g_1,g_2 be two independent identically distributed real random variables with symmetric distributions.*

Then $g_1 \pm g_2$ have symmetric distributions. Besides, if h is a Bernoulli random variable independent of g_1, then hg_1 has the same distribution as g_1.

Proof. The proof is clear. □

Now let us prove the proposition.

Proof. Define $\mathcal{E} := \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^3_*} \times \mathbb{R}^{\mathbb{Z}^3_*}$ equipped with the natural Banach space structure from the l^∞ norm.

We endow $\mathcal{E}$ with a probability measure μ_0 defined via the map

$$\omega \mapsto \Big(k_0(\omega), \big(l_n(\omega)\big)_{n\in\mathbb{Z}^3_*}, \big(h_n(\omega)\big)_{n\in\mathbb{Z}^3_*}\Big)$$

where (k_0, l_n, h_n) is a system of independent Bernoulli variables.

We define the operation $\odot$ by

$$h = \big(x, (y_n)_{n\in\mathbb{Z}^3_*}, (z_n)_{n\in\mathbb{Z}^3_*}\big) \qquad u(x) = a + \sum_{n\in\mathbb{Z}^3_*} \big(b_n \cos(n\cdot x) + c_n \sin(n\cdot x)\big)$$

$$h \odot u := ax + \sum_{n\in\mathbb{Z}^3_*} \big(b_n y_n \cos(n\cdot x) + c_n z_N \sin(n\cdot x)\big)$$

First of all we are going to evaluate

$$\begin{aligned}\mu\otimes\mu\Big(\{\big((v_0,v_1),(v_0',v_1')\big) \in \mathcal{H}^s\times\mathcal{H}^s : \|\langle t\rangle^{-\delta} S(t)(v_0-v_0', v_1-v_1')\|_{L^{p_1}(\mathbb{R}_t;L^{p_2}(\mathbb{T}^3))} > \lambda \\ \mid \|(v_0-v_0', v_1-v_1')\|_{\mathcal{H}^s} \leqslant \varepsilon, \quad \|(v_0+v_0', v_1+v_1')\|_{\mathcal{H}^s} \leqslant A\}\Big)\end{aligned} \tag{3.11}$$

Thanks to 3.2.2 we can tell that this quantity equals

$$\begin{aligned}\mu\otimes\mu\otimes\mu_0\otimes\mu_0\Big(\{\big((v_0,v_1),(v_0',v_1'),(h_0,h_1)\big) \in \mathcal{H}^s\times\mathcal{H}^s\times\mathcal{E} : \\ \|\langle t\rangle^{-\delta} S(t)\big(h_0\odot(v_0-v_0'), h_1\odot(v_1-v_1')\big)\|_{L^{p_1}(\mathbb{R}_t;L^{p_2}(\mathbb{T}^3))} > \lambda \\ \mid \|\big(h_0\odot(v_0-v_0'), h_1\odot(v_1-v_1')\big)\|_{\mathcal{H}^s} \leqslant \varepsilon, \quad \|\big(h_0\odot(v_0+v_0'), h_1\odot(v_1+v_1')\big)\|_{\mathcal{H}^s} \leqslant A\}\Big)\end{aligned} \tag{3.12}$$

Then we use the fact that the H^s norm of a function depends only on the absolute value of its Fourier coefficients, and we get

$$\begin{aligned}(3.12) = \mu\otimes\mu\otimes\mu_0\otimes\mu_0\Big(\{\big((v_0,v_1),(v_0',v_1'),(h_0,h_1)\big) \in \mathcal{H}^s\times\mathcal{H}^s\times\mathcal{E} : \\ \|\langle t\rangle^{-\delta} S(t)\big(h_0\odot(v_0-v_0'), h_1\odot(v_1-v_1')\big)\|_{L^{p_1}(\mathbb{R}_t;L^{p_2}(\mathbb{T}^3))} > \lambda \quad (3.13) \\ \mid \|(v_0-v_0', v_1-v_1')\|_{\mathcal{H}^s} \leqslant \varepsilon, \quad \|(v_0+v_0', v_1+v_1')\|_{\mathcal{H}^s} \leqslant A\}\Big)\end{aligned}$$

And thanks to the lemma 3.2.2 with $\mu_1 = \mu_0\otimes\mu_0$, $\mu_2 = \mu\otimes\mu$ we find :

$$\begin{aligned}(3.13) \leqslant \sup_{\|(v_0-v_0',v_1-v_1')\|_{\mathcal{H}^s}\leqslant\varepsilon} \mu_0\otimes\mu_0\big(\{(h_0,h_1)\in\mathcal{E} : \\ \|\langle t\rangle^{-\delta} S(t)\big(h_0\odot(v_0-v_0'), h_1\odot(v_1-v_1')\big)\|_{L^{p_1}(\mathbb{R}_t;L^{p_2}(\mathbb{T}^3))} > \lambda\}\big)\end{aligned} \tag{3.14}$$

Now, using the last corollary of the previous part we get the bound

$$\begin{aligned}(3.13) &\leqslant Csup_{\|(v_0-v_0',v_1-v_1')\|_{\mathcal{H}^s}\leqslant\varepsilon}\left(-\frac{c\lambda^2}{\|\big(h_0\odot(v_0-v_0'),h_1\odot(v_1-v_1')\big)\|^2_{\mathcal{H}^s}}\right)\\ &= Csup_{\|(v_0-v_0',v_1-v_1')\|_{\mathcal{H}^s}\leqslant\varepsilon}\left(-\frac{c\lambda^2}{\|(v_0-v_0',v_1-v_1')\|^2_{\mathcal{H}^s}}\right)\\ &\leqslant C\exp(-c\frac{\lambda^2}{\varepsilon^2})\end{aligned}$$

We use a very similar argument to prove that the other part is bounded by $C\exp(-c\frac{\Lambda^2}{A^2})$ and we get the bound that we first wanted. □

Thus we have obtained some inequalities that will be useful later. But now let us explain some properties of the constructed measures.

3.3 Properties of the measures

Thanks to the decompostion in Fourier series, we can identify $(u_0,u_1)\in\mathcal{H}^s$ with $\big(a_0,(b_{n,0},c_{n,0})_{n\in\mathbb{Z}^3_*},a_1,(b_{n,1},c_{n,1})_{n\in\mathbb{Z}^3_*}\big)\in(\mathbb{R}\times\mathbb{R}^{\mathbb{Z}^3_*}\times\mathbb{R}^{\mathbb{Z}^3_*})^2$

Thus we can see the measure $\mu_{(u_0,u_1)}$ as an infinite tensor product of measures on $(\mathbb{R}\times\mathbb{R}^{\mathbb{Z}^3_*}\times\mathbb{R}^{\mathbb{Z}^3_*})^2$.

$$\mu\sim\mu_{(0,0)}\otimes\bigotimes_{n\in\mathbb{Z}^3_*}\mu_{(n,0,b)}\otimes\bigotimes_{n\in\mathbb{Z}^3_*}\mu_{(n,0,c)}\otimes\mu_{(0,1)}\otimes\bigotimes_{n\in\mathbb{Z}^3_*}\mu_{(n,1,b)}\otimes\bigotimes_{n\in\mathbb{Z}^3_*}\mu_{(n,1,c)}$$

where all the $\mu_{(\cdot)}$ are the distributions of the corresponding random variables of the randomized Fourier serie.

