École Normale Supérieure de Rennes

Internship report

Control theory and long time study

Rudy Morel

Supervised by
Piermarco Cannarsa and
Alessio Porretta
University of Rome
Tor Vergata.

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Introduction

The control theory consists in studying controlled systems, these are dynamical systems on which we act on by a command, also called control. Depending on the situations, this control aims for bringing the system to a certain state as quickly as possible for instance or minimizing a certain energy of the system. In this last case, the general problem is the following. For some times going from 0 to T, we are given a dynamics :

$$\begin{cases} x_t = f(x,u) \\ x(0) = x_0, \end{cases}$$

and we are looking for a control u minimizing the quantity :

$$J^T(u) = \int_0^T L(x(t), u(t))dt.$$

Under some hypothesis we will precise later, there exists a unique control u and a unique trajectory, solution of the problem with time horizon T.

During my internship, my work was primarily based on the study of an article (cf [AP13]) in which we study the evolution of the couple (u, x) solution of the problem as the time horizon T goes to $+\infty$. Firstly, my work was to learn the basis of control theory, then to deeply understand the article and finally to prove a similar result of the one set out in the article but in another context.

This report is voluntarily precise on certain points and brief on others, so that it doesn't include all my work during two months. In the first section, we are interested in some convergence results for finite dimensionnal systems, many mecanical problems are taken into account by this theory. In the second section, we try to generalize this approach to prove similar results for infinite dimensionnal systems, which are PDE problems (see theorem 2.2). Finally, in the third section, we show that these results can be adapted to the one dimensionnal heat equation with boundary control. If the reader of this report wishes more details about the basis of optimal control theory than these provided in the first section, he can refer to the excellent book of Emmanuel Trélat [Tre05], a copy of which is on his personnal web page.

I would like to give my warmest thanks to my internship supervisors who seem to be not only excellent mathematicians, but also people with undeniable human qualities. During our appointments, they make me learn a lot of methods in analysis and control theory. I am extremely grateful they took so much time to help me and to talk with me about very interesting mathematical facts.

1 Control theory and convergence results for finite dimensionnal systems

In this section, we introduce the control theory and we restrict ourselves to linear case with quadratic cost. Once the framework and the basics of control theory are set, we prove convergence results as the time horizon goes to infinity.

1.1 Problem statement and basic properties

In the entire section, we are given two dimensions $n, m \geq 1$ and three matrices $A \in M_n(\mathbb{R})$, $B \in M_{n,m}(\mathbb{R})$ and $C \in M_n(\mathbb{R})$. We consider the following control system on the interval $[0, T]$ with $T \geq 0$:

$$\begin{cases} x_t + Ax = Bu \quad \text{in} \quad L^2(0,T;\mathbb{R}^n) \\ x(0) = x_0 \end{cases} \tag{1.1}$$

where the control u is taken in $\in L^2(0,T;\mathbb{R}^m)$, $x_0 \in \mathbb{R}^n$ and x_t is the derivative of x (considered as a distribution).

Here, taking the derivative in L^2 is more flexible than taking classic derivation and imposing u continuous et x continuously diffrentiable. Thus, we will be considering Sobolev spaces which have been well studied.

Some Cauchy-Lipschitz theorems give us the existence and uniqueness of a solution $x \in H^1(0,T;\mathbb{R}^n)$ of the system (1.1) for a given u (cf [Tre05],p197). Besides, one has the formula :

$$\forall t \in [0,T], \quad x(t) = e^{-At}x_0 + \int_0^t e^{-(t-s)A}Bu(s)ds. \tag{1.2}$$

Consider $z \in \mathbb{R}^n$ et $C \in M_n(\mathbb{R})$, the fonctional we want to minimize is :

$$J^T(u) = \int_0^T (|u(t)|^2 + |Cx(t) - z|^2)dt.$$

The aim is to bring the observation Cx as close as possible to the target z while minimizing the norm of the control which enables it.

Definition 1.1. *A control $u \in L^2(0,T;\mathbb{R}^m)$ is said to be optimal if it minimizes the fonctional J^T among the controls $L^2(0,T;\mathbb{R}^m)$. The trajectory x associated is also said to be optimal.*

Exemple. *Take the the harmonic oscillator in one dimension. Let's say we have an object attached to a spring with position $y(t)$. Assume that the object perfectly oscillates on the interval $]-\infty, 0[$. Starting at 0, we would like to change his phase and add $\frac{\pi}{2}$ to it. Assume that $y(0) = 0$ et $y_t(0) = 1$. Without constraints one would* *have $y(t) = sin(t)$. However, we apply a force u on the object to bring it as close as possible to the shifted trajectory : $\rho(t) := cos(t)$ in $[0,T]$. Thus, y verifies :*

$$y_{tt} + y = u.$$

For this particular problem we have then $n = 2$, $m = 1$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $x(t) = (y(t) - \rho(t), y_t(t) - \rho_t(t))^*$ et $z = (0,0)^*$.

We are also interested in the stationnary problem : finding $\bar{x} \in \mathbb{R}^n$ and $\bar{u} \in \mathbb{R}^m$ such that,

$$A\bar{x} = B\bar{u}$$

and minimizing the functional :

$$J_s(x,u) = |u|^2 + |Cx - z|^2.$$

For the evolution problem we have the following result.

Proposition 1.1. *There exists a unique optimal control $u \in L^2(0,T;\mathbb{R}^m)$ minimizing the functional J^T.*

Remark 1.1. *Thus, there exists a unique trajectory x associated to the unique optimal control u minimizing J^T. These functions depend on T.*

To prove it, we use the weak compactness of the bounded sets in $L^2(0,T;\mathbb{R}^m)$ for the existence and we use the strict convexity of J^T for the uniqueness. A precise proof can be found in [Tre05](theorem 4.9, p50).