Now we can apply the followng result proved by Kakutani.

Theorem 3.3.1. *(Kakutani) Consider the infinite tensor product of probability measure on $\mathbb{R}^{\mathbb{N}}$:*

$$\mu_i=\bigotimes_{\mathbb{R}^{\mathbb{N}}}\mu_{n,i}\qquad i=1,2$$

Then the measures μ_1 and μ_2 on $\mathbb{R}^{\mathbb{N}}$ endowed with its cylindrical Borel σ-algebra are absolutely continuous with respect to each other if and only if

1. *$\forall n\in\mathbb{N},\mu_{n,1}$ and $\mu_{n,2}$ are absolutely continuous with respect to each other: $\exists g_n\in L^1(\mathbb{R},d\mu_{n,2}),k_n\in L^1(\mathbb{R},d\mu_{n,1})$ such that*

$$d\mu_{n,1}=g_n d\mu_{n,2}\qquad d\mu_{n,1}=g_n d\mu_{n,2}$$

2. *the functions g_n are such that*

$$\prod_{n\in\mathbb{N}}\int_{\mathbb{R}}g_n^{\frac{1}{2}}d\mu_{n,2}=\prod_{n\in\mathbb{N}}\int_{\mathbb{R}}\sqrt{d\mu_{n,1}}\sqrt{d\mu_{n,2}}\tag{3.15}$$

is convergent.

Furthermore, if any of these conditions above is not satisfied then the two measures are mutually singular : $\exists A \in \mathbb{R}^{\mathbb{N}}$ *such that*

$$\mu_1(A) = 1, \qquad \mu_2(A) = 0$$

We admit this theorem. This result implies the following statement concerning the measures that we are studying in the context of the cubic wave equation.

Proposition 3.3.1. *Assume that the random variables* $(\alpha_j, \beta_{n,j}, \gamma_{n,j}), j = 0, 1, n \in \mathbb{Z}^3_*$ *used to obtain the randomization are independent centered Gaussian random variables. Let*

$$u_j(x) = a_{0,j} + \sum_{n \in \mathbb{Z}^3_*} \big(b_{n,j} \cos(n \cdot x) + c_{n,j} \sin(n \cdot x)\big), \quad j = 0, 1$$

$$\widetilde{u_j}(x) = \widetilde{a}_{0,j} + \sum_{n \in \mathbb{Z}^3_*} \big(\widetilde{b}_{n,j} \cos(n \cdot x) + \widetilde{c}_{n,j} \sin(n \cdot x)\big), \quad j = 0, 1$$

Then the measures $\mu_{(u_0,u_1)}$ *and* $\mu_{(\widetilde{u}_0,\widetilde{u}_1)}$ *are absolutely continuous with respect to each other if and only if neither of the coefficients of the Fouriers series above vanishes (or then they vanish simultaneously) and*

$$\sum_{j=0}^{1} \left[\left(|\frac{\widetilde{a}_{0,j}}{a_{0,j}}| - 1 \right)^2 + \sum_{n \in \mathbb{Z}^3_*} \left(|\frac{\widetilde{b}_{n,j}}{b_{n,j}}| - 1 \right)^2 + \sum_{n \in \mathbb{Z}^3_*} \left(|\frac{\widetilde{c}_{n,j}}{c_{n,j}}| - 1 \right)^2 \right] < +\infty$$

Furthermore, if this condition is not satisfied, then the two measures are mutually singular.

Proof. It is well known that if g is a normalized Gaussian random variable, the random variable αg is a Gaussian random variable centered, with variance α^2. In this case, if we eliminate the trivial contributions when the coefficents vanish simultaneously, the result amounts to proving that if $\mu_i = \bigotimes_{n \in \mathbb{N}} \mu_{n,i}$ where $\mu_{n,i}$ are Gaussian distributions of variance $x^2_{n,i}, x_{n,i} > 0$, then the measures μ_1 and μ_2 are absolutely continuous with respect to each other if and only if

$$\sum_n \left(\frac{x_{n,1}}{x_{n,2}} - 1 \right)^2 < +\infty$$

and here we have

$$d\mu_{n,i} = \frac{1}{x_{n,i}\sqrt{2\pi}} e^{-t^2/2x^2_{n,i}} dt$$

and

$$g_n = \frac{x_{n,2}}{x_{n,1}} e^{t^2/2x^2_{n,2} - t^2/2x^2_{n,1}}$$

Thus we have :

$$\int_{\mathbb{R}} g_n^{1/2} d\mu_{n,2} = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi x_{n,1} x_{n,2}}} e^{-t^2/4x_{n,2}^2 - t^2/4x_{n,1}^2} dt$$
$$= \left(\frac{2x_{n,1}x_{n,2}}{x_{n,1}^2 + x_{n,2}^2} \right)^{1/2} \tag{3.16}$$
$$= \left(\frac{x_{n,1}/x_{n,2} + x_{n,2}/x_{n,1}}{2} \right)^{-1/2}$$

and we deduce that if the infinite product (3.15) is convergent then necessarily the quotients $x_{n,1}/x_{n,2}$ tend to 1. Now, if we define $\varepsilon_n := x_{n,2}/x_{n,1} - 1$, we can use a classic Taylor expansion and we get :

$$\left(\frac{2x_{n,1}x_{n,2}}{x_{n,1}^2 + x_{n,2}^2} \right)^{1/2} = 1 - \frac{1}{4}\varepsilon_n^2 + \mathcal{O}(\varepsilon_n^3)$$

Finally, by taking the logarithm, we conclude that (3.15) is convergent if and only if

$$\sum_n \varepsilon_n^2 < +\infty$$

□

Then we have a last proposition.

Proposition 3.3.2. *If $(u_0, u_1) \in \mathcal{H}^s$ have all their Fourier coefficients different from zero and if the measure θ charges all open sets of $\mathbb{R}$then the support of $\mu_{(u_0,u_1)}$ is $\mathcal{H}^s$. In other words, under these assumptions, for any $(w_0, w_1) \in \mathcal{H}^s$ and any $\varepsilon > 0$*

$$\mu_{(u_0,u_1)}\big(\{(v_0, v_1) \in \mathcal{H}^s : \|(w_0, w_1) - (v_0, v_1)\|_{\mathcal{H}^s} < \varepsilon\}\big) > 0 \tag{3.17}$$

or, again in other words, the intersection of any set of $\mu_{(u_0,u_1)}$-measure 1 with any (non trivial) ball in $\mathcal{H}^s$ is non-empty.

Proof. According of the fact that Π_N is the orthogonal projection to the space of functions having Fourier modes $n \in \mathbb{Z}^3_*$ such that $|n| \leqslant N$ and $\Pi^N = Id - \Pi_N$, we have for fixed $< \varepsilon > 0$

$$\begin{aligned} &\mu_{(u_0,u_1)}\big(\{(v_0, v_1) \in \mathcal{H}^s : \|(w_0, w_1) - (v_0, v_1)\|_{\mathcal{H}^s} < \varepsilon\}\big) \\ &\quad \geqslant \mu_{(u_0,u_1)}\big(\{(v_0, v_1) \in \mathcal{H}^s : \|\Pi_N\big((w_0, w_1) - (v_0, v_1)\big)\|_{\mathcal{H}^s} < \varepsilon/2 \text{ and} \\ &\quad \|\Pi^N\big((w_0, w_1) - (v_0, v_1)\big)\|_{\mathcal{H}^s} < \varepsilon/2\}\big) \end{aligned} \tag{3.18}$$

and since the two events are independent (because the random variables of the randomization are independent) we deduce

$$\begin{aligned}
&\mu_{(u_0,u_1)}\big(\{(v_0,v_1)\in\mathcal{H}^s : \|(w_0,w_1)-(v_0,v_1)\|_{\mathcal{H}^s}<\varepsilon\}\big)\\
&\quad\geqslant \mu_{(u_0,u_1)}\big(\{(v_0,v_1)\in\mathcal{H}^s : \|\Pi_N\big((w_0,w_1)-(v_0,v_1)\big)\|_{\mathcal{H}^s}<\varepsilon/2\}\big)\\
&\quad\times \mu_{(u_0,u_1)}\big(\{(v_0,v_1)\in\mathcal{H}^s : \|\Pi^N\big((w_0,w_1)-(v_0,v_1)\big)\|_{\mathcal{H}^s}<\varepsilon/2\}\big)\\
&\quad\geqslant \mu_{(u_0,u_1)}\big(\{(v_0,v_1)\in\mathcal{H}^s : \|\Pi_N\big((w_0,w_1)-(v_0,v_1)\big)\|_{\mathcal{H}^s}<\varepsilon/2\}\big)\\
&\quad\times \mu_{(u_0,u_1)}\big(\{(v_0,v_1)\in\mathcal{H}^s : \|\Pi^N(w_0,w_1)\|_{\mathcal{H}^s}<\varepsilon/4 \text{ and } \|\Pi^N(v_0,v_1)\|_{\mathcal{H}^s}<\varepsilon/4\}\big)
\end{aligned}\tag{3.19}$$

Then we use

$$\lim_{N\to\infty}\|\Pi^N(u_0,u_1)\|_{\mathcal{H}^s}=\lim_{N\to\infty}\|\Pi^N(w_0,w_1)\|_{\mathcal{H}^s}=0$$

and we apply (3.8) to $\Pi^N(v_0,v_1)$ to obtain

$$\begin{aligned}
&\mu_{(u_0,u_1)}\big(\{(v_0,v_1)\in\mathcal{H}^s : \|\Pi^N(v_0,v_1)\|_{\mathcal{H}^s}\geqslant\varepsilon/4\}\big)\\
&\qquad\qquad\leqslant Ce^{-c\varepsilon^2/\|\Pi^N(u_0,u_1)\|^2_{\mathcal{H}^s}}\to 0 \text{ as } N\to\infty
\end{aligned}$$

Thus we deduce that for N big enough (depending on $\varepsilon>0$ and $(u_0,u_1),(w_0,w_1)$),

$$\mu_{(u_0,u_1)}\big(\{(v_0,v_1)\in\mathcal{H}^s : \|\Pi^N(w_0,w_1)\|_{\mathcal{H}^s}<\varepsilon/4 \text{ and } \|\Pi^N(v_0,v_1)\|_{\mathcal{H}^s}<\varepsilon/4\}\big)\geqslant 1/2.$$

Now taking such N we obtain

$$\begin{aligned}
&\mu_{(u_0,u_1)}\big(\{(v_0,v_1)\in\mathcal{H}^s : \|(w_0,w_1)-(v_0,v_1)\|_{\mathcal{H}^s}<\varepsilon\}\big)\\
&\qquad\geqslant \frac{1}{2}\mu_{(u_0,u_1)}\big(\{(v_0,v_1)\in\mathcal{H}^s : \|\Pi_N\big((w_0,w_1)-(v_0,v_1)\big)\|_{\mathcal{H}^s}<\varepsilon/2\}\big)
\end{aligned}\tag{3.20}$$

We have now reduced the study to a finite-dimensional problem. Because of the assumptions of all Fourier modes of (u_0,u_1) are non-vanishing and the distribution θ of our random variables charges all open sets of $\mathbb{R}$, (3.17) follows easily. $\square$

Now that we have these results and properties from the measures of $\mathcal{M}^s=\bigcup_{(u_0,u_1)\in\mathcal{H}^s}\{\mu_(u_0,u_1)\}$, we can use them to work on the initial problem (3.1). Let us enounce our first theorem about it.

3.4 Theorem

Theorem 3.4.1. *Fix $\mu\in\mathcal{M}^s$, $0\leqslant s<1$. Then there exists a full μ-measure set $\Sigma\subset\mathcal{H}^s(\mathbb{T}^3)$ such that for every $(v_0,v_1)\in\Sigma$ there exists a unique global solution v of the non-linear wave equation*

$$\partial_t^2 v-\Delta v=-v^3 \qquad \big(v(0),\partial_t v(0)\big)=(v_0,v_1)$$

satisfying

$$\big(v(t), \partial_t v(t)\big) \in \big(S(t)(v_0, v_1), \partial_t S(t)(v_0, v_1)\big) + \mathcal{C}\big(\mathbb{R}_t, H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)\big).$$

Furthermore, if we denote by $\phi(t)(v_0, v_1) := \big(v(t), \partial_t v(t)\big)$ *the flow thus defined, the set* Σ *is invariant under the map* $\phi(t)$*, that is*

$$\forall t \in \mathbb{R}, \phi(t)(\Sigma) = \Sigma.$$

Finally, for any $\varepsilon > 0$ *there exists* $C, \delta > 0$ *such that for* μ*-almost every* $(v_0, v_1) \in \mathcal{H}^s$*, there exists* $M > 0$ *such that the global solution previously constructed satisfies*

$$v(t) = S(t)\Pi^0(v_0, v_1) + w(t)$$

with

$$\|\big(w(t), \partial_t w(t)\big)\|_{\mathcal{H}^1(\mathbb{T}^3)} \leqslant C(m + |t|)^{(1-s)/s+\varepsilon} \quad \text{if } s > 0 \tag{3.21}$$

$$\|\big(w(t), \partial_t w(t)\big)\|_{\mathcal{H}^1(\mathbb{T}^3)} \leqslant Ce^{C(t+M)^2} \quad \text{if } s = 0 \tag{3.22}$$

and

$$\mu\big(\{(v_0, v_1) \in \mathcal{H}^s : M > \lambda\}\big) \leqslant Ce^{-\lambda^\delta}$$

This theorem will be proved, in the case s¿0, in the following sections.