The existence and uniqueness of such an optimal trajectory is linked to the existence of an adjoint state.

Proposition 1.2. *The control u giving the trajectory x is optimal if and only if there exists an adjoint state $p \in L^2(0,T;\mathbb{R}^n)$ verifying :*

$$\begin{cases} -p_t + A^*p = C^*(Cx - z) \quad in \quad L^2(0,T;\mathbb{R}^n) \\ p(T) = 0 \end{cases} \tag{1.3}$$

with the law :

$$u = -B^*p. \tag{1.4}$$

Proof. We should see what characterizes a minimum u of the cost function J^T. Consider $u \in L^2(0,T;\mathbb{R}^n)$ and $\delta u \in L^2(0,T;\mathbb{R}^n)$ a perturbation of u. We define δx the perturbation of x associated. It is the solution of : $(\delta x)' + A\delta x = B\delta u$, $\delta x(0) = 0$. According to the formula (1.2), δx is negligible with respect to δu in L^2

when δu is small. We notice that the trajectory associated to the control $u + \delta u$ is $x + \delta x$ by linearity. Thus one has :

$$\begin{aligned} J^T(u+\delta u) &= \frac{1}{2}\int_0^T |u+\delta u|^2 dt + \frac{1}{2}\int_0^T |Cx + C\delta x - z|^2 dt \\ &= J^T(u) + \int_0^T \langle u, \delta u\rangle\, dt + \int_0^T \langle Cx - z, C\delta x\rangle\, dt + o(\delta u). \end{aligned}$$

We can prove that J^T is strictly convex, especially because for all t the function associating $x(t)$ to a control u is convex. In this case, u is the optimal control (giving the optimal trajectory x) if and only if :

$$\forall \delta u \in L^2(0,T;\mathbb{R}^m), \quad \int_0^T \langle u, \delta u\rangle\, dt + \int_0^T \langle Cx - z, C\delta x\rangle\, dt = 0. \tag{1.5}$$

Let's introduce p the solution of the problem (1.3). We have :

$$p(t)^* = \Lambda e^{tA} + \int_0^t (Cx(s) - z)^* C e^{(t-s)A} ds \tag{1.6}$$

where $\Lambda = \int_0^T (Cx(s) - z)^* C e^{-sA}$.
We develop the second term in (1.5) :

$$\begin{aligned} \int_0^T \langle Cx - z, C\delta x\rangle\, dt &= \int_0^T \left\langle Cx(t) - z, Ce^{-tA}\left(\int_0^t e^{sA} B\delta u(s) ds\right)\right\rangle dt \\ &= \int_0^T \left\langle e^{sA} B\delta u(s), \int_s^T e^{-tA^*} C^*(Cx(t) - z) dt\right\rangle ds \\ &= \left\langle \int_0^T e^{-tA^*} C^*(Cx(t) - z) dt, \int_0^T e^{sA} B\delta u(s) ds\right\rangle \\ &- \int_0^T \left\langle Ce^{sA} B\delta u(s), \int_0^s (Cx(t) - z) e^{-tA} dt\right\rangle ds \\ &= \left\langle \Lambda^*, \int_0^T e^{sA} B\delta u(s) ds\right\rangle - \int_0^T \left\langle e^{sA} B\delta u(s), \int_0^s e^{-tA^*} C^*(Cx(t) - z) dt\right\rangle ds \\ &= \int_0^T \left\langle \Lambda^* - \int_0^s e^{-tA^*} C^*(Cx(t) - z) dt, e^{sA} B\delta u(s)\right\rangle ds \\ &= \int_0^T p(s)^* B\delta u(s) ds. \end{aligned}$$

So that the assertion (1.5) is equivalent to :

$$\forall \delta u \in L^2(0,T;\mathbb{R}^m), \quad \int_0^T \langle u + B^* p, \delta u\rangle\, dt = 0. \tag{1.7}$$

Therefore, u is optimal if and only if $u = -B^* p$, which ends the proof. $\square$

The point of this characterization is that it provides coupled equations on x and p allowing us to establish inequalities on the norms L^2 of x and p as we will proceed in the report. The relation established between p and u we may also have estimations on u.

Remark 1.2. *The construction of the adjoint state intervins in many control problems, which are not necessarily linear, with non-necessarily quadratic functional. A general method to obtain the equation verified by the adjoint state is the so-called Pontryagin's maximum principle.*

We need some hypothesis on the matrices A, B, et C, leading to useful inequalities for the following proves and enabling us to solve the stationnary problem.

Definition 1.2. *The couple (A, C) is said to be observable in time T if there exists a constant $\gamma > 0$ such that for every solution p of*

$$\begin{cases} -p' + Ap = 0, \\ p(T) = p_0 \end{cases}$$

one has :

$$|p_0|^2 \leq \gamma \int_0^T |Cp|^2 dt.$$

Remark 1.3. *The definition of observability for (A, C) is equivalent to the Kalman condition : $rg[C\ CA\ CA^2 ...\ CA^{n-1}] = n$. In particular, this definition doesn't depend on the time T and $(-A, C)$ is also observable (that we could have noticed directly on our definition). The Kalman condition also characterizes the controllability of (A^*, C^*) which is another field of the control theory in which we don't want to minimize a functional but we study the set of accessible states, which means the $x(T)$ with u varying.*

In what follows we assume :

$$\begin{cases} (A, C) \quad \text{is observable,} \\ (A^*, B^*) \quad \text{is observable.} \end{cases} \tag{1.8}$$

Thanks to this hypothesis we deduce the next property, useful in what follows.

Proposition 1.3. *We assume that $T \geq 1$. For all $f \in L^2(0, T; \mathbb{R}^n)$ and all y solution of*

$$\partial_t y + Ay = f \ \ sur\ [0, T]$$

one has :

$$y(T) \leq \gamma \int_{T-1}^T (|Cy|^2 + |f|^2) dt.$$

Proof. Consider $f \in L^2(0,T;\mathbb{R}^n)$ and y a solution of the system. Let w be the solution of :

$$\begin{cases} \partial_t w + Aw = f \quad \text{in} \quad L^2(T-1,T;\mathbb{R}^n) \\ w = 0 \quad \text{in} \quad [0,T-1] \end{cases}$$

Then, $\tilde{y} := y - w$ verifies :

$$\partial_t \tilde{y} + A\tilde{y} = 0 \text{ dans } L^2(0,T;\mathbb{R}^n).$$

Besides, by energy estimates on w, one has :

$$\forall \tau \in [T-1,T],\ |w(\tau)|^2 \leq c \int_{T-1}^{\tau} |f|^2 dt.$$

So that, thanks to the observability inequality on $\tilde{y}$ one gets :

$$\begin{aligned} |y(T)|^2 &\leq 2\ (|\tilde{y}(T)|^2 + |w(T)|^2) \\ &\leq c\ \left(\int_{T-1}^{T} |C\tilde{y}|^2 dt + \int_{T-1}^{T} |f|^2 dt \right) \\ &\leq c\ \left(2\int_{T-1}^{T} |Cy|^2 dt + 2||C\| \int_{T-1}^{T} (\int_{T-1}^{\tau} |f|^2 ds) d\tau + \int_{T-1}^{T} |f|^2 dt \right) \\ &\leq c\ \int_{T-1}^{T} (|Cy|^2 + |f|^2) dt \end{aligned}$$

□

An immediate consequence of this inequality is obtained by taking y constant, then $f = Az$ is constant. One has the stationnary inequality due to the observability of (A,C) :

$$\forall y \in \mathbb{R}^n,\ |y|^2 \leq c\ (|Ay|^2 + |Cy|^2). \tag{1.9}$$

This enables us to solve the stationnary problem.

Theorem 1.1. *There exists a unique $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ verifying $A\bar{x} = B\bar{u}$ and minimizing the functional J_s.*

Proof. Firstly, we proove the existence. We write $S = \{(x,u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Ax = Bu\}$. This set is closed and non-empty because $(0,0) \in S$. According to (1.9), the level set $\{J_s \leq J_s(0,0)\} \cap S$ is non-empty and bounded because :

$$\forall x \in S,\ |x|^2 \leq c\ (|Ax|^2 + |Cx|^2) \leq c\ (\|B\|^2 |u|^2 + |Cx|^2) \leq c\ (|u|^2 + |Cx - z|^2 + |z|^2).$$

Thus, this level set is compact because J_s is continuous. We deduce the existence of a minimum.

Then we proove the uniqueness. We notice that J^T can be written as $J_s(x,u) = f(u) + g(Cx)$, where f and g are strictly convex functions. Thus, if f has a minimum at (x_1, u_1) and (x_2, u_2) one has : $u_1 = u_2$ and $Cx_1 = Cx_2$. This implies $Ax_1 = Ax_2 = Bu_1$. Therefore, according to the stationnary inequality (1.8) applied to $x_1 - x_2$ one gets $x_1 = x_2$, hence the uniqueness. □

Since $(\bar{x}, \bar{u})$ minimizes J_s, the differential is zero on S :

$$\forall (v, \varphi) \in \mathbb{R}^n \times \mathbb{R}^m \text{ such that } A\varphi = Bv, \quad \langle \bar{u}, v\rangle + \langle C\bar{x} - z, C\varphi\rangle = 0. \tag{1.10}$$

1.2 Average convergence

Taking $v = 0$ in the expression (1.10) one gets : $C^*(C\bar{x} - z) \in \mathrm{Ker}(A)^\perp = \mathrm{Im}(A^*)$. So that we can choose some $\bar{p} \in \mathbb{R}^n$ such that :

$$A^*\bar{p} = C^*(C\bar{x} - z).$$

For such a $\bar{p}$ we have, according to (1.10):

$$\forall (v, \varphi) \in \mathbb{R}^n \times \mathbb{R}^m \text{ tel que } A\varphi = Bv, \quad \langle \bar{u}, v\rangle + \langle \bar{p}, Bv\rangle = 0. \tag{1.11}$$

Theorem 1.2. *Under the observability hypothesis* (1.2) *we have :*

$$\frac{1}{T} \min_{u \in L^2} J^T \underset{T\to+\infty}{\longrightarrow} \min_{(x,u)\in S} J_s$$

and

$$\frac{1}{T}\int_0^T (|u(t) - \bar{u}|^2 + |C(x(t) - \bar{x})|^2) dt = \underset{T\to\infty}{\mathcal{O}}(\frac{1}{T}).$$

Remark 1.4. *This results tells us that in a long-term perspective, the optimal way of controlling the system* (1.1) *is the same in average.*

Proof. One has the coupled system :

$$\begin{cases} \partial_t(x - \bar{x}) + A(x - \bar{x}) = B(u - \bar{u}) \\ -\partial_t(p - \bar{p}) + A^*(p - \bar{p}) = C^*C(x - \bar{x}). \end{cases} \tag{1.12}$$