4 Resolution of the problem in the case $s > 0$

4.1 Existence and unicity

Proposition 4.1.1. *Consider the problem*

$$\partial_t^2 v - \Delta v = -(f + v)^3 \tag{4.1}$$

(here f is the linear evolution of the problem).There exists a constant C such that for every time interval $I = [a, b]$ *of size 1, every* $\Lambda \geqslant 1$*, and every* $(v_0, v_1, f) \in H^1 \times L^2 \times L^3(I, L^6)$ *satisfying*

$$\|v_0\|_{H^1} + \|v_1\|_{L^2} + \|f\|^3_{L^3(I,L^6)} \leqslant \Lambda$$

there exists an unique solution of (4.1) *on the time interval* $[a, a + \Lambda^{-2}]$ *with initial data*

$$v(a, x) = v_0(x), \quad \partial_t v(a, x) = v_1(x)$$

Moreover the solution satisfies $\|(v, \partial_t v)\|_{L^\infty([a,a+\Lambda^{-2}], H^1 \times L^2)} \leqslant C\Lambda$*,* $(v, \partial_t v)$ *is unique in the class* $L^\infty([a, a + \Lambda^{-2}], H^1 \times L^2)$ *and the dependence in time is conitnuous.*

Proof. By translation, we can assume that $I = [0,1]$. Now to define $S(t)(v_0, v_1)$ we study the problem

$$\partial_t^2 v - \Delta v = 0$$

and we rewrite it with Fourier to obtain

$$\partial_t^2 \widehat{v} - |\xi|^2 \widehat{v} = 0$$

which is a classic ODE with solution

$$\widehat{v}(t,\xi) = \widehat{v_0}(\xi)\cos(t|\xi|) + \widehat{v_1}\frac{\sin(t|\xi|}{|\xi|}$$

Now we just have to apply the inverse Fourier transformation to have v.

Property 4.1.1. *Let $m(\xi)$ be a Fourier multiplicator, then the operator T : $f \mapsto \mathcal{F}^{-1}(\xi \mapsto m(\xi)\widehat{f}(\xi))$ can be written as $f \mapsto m(\sqrt{-\Delta})(f)$.*

This property is admitted.

Thus we can define the free evolution by

$$S(t)(v_0, v_1) := \cos(t\sqrt{-\Delta})(v_0) + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}(v_1). \tag{4.2}$$

According to Duhamel, we can rewrite the probem as

$$v(t) = S(t)(v_0, v_1) - \int_0^1 \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}}\big((f(\tau) + v(\tau))^3\big)d\tau \tag{4.3}$$

To prove it, we use Fourier again, the problem becomes

$$\widehat{w}'' + |\xi|^2\widehat{w} = \widehat{(f+w)^3} \qquad \widehat{w}(0) = \widehat{w}'(0) = 0$$

and we know thanks to the Cauchy theory that there exists a unique solution. Now define

$$\widehat{u}(t,\xi) := -\int_0^t \frac{\sin((t-\tau)|\xi|)}{|\xi|}\big(\widehat{(f(\tau) + w(\tau))^3}\big)d\tau$$

and let us show that it is a solution.

- $\widehat{u}(0,\xi) = 0$
- $\widehat{u}'(t,\xi) = \int_0^t \cos\big((t-s)|\xi|\big)\big(\widehat{f(s) + w(s)}\big)^3 ds \Rightarrow \widehat{u}'(0,\xi) = 0$
- $\widehat{u}''(t,\xi) = -\big(\widehat{f}(t) + \widehat{w}(t)\big)^3 + |\xi| \int_0^t \sin\big((t-s)|\xi|\big)\big(\widehat{f(s) + w(s)}\big)^3 ds$

It is now clear that we have

$$\widehat{u}'' + |\xi|^2\widehat{u} = -\big(\widehat{f+w}\big)^3$$

Let us now set

$$\phi_{u_0,u_1,f}(v) = S(t)(v_0,v_1) - \int_0^1 \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}}\big((f(\tau)+v(\tau))^3\big)d\tau$$

We can use , for $T \in (01]$ the Sobolev embedding $\mathcal{H}^1(\mathbb{T}^3) \subset \L^6(\mathbb{T}^3)$ to get ($S(t)$ is bilinear and continuous and the functions in the sum are bounded because $\mathbb{T}^3$ is compact) :

$$\begin{aligned}\phi_{u_0,u_1,f}(v) &\leqslant C\big(\|v_0\|_{H^1} + \|v_1\|_{L^2} + T \sup_{\tau\in[0,T]} \|f(\tau)+v(\tau)\|^3_{L^6}\big)\\ &\leqslant C\big(\|v_0\|_{H^1} + \|v_1\|_{L^2} + \sup_{\tau\in[0,T]} \|f(\tau)\|^3_{L^6}\big) + CT\|v\|^3_{L^\infty([0,T];H^1)}\big)\end{aligned}$$

This estimates gives us that for $T \approx \Lambda^{-2}$ the map $\phi_{u_0,u_1,f}$ sends the ball $\{v : \|v\|^3_{L^\infty([0,T];H^1)} \leqslant C\Lambda\}$ onto itself. A similar argument can show that this is a contraction of the same ball and thus we only have to apply the Picard theorem. We can get the estimate of $\|\partial_t v\|_{L^2}$ by differentiating in t the Duhamel formula (4.3). ☐

Proposition 4.1.2. *Assume that $s > 0$ and fix $\mu \in \mathcal{M}^s$. Then for μ-almost every $(v_0, v_1) \in \mathcal{H}^s(\mathbb{T}^3)$, there exists a unique global solution*

$$\big(v(t), \partial_t v(t)\big) \in \big(S(t)(v_0,v_1), \partial_t S(t)(v_0,v_1)\big) + \mathcal{C}\big(\mathbb{R}_t, H^1(\mathbb{T}^3)\times L^2(\mathbb{T}^3)\big)$$

of the non-linear wave equation

$$(\partial_t - \Delta)v + v^3 = 0$$

with initial data

$$v(0,x) = v_0(x) \qquad \partial_t v(0,x) = v_1(x)$$

Proof. We look for v in the form $v(t) = S(t)(v_0,v_1) + w(t)$. Then w solves

$$(\partial_t^2 - \Delta)w + \big(S(t)(v_0,v_1)+w\big)^3 = 0, \quad w\big|_{t=0}, \quad \partial_t w\big|_{t=0} = 0 \tag{4.4}$$

Then thanks to Corollary 3.2.2 gives us that if $\delta > 1 = \frac{1}{p}$, then μ-almost surely

$$\|\langle t\rangle^{-\delta} S(t)(v_0,v_1)\|_{L^p(\mathbb{R}_t;\mathcal{W}^{s,p}(\mathbb{T}^3))} < \infty.$$

Let us take p big enough such as $\frac{3}{p} < s$, thus $\frac{1}{p} - \frac{s}{3} < 0$. Then thanks to corollary 2.1.1 we know that $\mathcal{W}^{s,p}(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$.

Now let us set

$$f(t) = \|S(t)(v_0,v_1)\|_{L^\infty(\mathbb{T}^3)}, \qquad g(t) = \|S(t)(v_0,v_1)\|^3_{L^6(\mathbb{T}^3)}$$

Then thanks to the previous Sobolev embedding, we get

$$\int_0^T \|S(t)(v_0,v_1)\|^p_{L^\infty(\mathbb{T}^3)}dt \leqslant \int_0^T \|S(t)(v_0,v_1)\|^p_{\mathcal{W}^{s,p}(\mathbb{T}^3)}dt < \infty$$

(notice that here we use the fact that $\langle t\rangle^{-\delta}$ is not important because $[0,T]$ is compact). Thus we deduce that $f \in L^p_{loc}(\mathbb{R}_t) \subset L^1_{loc}(\mathbb{R}_t)$ ($\mathbb{T}^3$ is a compact variety). Besides we just have to notice that $g(t) \leqslant f(t)^3$ to get

$$f(t) = \|S(t)(v_0,v_1)\|_{L^\infty(\mathbb{T}^3)} \in L^1_{loc}(\mathbb{R}_t), \quad g(t) = \|S(t)(v_0,v_1)\|^3_{L^6(\mathbb{T}^3)} \in L^1_{loc}(\mathbb{R}_t). \tag{4.5}$$

Thus we have the local existence thanks to propostion 4.1.1 and the second estimate of (4.5). We also deduce that as long as the $H^1 \times L^2$ norm of $(w, \partial_t w)$ remains bounded, the solution w of (4.4) exists.

Set

$$\mathcal{E}(w(t)) = \frac{1}{2}\int_{\mathbb{T}^3} \Big((\partial_t w)^2 + |\nabla_x w|^2 + \frac{1}{2}w^4\Big)dx.$$

We can derive in the sum because of the boundedness of the $H^1 \times L^2$ norm and we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(w(t)) &= \int_{\mathbb{T}^3} \big(\partial_t w \partial_t^2 w + \nabla_x \partial_t w \cdot \nabla_x w + \partial_t w w^3\big)dx \\
&= \int_{\mathbb{T}^3} \partial_t w(\partial_t^2 w - \Delta w + w^3)dx \\
&= \int_{\mathbb{T}^3} \partial_t w\big(w^3 - (S(t)(v_0,v_1) + w)^3\big)dx
\end{aligned}$$

And thanks to Cauchy-Schwarz and Hölder inequalities :

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(w(t)) \leqslant\, & C\big(\mathcal{E}(w(t))\big)^{1/2}\|w^3 - \big(S(t)(v_0,v_1)+w\big)^3\|_{L^2(\mathbb{T}^3)} \\
\leqslant\, & C\big(\mathcal{E}(w(t))\big)^{1/2}\big(\|S(t)(v_0,v_1)\|^3_{L^6(\mathbb{T}^3)} + \|S(t)(v_0,v_1)\|_{L^\infty(\mathbb{T}^3)}\|w^2\|_{L^2(\mathbb{T}^3)}\big) \\
\leqslant\, & C\big(\mathcal{E}(w(t))\big)^{1/2}\big(g(t) + f(t)(\mathcal{E}(w(t)))^{1/2}\big)
\end{aligned}$$

Thus, according to the Gronwall lemma and (4.5), w exists globally in time. □

4.2 Construction of an invariant set

The previous section has shown the existence on a set of full μ-measure. However we may have a dynamics which sends for some $t \neq 0$ this set of full measure, where the global existence holds, to a set of small measure.

The aim is now to establish a global dynamics on an invariant set of full measure in the context of the argument of the previous section.

Definition 4.2.1.

$$\begin{aligned}
\Theta := \{(v_0,v_1) \in \mathcal{H}^s :& \|S(t)(v_0,v_1)\|^3_{L^6(\mathbb{T}^3)} \in L^1_{loc}(\mathbb{R}_t), \\
& \|S(t)(v_0,v_1)\|_{L^\infty(\mathbb{T}^3)} \in L^1_{loc}(\mathbb{R}_t)\}
\end{aligned}$$

and $\Sigma := \Theta + \mathcal{H}^1$.

Notice that Σ is of full μ-measure for every $\mu \in \mathcal{M}^s = \bigcup_{(u_0,u_1)\in\mathcal{H}^s}\{\mu_(u_0,u_1)\}$ since so is Θ.

Proposition 4.2.1. *Assume that $s > 0$ and fix $\mu \in \mathcal{M}^s$. Then, for every $(v_0, v_1) \in \Sigma$, there exists a unique global solution*

$$\big(v(t), \partial_t v(t)\big) \in \big(S(t)(v_0, v_1), \partial_t S(t)(v_0, v_1)\big) + \mathcal{C}\big(\mathbb{R}_t, H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)\big)$$

of the non-linear wave equation

$$(\partial_t^2 v - \Delta)v + v^3 = 0, \qquad \big(v(0,x), \partial_t v(0,x)\big) = \big(v_0(x), v_1(x)\big). \tag{4.6}$$

Moreover, $(v(t), \partial_t v(t)) \in \Sigma$ for every $t \in \mathbb{R}$, and thus by time reversibility Σ is invariant under the flow of (4.6).