Integrating $\langle x - \bar{x}, \partial_t(p - \bar{p})\rangle$ from 0 to T one gets :

$$\int_0^T |C(x - \bar{x})|^2 dt = \langle x(T) - \bar{x}, \bar{p}\rangle + \langle x_0 - \bar{x}, p(0) - \bar{p}\rangle + \int_0^T \langle B(u - \bar{u}), p - \bar{p}\rangle\, dt. \tag{1.13}$$

So that using $u=-B^*p$ (cf proposition 1.2) and $\langle \bar{u},\bar{u}\rangle + \langle \bar{p}, B\bar{u}\rangle = 0$ (1.11), one obtains :

$$\int_0^T (|u-\bar{u}|^2 + |C(x-\bar{x})|^2)dt = \tag{1.14}$$

$$\langle x(T)-\bar{x},\bar{p}\rangle + \langle x_0-\bar{x}, p(0)-\bar{p}\rangle - \int_0^T \langle u, \bar{u}+B^*\bar{p}\rangle\, dt. \tag{1.15}$$

The aim is now to control this expression. We use the observability hypothesis (1.8) which writes :

$$|x(T)-\bar{x}|^2 \leq c\int_0^T (|u-\bar{u}|^2 + |C(x-\bar{x})|^2)dt \tag{1.16}$$

and

$$|p(0)-\bar{p}|^2 \leq c\int_0^T (|u-\bar{u}|^2 + |B^*(p-\bar{p})|^2)dt. \tag{1.17}$$

Combining these three inequalities, one gets :

$$\int_0^T (|u-\bar{u}|^2 + |C(x-\bar{x})|^2)dt \leq cT \tag{1.18}$$

where the constant c does not depend on T.

Now, we want to improve this estimation. Firstly, it enables us to deduce that the average $\frac{1}{T}\int_0^T udt$ and $\frac{1}{T}\int_0^T Cxdt$ are bounded in T.

Integrating the evolution equation one obtains :

$$\frac{1}{T}\int_0^T Axdt = \frac{1}{T}\int_0^T Budt - \frac{1}{T}(x(T)-x_0). \tag{1.19}$$

However, the second term on the right goes to 0 when T goes to $+\infty$ thanks to (1.16) and (1.18) so $\frac{1}{T}\int_0^T Axdt$ is also bounded. Thus, applying the stationnary inequlity (1.9) to $\frac{1}{T}\int_0^T xdt$ we deduce $\frac{1}{T}\int_0^T xdt$ is bounded in T.

Let φ and v be eigenvalues at $+\infty$ of the average $\frac{1}{T}\int_0^T xdt$ and $\frac{1}{T}\int_0^T udt$, for the sequence $(T_n)_n$. According to (1.19) we have then : $A\varphi = Bv$, which enables us to deduce from (1.11) that :

$$\frac{1}{T_n}\int_0^{T_n} u\cdot(\bar{u}+B^*\bar{p})dt \underset{n\to+\infty}{\longrightarrow} v\cdot\bar{u} + v\cdot B^*\bar{p} = 0.$$

Then, going back to (1.14), (1.15) we have :

$$\frac{1}{T_n}\int_0^{T_n} (|u-\bar{u}|^2 + |C(x-\bar{x})|^2)dt \leq \frac{c}{\sqrt{T_n}} - \frac{1}{T_n}\int_0^{T_n} u\cdot(\bar{u}+B^*\bar{p})dt \underset{n\to+\infty}{\longrightarrow} 0.$$

By the Cauchy-Schwarz inequality we get :

$$\frac{1}{T_n}\int_0^{T_n} Cxdt \underset{n\to+\infty}{\longrightarrow} C\bar{x} = \varphi \quad \text{et} \quad \frac{1}{T_n}\int_0^{T_n} udt \underset{n\to+\infty}{\longrightarrow} \bar{u} = v.$$

This is true for all eigenvalues φ and v so we deduce the convergence :

$$\frac{1}{T}\int_0^{T} Cxdt \underset{T\to+\infty}{\longrightarrow} C\bar{x} \quad \text{et} \quad \frac{1}{T}\int_0^{T} udt \underset{T\to+\infty}{\longrightarrow} \bar{u}.$$

Therefore we have : $\frac{1}{T}\int_0^T Budt \underset{T\to+\infty}{\longrightarrow} B\bar{u}$. According to (1.19) we also deduce that : $\frac{1}{T}\int_0^T Axdt \underset{T\to+\infty}{\longrightarrow} A\bar{x}$. Recalling the stationnary inequality (1.9), these convergence bring finally :

$$\frac{1}{T}\int_0^{T} xdt \underset{T\to+\infty}{\longrightarrow} \bar{x}.$$

This result is a first result of convergence in average.

Similarly, we can prove convergence results for the average of the adjoint state p using for this the observability hypothesis (1.8) on (A^*, B^*). We can prove that $\frac{1}{T}\int_0^T pdt$ converges to a certain limit we note $\tilde{p}$. Besides, taking the limit in :

$$A^*\left(\frac{1}{T}\int_0^T pdt\right) = \frac{1}{T}\int_0^T C^*(Cx-z)dt - \frac{p(0)}{T}$$

we have that $\tilde{p}$ verifies : $A\tilde{p} = C^*(C\bar{x}-z)$.