Proof. By assumption, we can write $(v_0, v_1) = (\widetilde{v_0}, \widetilde{v_1}) + (w_0, w_1)$ with $(\widetilde{v_0}, \widetilde{v_1}) \in \Theta$ and $(w_0, w_1) \in \mathcal{H}^1$. We look for v in the form $v(t) = S(t)(\widetilde{v_0}, \widetilde{v_1}) + w(t)$. Then w solves

$$(\partial_t^2 v - \Delta)w + (S(t)(\widetilde{v_0}, \widetilde{v_1}) + w(t))^3 = 0, \quad w\big|_{t=0} = w_0, \quad \partial_t w\big|_{t=0} = w_1.$$

Now, exactly as in the proof of proposition 4.1.2, we obtain

$$\frac{d}{dt}\mathcal{E}(w(t)) \leqslant C\big(\mathcal{E}(w(t))\big)^{1/2}\big(g(t) + f(t)(\mathcal{E}(w(t)))^{1/2}\big)$$

where

$$f(t) = \|S(t)(\widetilde{v_0}, \widetilde{v_1})\|_{L^\infty(\mathbb{T}^3)}, \qquad g(t) = \|S(t)(\widetilde{v_0}, \widetilde{v_1})\|^3_{L^6(\mathbb{T}^3)}$$

Therefore thanks to the Gronwall lemma and using that $\mathcal{E}(w(0))$ is well-defined, we obtain the global existence for w. Thus the solution of (4.6) can be written as

$$v(t) = S(t)(\widetilde{v_0}, \widetilde{v_1}) + w(t), \qquad (w, \partial_t w) \in \mathcal{C}(\mathbb{R}; \mathcal{H}^1).$$

Besides, coming back to the definition of Θ, we observe that

$$S(t)(\Theta) = \Theta.$$

By this we mean

$$\{(S(t)(v_0, v_1), \partial_t S(t)(v_0, v_1)) : (v_0, v_1) \in \Theta\} = \Theta$$

In other words, if one wants to explain it heuristically : if we take at first $(v_0, v_1) \in \Theta$ as initial data, and then we stop at time t, we obtain $(S(t)(v_0, v_1), \partial_t S(t)(v_0, v_1))$. Now if we take $(S(t)(v_0, v_1), \partial_t S(t)(v_0, v_1))$ as initial data, they are still in Θ, it is a kind of invariance by translation. $\square$

4.3 Bounds on the growth of the Sobolev norm

We follow the high-low decomposition method of Bourgain [1] in this section.

Proposition 4.3.1. *Let $1 > s > 0$ and $\mu \in \mathcal{M}^s$. Consider the flow of* (4.6) *established in the previous section.*

Then, for any $\varepsilon > 0$ there exist $C, \delta > 0$ such that for every $(v_0, v_1) \in \Sigma$, there exists $M > 0$ such that the global solution of (4.6) *satisfies :*

$$v(t) = S(t)(v_0, v_1) + w(t), \quad \|(w(t), \partial_t w(t))\|_{\mathcal{H}^1(\mathbb{T}^3)} \leqslant C(M + |t|)^{(1-s)/s+\varepsilon}$$

with $\mu(M > \lambda) \leqslant Ce^{-\lambda^\delta}$.

Proof. We only give a proof for positive times, the analysis for negativee times being analogous. Let be $\varepsilon > 0, \delta > \frac{1}{2}, \widetilde{\delta} > \frac{1}{3}$. Now we introduce the sets

- $F_N := \{(v_0, v_1) \in \Sigma : \|\Pi_N(v_0, v_1)\|_{\mathcal{H}^s} \leqslant N^{1-s+\varepsilon}\}$
- $G_N := \{(v_0, v_1) \in \Sigma : \|\Pi_N(v_0)\|_{L^4(\mathbb{T}^")} \leqslant N^\varepsilon\}$
- $H_N := \{(v_0, v_1) \in \Sigma : \|\langle t\rangle^{-\delta} S(t)(\Pi^N(v_0, v_1))\|_{L^2(\mathbb{R}_t; L^4(\mathbb{T}^3))} \leqslant N^{\varepsilon - s}\}$
- $K_N := \{(v_0, v_1) \in \Sigma : \|\langle t\rangle^{-\widetilde{\delta}} S(t)(\Pi^N(v_0, v_1))\|_{L^3(\mathbb{R}_t; L^6(\mathbb{T}^3))} \leqslant N^{\varepsilon - s}\}$

Lemma 4.3.1. *Let $\delta > 1$ and $\widetilde{\delta} > 1$. There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leqslant \varepsilon_0$, there exist $C, c > 0$ such that $\forall N \geqslant 1$,*

$$\mu(F_N^c) \leqslant Ce^{-cN^{2\varepsilon}}, \mu(G_N^c) \leqslant Ce^{-cN^{2\varepsilon}}, \mu(H_N^c) \leqslant Ce^{-cN^{2\varepsilon}}, \mu(K_N^c) \leqslant Ce^{-cN^{2\varepsilon}}$$

Proof. For $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)$, we have

$$\|\Pi_N(u_0, u_1)\|_{\mathcal{H}^1} \leqslant CN^{1-s}\|(u_0, u_1)\|_{\mathcal{H}^s},$$
$$\|\Pi^N(u_0, u_1)\|_{\mathcal{H}^0} \leqslant CN^{-s}\|(u_0, u_1)\|_{\mathcal{H}^s}.$$

Indeed

$$\begin{aligned} N^{-2}\|\Pi_N(u_0)\|_{\mathcal{H}^1}^2 &= \sum_{|n| \leqslant N} \frac{|n|}{N^2}|a_n|^2 \\ &\leqslant \sum_{|n| \leqslant N} \left(\frac{|n|}{N^2}\right)|a_n|^2 \quad (\frac{|n|}{N^2} \leqslant 1, s \in (0,1)) \\ &\leqslant CN^{-2s}\|(u_0, u_1)\|_{\mathcal{H}^s} \end{aligned}$$

and so come these inequalities.

Then, according to (3.8) and the first inequality we get

$$\begin{aligned}\mu(F_N^c) &\leqslant \mu\big(\{(v_0,v_1)\in\mathcal{H}^s : CN^{1-s}\|(v_0,v_1)\|_{\mathcal{H}^s} > N^{1-s+\varepsilon}\}\big)\\ &= \mu\big(\{(v_0,v_1)\in\mathcal{H}^s : \|(v_0,v_1)\| > \frac{N^\varepsilon}{C}\}\big)\\ &\leqslant C_1\exp\left(-\frac{cN^{2\varepsilon}}{C^2\|(u_0,u_1)\|^2_{\mathcal{H}^s(\mathbb{T}^3)}}\right)\\ &= \widetilde{C}e^{-cN^{2\varepsilon}}\end{aligned}$$

ad we use (3.7) to obtain

$$\mu\big(\{(v_0,v_1)\in\mathcal{H}^s : \|\Pi_N(v_0)\|_{L^4} > N^\varepsilon\}\big) \leqslant Ce^{-cN^{2\varepsilon}}.$$

Then notice that we have the Sobolev embedding $H^1 \subset \mathrm{L}^4$ because we can write $H^1 = \mathcal{W}^{1,2}$ and $\frac{1}{2}-\frac{1}{3} > 0$.