Therefore $\tilde{p}$ verifies the same equation (1.11) than $\bar{p}$, so that we can apply all the above reasonning choosing now $\bar{p} = \hat{p}$. As we have $u = -B^*p$, we get by integrating and taking the limit : $\bar{u} = -B^*\bar{p}$. Thus, going back to (1.14), (1.15), this choice of $\bar{p}$ makes the second term equal zero :

$$\int_0^T (|u(t)-\bar{u}|^2 + |C(x(t)-\bar{x})|^2)dt \leq c \tag{1.20}$$

where the constant c does not depend on T.

Finally, by the Cauchy-Schwarz inequality :

$$\frac{1}{T}\min_{u\in L^2} J^T \underset{T\to+\infty}{\longrightarrow} \min_{(x,u)\in S} J_s.$$

□

1.3 Pointwise convergence

In order to obtain more precise convergence results on the trajectory x and the control u we first consider $z=0$ and we prove that the optimal control u is linked directly to the trajectory x by a linear fonction $\mathcal{E}$: $u = -B^*\mathcal{E}(T-t)x$. Thus we can obtain pointwise estimations by controling the norm of $\mathcal{E}$.

Proposition 1.4. *If $z = 0$ then we have the feedback law :*

$$u(t) = -B^*\mathcal{E}(T-t)x(t) \tag{1.21}$$

where $\mathcal{E} \in \mathcal{C}^1([0, +\infty[, M_n(\mathbb{R}))$ is solution of the Riccati equation :

$$\begin{cases} \mathcal{E}_t = C^*C - (\mathcal{E}A + A^*\mathcal{E}) - \mathcal{E}BB^*\mathcal{E} \quad in \quad]0, +\infty[\\ \mathcal{E}(0) = 0, \end{cases} \tag{1.22}$$

with $\mathcal{E}(t)$ symmetric for all $t \geq 0$.

Proof. By Cauchy-Lipschitz theorem, there is a unique solution $\mathcal{C}^1$ of (1.22) defined on a maximal interval I including 0. For all t in this definition interval, $\mathcal{E}(t)$ is symmetric. This can be proved by local uniqueness.

To prove locally the formula (1.21) we use the characterization given by the proposition 1.2. We write y the solution of the problem :

$$\begin{cases} y_t + Ay = -BB^*\mathcal{E}(T-t)y \\ y(0) = x_0. \end{cases}$$

We define $v := -B^*\mathcal{E}(T-t)y$ and $q := \mathcal{E}(T-t)y$.
We have $q(T) = 0$. Besides :

$$\begin{aligned} \forall t \in I, \quad q_t(t) &= \mathcal{E}_t(T-t)y(t) - \mathcal{E}(T-t)y_t(t) \\ &= C^*Cy(t) - \mathcal{E}(T-t)Ay(t) - A^*\mathcal{E}(T-t)y(t) \\ &- \mathcal{E}(T-t)BB^*\mathcal{E}(T-t)y(t) - \mathcal{E}(T-t)Bv(t) + \mathcal{E}(T-t)Ay(t) \\ &= CC^*y(t) + A^*q(t). \end{aligned}$$

Thus, the triplet (y, v, q) verifies the hypothesis of proposition 1.2 (with $z = 0$). So v is optimal, and by uniqueness, $v = u$ then $y = x$ and finally $q = p$.

It remains to be proven that $\mathcal{E}$ is defined on $[0, +\infty[$ thanks to the explosion theorem. Consider $t_0 \leq T$, and consider the problem of control in $[0, t_0]$. As we saw, the optimal trajectory and the adjoint state are solutions of :

$$\begin{cases} x_t + Ax = -BB^*p \\ -p_t + A^*p = C^*Cx \\ x(0) = x_0, \ p(t_0) = 0, \end{cases}$$

and we have $x_0^*\mathcal{E}(t_0)x_0 = \langle x_0, p(0)\rangle$.

Consider x_0 with $|x_0| \leq 1$. By the well-posedness of the problem, we have the inequality : $|p(0)| \leq c|x_0|$. Therefore, $p(0)$ is uniformly bounded on $t_0 \in [0, T]$. Thus, $x_0^*\mathcal{E}(t_0)x_0$ is uniformly bounded for any $|x_0| \leq 1$ and $t_0 \leq T$. Since $\mathcal{E}$ is symmetric, it proves that $\mathcal{E}(t)$ is bounded for finite time horizon. So the function $\mathcal{E}$ is defined on $[0, +\infty[$ because we have taken arbitrary T. □

Proposition 1.5. *The function $\mathcal{E}$ verifies the following properties :*

(i) $\forall t_1 \le t_2,\ \mathcal{E}(t_1) \le \mathcal{E}(t_2)$,

(ii) there exists $M > 0$ such that : $\forall t \in]0, +\infty[,\ 0 < \mathcal{E}(t) \le M$,

(iii) when $T \to +\infty$, $\mathcal{E}$ converges to a symmetric matrix written $\hat{E}$, solution of

$$\hat{E}A + A^*\hat{E} + \hat{E}BB^*\hat{E} = C^*C, \tag{1.23}$$

(iv) the following linear system is asymptotically stable

$$\begin{cases} \hat{E}_t + (A + BB^*\hat{E})x = 0 \\ x(0) = x_0, \end{cases} \tag{1.24}$$

(v) the convergence of $\mathcal{E}$ to $\hat{E}$ is exponential :

$$\exists c, \mu > 0,\ \forall t > 0,\ \|\mathcal{E}(t) - \hat{E}\| \le ce^{-2\mu t},$$