Thus we use the other inequality, remark 3.2.1 and corollary 3.2.3 to get

$$\begin{aligned}\mu(K_N^c) &\leqslant C\exp\left(-\frac{cN^{2(\varepsilon-s)}}{N^{-2s}\|\Pi^N(u_0,u_1)\|_{\mathcal{H}^s}}\right)\\ &= \widetilde{C}e^{-cN^{2\varepsilon}}\end{aligned}$$

This completes the proof of Lemma 4.3.1. □

Now let $N\in\mathbb{N}^*$. We define

$$E_N := F_N\cap G_N\cap H_N\cap K_N.$$

According to the precedent lemma we have

$$\mu(E_N^c) \leqslant e^{-cN^\kappa}, \qquad \kappa > 0.$$

We now fix $\varepsilon_1 > 0$, and then $\varepsilon > 0$ small enough such that

$$\frac{1-s+\varepsilon}{s-2\varepsilon} \leqslant \frac{1-s}{s}+\varepsilon_1 \quad \text{and } \varepsilon < \frac{s}{2} \tag{4.7}$$

We finally fix $\delta > \frac{1}{2}, \widetilde{\delta} > \frac{1}{3}$ such that

$$(\delta - \frac{1}{2})s < 2\delta\varepsilon, \quad \text{and } \widetilde{\delta} < 1 \tag{4.8}$$

Lemma 4.3.2. *For every $c > 0$ there exists $C > 0$ such that for every $t \geqslant 1$ and every $N\in\mathbb{N}^*$ such that $t \leqslant cN^{s-2\varepsilon}$, and every $(v_0,v_1)\in E_N$ the solution of (4.6) with data (v_0,v_1) satisfies :*

$$\|v(t) - S(t)\Pi^0(v_0,v_1)\|_{\mathcal{H}^1(\mathbb{T}^3)} \leqslant CN^{1-s+\varepsilon}.$$

In particular, thanks to (4.7), if $t\approx N^{s-2\varepsilon}$ then

$$\|v(t) - S(t)\Pi^0(v_0,v_1)\|_{\mathcal{H}^1(\mathbb{T}^3)} \leqslant t^{(1-s)/s+\varepsilon_1}$$

Proof. For $(v_0, v_1) \in E_N$ we can decompose $(v_0, v_1) = \Pi_N(v_0, v_1) + \Pi^N(v_0, v_1)$ and thus the solution of (4.6) can be written as

$$v(t) = S(t)\Pi^N(v_0, v_1) + w_N$$

where w_N solves :

$$(\partial_t^2 - \Delta)w_N + \big(w_N - S(t)\Pi^N(v_0, v_1)\big)^3 = 0, \quad \big(w(0), \partial_t w(0)\big) = \Pi_N(v_0, v_1). \tag{4.9}$$

We use the energy estimates of the previous sections to get

$$\frac{d}{dt}\mathcal{E}(w_N(t)) \leqslant C\big(\mathcal{E}(w_N(t))\big)^{1/2}\Big(g_N(t) + f_N(t)\big(\mathcal{E}(w_N(t))\big)^{1/2}\Big) \tag{4.10}$$

where

$$f_N(t) = \|S(t)\Pi^N(v_0, v_1)\|_{L^\infty(\mathbb{T}^3)}, \qquad g_N(t) = \|S(t)\Pi^N(v_0, v_1)\|^3_{L^6(\mathbb{T}^3)}$$

Notice that (4.10) can be written as

$$\frac{d}{dt}\big(\mathcal{E}(w_N(t))^{1/2}\big) \leqslant C\Big(g_N(t) + f_N(t)\big(\mathcal{E}(w_N(t))\big)^{1/2}\Big).$$

Now by integrating (4.10) we know that $\mathcal{E}(w_N(t))^{1/2}$ will be smaller than the solution of the problem

$$u' = C(f_N u + g_N).$$

And thanks to the Duhamel formula we obtain

$$\mathcal{E}(w_N(t))^{1/2} \leqslant Ce^{C\int_0^t f_N(s)ds}\left(\mathcal{E}(w_N(0))^{1/2} + \int_0^t g_N(s)ds\right) \tag{4.11}$$

We now observe that for $(v_0, v_1) \in E_N$

$$\begin{aligned}
\Big|\int_0^t g_N(s)ds\Big| &= \Big|\int_0^t \|S(s)\Pi^N(v_0, v_1)\|^3_{L^6(\mathbb{T}^3)}ds\Big| \\
&\leqslant \int_0^t C\|\langle s\rangle^{-\widetilde{\delta}}S(s)\Pi^N(v_0, v_1)\|^3_{L^3(\mathbb{R}_t;L^6(\mathbb{T}^3))}\langle s\rangle^{3\widetilde{\delta}}ds \\
\Big((v_0, v_1) \in E_N\Big) \quad &\leqslant CN^{3(\varepsilon - s)}\langle t\rangle^{3\widetilde{\delta}} \\
\Big(t \approx N^{s-2\varepsilon}\Big) \quad &\leqslant CN^{3(\varepsilon - s + \widetilde{\delta}(s-2\varepsilon))} \\
&\leqslant C
\end{aligned}$$

provided $\varepsilon - s + \widetilde{\delta}(s - 2\varepsilon) \leqslant 0$, which is true because of the hypothesis $\widetilde{\delta} > 1$.

Let us now prove the following statement :

$$\|\langle s\rangle^\delta\|_{L^2([0,t])} \leqslant \langle t\rangle^{\delta+1/2}$$

Indeed we have :

$$\begin{aligned}
\|\langle s\rangle^{\delta}\|_{L^2([0,t])} &= \left(\int_0^t (1+|s|)^{2\delta} ds\right)^{\frac{1}{2}} \\
&\leqslant \left(\frac{1}{2\delta+1}(1+|t|)^{2\delta+1}\right)^{\frac{1}{2}} \\
&\leqslant (1+|t|)^{\delta+\frac{1}{2}} = \langle t\rangle^{\delta+\frac{1}{2}}
\end{aligned}$$

Next we have (using the Cauchy-Schwarz inequality in time) that for $(v_0, v_1) \in E_N$

$$\begin{aligned}
\Big|\int_0^t f_N(s) ds\Big| &= \Big|\int_0^t \langle s\rangle^{-\delta} f_N(s)\langle s\rangle^{\delta} ds\Big| \\
&\leqslant \|\langle s\rangle^{-\delta} f_N\|_{L^2([0,t])}\|\langle s\rangle^{\delta}\|_{L^2([0,t])} \\
&\leqslant \|\langle s\rangle^{-\delta} f_N\|_{L^2(\mathbb{R})}\langle t\rangle^{\delta} \\
((v_0,v_1)\in E_N) \qquad &\leqslant CN^{-s+\varepsilon}\langle t\rangle^{\delta} \\
&\leqslant CN^{-s+\varepsilon+(\delta+1/2)(s-2\varepsilon)} \\
&\leqslant C
\end{aligned}$$

provided $-s+\varepsilon+(\delta+1/2)(s-2\varepsilon) \leqslant 0$, which is satisfied because of the hypothesis on δ.