Proof. (i) The proof of this point will be the occasion to prove an important property satisfied by the function $\mathcal{E}$: the minimum of J^T for the problem on $[0, T]$ is equal to $x_0^*\mathcal{E}(T)x_0$. Indeed, by a similar calculus to (1.13) one has :

$$\langle x_0, p(0)\rangle = \int_0^T -(\langle x_t, p\rangle + \langle p_t, x\rangle)dt = \int_0^T (|u|^2 + |Cx|^2)dt,$$

which means :

$$x_0^*\mathcal{E}(T)x_0 = \min_{u \in L^2(0,T;\mathbb{R}^m)} J(u).$$

Consider $t_1 \le t_2$. We write x_2 and u_2 the optimal trajectory and control on $[0, t_2]$. In particular, they verify the equations (1.1) in $[0, t_1]$ so that :

$$\min_{u \in L^2(0,t_2;\mathbb{R}^m)} J^{t_2}(u) = J^{t_2}(u_2) \ge J^{t_1}(u_1) \ge \min_{u \in L^2(0,t_1;\mathbb{R}^m)} J^{t_1}(u).$$

therefore, by property of $\mathcal{E}(t)$:

$$\forall x_0 \in R^n, \quad x_0^*\mathcal{E}(t_2)x_0 \ge x_0^*\mathcal{E}(t_1)x_0.$$

(ii) Consider $x_0 \in \mathbb{R}^n$, we have : $x_0^*\mathcal{E}(T)x_0 = \min_{u \in L^2(0,T;\mathbb{R}^m)} J^T(u) = \int_0^T (|u|^2 + |Cx|^2)dt \ge 0$. We assume $x_0^*\mathcal{E}(T)x_0 = 0$ then x verifies :

$$\begin{cases} x_t + Ax = 0 \quad \text{in} \quad L^2(0, T; \mathbb{R}^n) \\ Cx = 0. \end{cases}$$

The couple $(-A, C)$ being observable (cf remark 1.3), we have : $|x(T)|^2 \leq c\int_0^T |Cx|^2 dt = 0$. So, $x = 0$ by uniqueness and $x_0 = 0$. Thus, for all $t > 0$, $\mathcal{E}(t)$ is positive-definite.

In order to prove that $(\mathcal{E}(t))_{t\geq 0}$ is uniformly bounded, we use a controllability result : for all $x_0 \in \mathbb{R}^n$ there exists a command u bringing x_0 to 0 during a time equal to 1.
Indeed, we consider the matrix $G = \int_0^1 e^{-(1-s)A}BB^*(e^{(1-s)A})^* ds$. We can prove that G is invertible because the condition $Gx \cdot x = 0$ implies $\forall s \in [0,1], (e^{(1-s)A}B)^*x = 0$ so $\forall i \in \mathbb{N}, (A^iB)^*x = 0$. Thanks to the Kalman condition (cf remark 1.3) we conclude that $x = 0$. Then we easily compute that $u : t \mapsto (e^{-tA}B)^*G^{-1}x_0$ is appropriate (cf (1.2)). We notice that the control u depends linearly on x_0.
Let's go back to the inequality we want to prove on $\mathcal{E}(T)$. If $T \geq 1$ Then we take the command bringing x_0 on 0 in time 1 and we extend it by zero from 1 to T. In this case, the associated trajectory goes from x_0 to 0 in the time interval $[0,1]$ and is constantly equal to 0 in $[1,T]$. Then we have

$$x_0^*\mathcal{E}(T)x_0 = \int_0^1 (|(e^{-tA}B)^*G^{-1}x_0|^2 + |Cx|^2)dt \leq M|x_0|^2,$$

where M is obtained by controlling the expression of x (1.2) and does not depend on T. If $T \leq 1$ then we use the point (i) to prove : $x_0^*\mathcal{E}(T)x_0 \leq x_0^*\mathcal{E}(1)x_0 \leq M|x_0|^2$.

(iii) According to (ii) we know that for all $x \in \mathbb{R}^n$, $x^*\mathcal{E}(t)x$ converges when t goes to $+\infty$. By the symmetry of $\mathcal{E}(t)$ we also have the convergence of $x^*\mathcal{E}(T)y$ for any $x, y \in \mathbb{R}^n$. This enables us to define a symmetric matrix $\hat{E}$ by $x^*\hat{E}y = \lim\limits_{t\to+\infty} x^*\mathcal{E}(t)y$. We have : $\|\mathcal{E}(t) - \hat{E}\| \underset{t\to+\infty}{\rightarrow} 0$. This implies :

$$\int_0^{+\infty} \mathcal{E}_t(t)dt = \hat{E}.$$

Since $\mathcal{E}$ is continuously differentiable, we deduce $\|\mathcal{E}_t(t)\| \underset{t\to+\infty}{\rightarrow} 0$. Therefore, taking the limit, $\hat{E}$ verifies the equation (1.23) .

(iv) According to a classical result in differential equation, it is sufficient to exhibit a strict Lyapunov function of the system (1.24). We can prove that $V(x) = x^*\hat{E}x$ is appropriate.

$\square$

Theorem 1.3. *Under the hypothesis* (1.8)*, there exists* $\lambda, K > 0$ *such that :*

$$\forall t \in [0,T], \quad |u(t) - \bar{u}| + |x(t) - \bar{x}| \leq K(e^{-\lambda t} + e^{-\lambda(T-t)}).$$

Remark 1.5. *This convergence result is better than* 1.2. *It shows that the optimal control and trajectory are divided in three parts. The first and the third one, at the beginning and at the end, are quick transition parts. The second part is longer, it is a quasi-stationnary part. Such a property is called Turnpike property and was firstly introduced in econometrics. This terminology is justified by the definition of the word Turnpike in English.*

Proof. We may use the previous results obtained with $z = 0$, for the case $z \neq 0$ where we do not necesseraly have $p(t) = \mathcal{E}(T-t)x(t)$. So we try to estimate the difference : $h(t) := p(t) - \bar{p} - \mathcal{E}(T-t)(x(t) - \bar{x})$, where we chose $\bar{p}$ verifying $\bar{u} = -B^*\bar{p}$ like in the proof of the convergence of the average.

Knowing that $x - \bar{x}$ and $p - \bar{p}$ satisfy

$$\begin{cases} (x-\bar{x})_t + A(x-\bar{x}) = B(u-\bar{u}) \\ -(p-\bar{p})_t + A^*(p-\bar{p}) = C^* j_V C(x-\bar{x}). \end{cases}$$

and according to the equations satisfyed by $\mathcal{E}, p$ and x, we have :

$$\begin{aligned} h_t &= (p-\bar{p})_t + \mathcal{E}_t(T-t)(x-\bar{x}) - \mathcal{E}(T-t)(x-\bar{x})_t \\ &= A^*(p-\bar{p}) - C^*C(x-\bar{x}) + C^*C(x-\bar{x}) - \mathcal{E}(T-t)A(x-\bar{x}) \\ &+ A^*\mathcal{E}(T-t)(x-\bar{x}) - \mathcal{E}(T-t)BB^*\mathcal{E}(T-t)(x-\bar{x}) \\ &+ \mathcal{E}(T-t)A(x-\bar{x}) + \mathcal{E}(T-t)BB^*(p-\bar{p}) \\ &= A^*h + \mathcal{E}(T-t)BB^*h. \end{aligned}$$

So that h is solution of the linear system :

$$\begin{cases} h_t = (A^* + \mathcal{E}(T-t)BB^*)h \quad \text{dans} \quad L^2(0,T;\mathbb{R}^n) \\ h(T) = -\bar{p}. \end{cases}$$

We write $M = A + BB^*\hat{E}$. Then the system writes :

$$\begin{cases} h_t = M^*h + (\mathcal{E}(T-t) - \hat{E})h \\ h(T) = -\bar{p}. \end{cases}$$

We can express h as :

$$\forall t \in [0,T], \quad h(t) = -e^{-(T-t)M^*}\bar{p} + \int_t^T e^{-(s-t)M^*}(\mathcal{E}(T-s) - \hat{E})BB^*h(s)ds.$$

Therefore, we can use the lemma (points (iv) and (v)):

$$\begin{aligned} \forall t \in [0,T], \quad |h(t)| &\leq |\bar{p}|e^{-\mu(T-t)} + c\int_t^T e^{-\mu(s-t)}e^{-2\mu(T-s)}\|h(s)\|ds \\ &\leq ce^{-\mu(T-t)}\left(1 + \int_t^T e^{-\mu(T-s)}|h(s)|ds\right). \end{aligned}$$

Using the Gronwall lemma we obtain :

$$\forall t \in [0,T], \quad |h(t)| \leq ce^{-\mu(T-t)}.$$

Thus, there is not feedback law giving p in function of x like in the case $z=0$ but still, we estimated the difference with this model. In order to obtain an estimation on $x-\bar{x}$, we go back to the evolution problem, we have :

$$(x-\bar{x})_t + (A+BB^*\hat{E})(x-\bar{x}) = BB^*(\hat{E}-\mathcal{E}(T-t))(x-\bar{x}) - BB^*h(t).$$

So that $x-\bar{x}$ verifies :

$$\forall t \in [0,T], \quad x(t)-\bar{x} = e^{-tM}(x_0-\bar{x}) + \int_0^t e^{-(t-s)M}K(s)ds$$

where $K(s) = BB^*(\hat{E}-\mathcal{E}(T-s))(x(s)-\bar{x}) - BB^*h(s)$. Using the inequality on h and the lemma (points (iv) et (v)) we finally obtain :

$$\forall t \in [0,T], \quad |x(t)-\bar{x}| \leq ce^{-\mu t} + \int_0^t (e^{-\mu(t-s)}e^{-2\mu(T-s)}|x(s)-\bar{x}| + e^{-\mu(t-s)}e^{-\mu(T-s)})ds.$$

But since $|x-\bar{x}|$ is bounded uniformly in T we get :

$$\forall t \in [0,T], \quad \|x(t)-\bar{x}\| \leq c(e^{-\mu t} + e^{-\mu(T-t)}).$$

However, we have by definition $p(t)-\bar{p} = \mathcal{E}(T-t)(x(t)-\bar{x}) + h(t)$ so $|p-\bar{p}| \leq c|x(t)-\bar{x}| + |h(t)| \leq c(e^{-\mu t} + e^{-\mu(T-t)})$, which ends the proof. □

2 Convergence results for PDE

In this section we try to establish similar results to PDE. In order to do so we will try to apply the same methods.

2.1 From the finite dimensionnal to the infinite dimensionnal problems

The PDE control problems are called infinite dimensionnal control problems. We will justify later such a terminology. Before this, we illustrate the kind of problems we encounter with an example.

Let's take the one dimensionnal controlled heat equation :

$$\begin{cases} u_t = u_{xx} + g \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

where u : $]0,T[\times]0,1[\to \mathbb{R}$ is the solution and g :$]0,T[\times]0,1[\to \mathbb{R}$ the control. We can prove that if $g \in L^2(]0,T[\times]0,1[)$, this problem admits a unique solution u satisfying $u, u_t \in L^2(]0,T[\times]0,1[)$ and for almost every t, $u(t,\cdot) \in H_0^1(0,1)$. Here, the derivative in time is partial.