For $(v_0, v_1) \in E_N$ we have

$$\begin{aligned}
\mathcal{E}(w_N(0))^{1/2} &= \mathcal{E}\big(\Pi_N(v_0)\big)^{1/2} \\
&= \left(\int_{\mathbb{T}^3} \Pi_N(v_1)^2 + |\nabla_x \Pi_N(v_0)|^2 + \frac{1}{2}\Pi_N(v_0)^4 dx\right)^{\frac{1}{2}} \\
&\leqslant C\big(\|\Pi_N(v_0,v_1)\|_{\mathcal{H}^1} + \|\Pi_N(v_0)\|_{L^4}^2 \\
&\leqslant CN^{1-s+\varepsilon}
\end{aligned}$$

Thus coming back to (4.11) we obtain

$$\big(\mathcal{E}(w_N(t))\big)^{1/2} \leqslant Ce^{C}(CN^{1-s+\varepsilon}+C) \leqslant CN^{1-s+\varepsilon}.$$

Recall that

$$v(t) = w_N(t) + S(t)\Pi^N(v_0,v_1) = S(t)\Pi^0(v_0,v_1) + w_N(t) - S(t)\Pi_N\Pi^0(v_0,v_1).$$

We find that the linear energy $\|\nabla_x u\|_{L^2(\mathbb{T}^3)}^2 + \|\partial_t u\|_{L^2(\mathbb{T}^3)}^2$ of a solution to the linear wave equation is independant of time, and if $(u, \partial_t u) = \Pi^0(u, \partial_t u)$ (i.e $(u, \partial_t u)$ is orthogonal to constants) then this energy controls the $\mathcal{H}^1(\mathbb{T}^3)$ norm. We deduce that for $(v_0, v_1) \in E_N \subset F_N$:

$$\|S(t)\Pi_N\Pi^0(v_0,v_1)\|_{\mathcal{H}^1(\mathbb{T}^3)} \leqslant CN^{1-s+\varepsilon}$$

and therefore
$$\|v(t)-S(t)\Pi^0(v_0,v_1)\|_{\mathcal{H}^1(\mathbb{T}^3)} \leqslant CN^{1-s+\varepsilon}$$
and this completes the proof of the lemma. □

Now let us end the proof of the proposition.

We set
$$E^N = \bigcap_{M\geqslant N} E_M$$
where the intersection is taken over the dyadic values of M, i.e $M = 2^j$ with $j \in \mathbb{N}$. Then
$$\begin{aligned} \forall M, \mu(E_M^c) \leqslant Ce^{-cM^\kappa} \underset{M\to\infty}{\longrightarrow} 0 \\ \Longrightarrow \mu((E^N)^c) \underset{N\to\infty}{\longrightarrow} 0 \\ \Longrightarrow \mu(E^N) \underset{N\to\infty}{\longrightarrow} 1 \end{aligned}$$
We use the lemma 4.3.2 to conclude that there exists $C > 0$ such that for every $t \geqslant 1$, every N, and every $(v_0, v_1) \in E^N$
$$\|v(t)-S(t)\Pi^0(v_0,v_1)\|_{\mathcal{H}^1(\mathbb{T}^3)} \leqslant C(N^{1-s+\varepsilon} + t^{\frac{1-s}{s}+\varepsilon_1}).$$
Finally we set
$$E = \bigcup_{N=1}^{\infty} E^N.$$

We have thus shown μ-almost sur bounds on the possible growth of the Sobolev norms of the solutions established in the previous section for data in E, which is of full μ-measure.

This completes the proof of the proposition. □

5 Probabilistic continuity of the flow

In this section we will see an interesting result about the continuity of the flow. Indeed it is possible to show that we don't have the deterministic continuity of the flow. However, we are able to prove that it is still continuous *in probability*. Consequently the Cauchy problem (3.1) is globally well posed in the following Hadamard-probabilistic sense.

Theorem 5.0.1. *Fix* $s \in (0,1)$*, let* $A > 0$*, let* $B_A := \{V \in \mathcal{H}^s : \|V\|_{\mathcal{H}^s} \leqslant A\}$ *and let* $T > 0$*. Let* $\mu \in \mathcal{M}^s = \bigcup_{(u_0,u_1)\in\mathcal{H}^s}\{\mu_(u_0,u_1)\}$ *and suppose that the joint distribution* θ *is symmetric. Let* $\phi(t)$ *be the flow of the cubic wave equations defined* μ*-almost everywhere in the theorem 3.4.1.*

Then, for $\eta, \varepsilon > 0$*, we have :*

$$\begin{aligned}\mu \otimes \mu\big(\{(V,V') \in &\mathcal{H}^s \times \mathcal{H}^s : \|\phi(t)(V) - \phi(t)(V')\|_{X_T} > \varepsilon \big| \\ &\|V - V'\|_{\mathcal{H}^s} < \eta \text{ and } (V,V') \in B_A \times B_A\}\big) \leqslant g(\varepsilon,\eta),\end{aligned} \tag{5.1}$$

where $X_T := \big(\mathcal{C}([0,T];\mathcal{H}^s) \cap L^4([0,T]\times\mathbb{T}^3)\big) \times \mathcal{C}([0,T];\mathcal{H}^{s-1})$ *and* $g(\varepsilon,\eta)$ *is such that*

$$\forall \varepsilon > 0, \quad \lim_{\eta\to 0} g(\varepsilon,\eta) = 0.$$

Moreover, if in addition the support of μ *is the whole* $\mathcal{H}^s$ *(which is true iff in the definition of* μ *we have* $a_i, b_{n,i}, c_{n,i} \neq 0$ *for all* $n \in \mathbb{Z}^3$ *and the support of the distribution function of the random variables is* $\mathbb{R}$*), then there exists* $\varepsilon > 0$ *such that the left-handed side in* (5.1) *is positive.*

This theorem shows that as soon as $\eta \ll \varepsilon$, among the initial data which are η-close to each other, the probability of finding two for which the coresponding solutions to (3.1) do not remain ε-close to each other, is very small.

6 Aknowledgements

I would like to thank Mr. Nicola Visciglia for the internship that he allows me to do under his direction, for the very interesting article he gave me to study and for his availability when I needed him. I also would like to thank Mr. Arnaud Debussche who helped me to find an internship. Finally I would like to thank the Univeristy of Pisa which gave me all I needed to work in good work conditions.

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