Let consider u as a fonction from $]0,T[$ to $H_0^1(0,1)$, $t \mapsto (x \mapsto u(t,x))$. We equip $H_0^1(0,1)$ with the usual norm $(|u|_{L^2} + |u_t|_{L^2})$, then we have $u \in L^2(]0,T[; H_0^1(0,1))$, in the sense of L^p spaces valued in Hilbert spaces and we can derivate u as a distribution valued in a Hilbert space.

For the space variable, we have for almost all $t \in]0,T[$, $u(t,\cdot) \in H_0^1(0,1)$, $u_x(t,\cdot) \in L^2(0,1)$ et $u_{xx}(t,\cdot) \in H^{-1}(0,1)$. Thus, three spaces appear. We write $X = H_0^1(0,1)$, $H = L^2(0,1)$ and by definition $X' = H^{-1}(0,1)$ endowed with its dual norm. Identifying H with his dual we have the following continuous inclusions :

$$X \subset H \subset X'.$$

Let us remark that the first injection is compact thanks to the Rellich theorem.

Finally, to imitate the form of the problem in finite dimension, we define $A : X \to X'$ de Laplacian operator.

The controlled heat equation in one dimension for $u \in L^2(0,T;X)$ is therefore the following :

$$\begin{cases} u_t - Au = g \quad \text{dans} \quad L^2(0,T;X') \\ u(0,x) = u_0(x). \end{cases}$$

2.2 Framework and hypothesis for infinite dimensionnal problems

Considering the previous example, let assume we are given X, H et X' (dual of X) three Hilbert spaces with continuous inclusion $X \subset H \subset X'$, the first one being dense and compact. We identify H with its dual.

Remark 2.1. *We are now working on infinite dimensional Hilbert spaces rather than with $\mathbb{R}^n$ or $\mathbb{R}^m$ like in section I, hence the terminology. This triplet is sometimes called Hilbert triplet, it is the general statement. In order to keep some instinct on the problem, one could always think to X as $H_0^1(0,1)$, to H as $L^2(0,1)$ and to X' as $H^{-1}(0,1)$, as it was the case in the previous example.*

Let us consider $A \in \mathcal{L}(X,X')$ such that $\mathrm{Im}(A)$ is closed in X', $B \in \mathcal{L}(U,H)$ the control operator where U is a Hilbert space and $C \in \mathcal{L}(X,V)$ the observation operator where V is a Hilbert space. The space U is the space of the eligible control and V the space of observations. We write $||||_X$, $||_H$, $||||_{X'}$, $||_V$, $||_{V'}$, $||_U$

and $||_{U'}$ the norms on these spaces and we write $\langle y, x\rangle_{X',X}$ for $y \in X', x \in X$ the evaluation of the linear form y on x.

The transposed operators of B and C, written $B^* \in \mathcal{L}(H, U')$ and $C^* \in \mathcal{L}(V', X')$, are defined by the formula :

$$\forall x \in X, y \in U', \langle By, x\rangle_{X',X} = \langle B^* x, y\rangle_{U',U} .$$

Therefore, the control problem is to find a command $u \in L^2(0,T;U)$ and a trajectory $x \in L^2(0,T;X)$ verifying

$$\begin{cases} x_t + Ax = Bu & \text{in} \quad L^2(0,T;X') \\ x(0) = x_0 \in H \end{cases} \tag{2.1}$$

and minimizing :

$$J^T(u) = \frac{1}{2}\int_0^T (|u(t)|_U^2 + |Cx(t) - z|_V^2)dt$$

where $z \in V$ is the target.

According to [Lio71] (th1.2, p102) we know that for $u \in L^2(0,T;U)$, the problem (2.1) admits a unique solution $x \in W(0,T) \coloneqq \{x \in L^2(0,T;X) \mid x_t \in L^2(0,T;X')\}$ and the linear application $(u, x_0) \mapsto x \in W(0,T)$ is continuous, where $W(0,T)$, endowed with the norm $(|x|^2_{L^2(0,T;X)} + |x_t|^2_{L^2(0,T;X')})^{\frac{1}{2}}$ is a Hilbert space. Besides, we know that every $x \in W(0,T)$ is continuous in H, which enables us to consider $x(0)$. Finally, the inclusion $W(0,T) \to \mathcal{C}(0,T;H)$ is continuous (see [JLL72a], p20, 3.3 because one has $[X, X'] = H$, cf [JLL72a], p18, proposition. 2.2,).

As for finite dimensionnal systems, we have the following proposition.

Proposition 2.1. *There exists a unique optimal control $u \in L^2(0,T;U)$ minimizing the functional J^T.*

Besides, it is characterized by the existence of an adjoint state.

Proposition 2.2. *The couple (u, x) is optimal if and only if there exists an adjoint vector $p \in L^2(0,T;X)$ solution of :*

$$\begin{cases} -p_t + A^* p = C^* j_V(Cx - z) & \textit{in} \quad L^2(0,T;X') \\ p(T) = 0 \end{cases}$$

verifying :

$$u = -j_U B^* p$$

where $j_U : U' \to U$ and $j_V : V \to V'$ are the classical projection and injection.

The proof of this is the same than in finite dimension (see cf [Lio71] p144 theorem 2.1).

In terms of hypothesis, we impose an inequality on A which will provide energy estimates :

$$\exists \lambda, \mu, \forall x \in X, \quad \langle Ax, x \rangle_{X',X} + \mu |x|_H^2 \geq \lambda \|x\|_X^2. \tag{2.2}$$

As in section I we impose observability conditions.

Definition 2.1. *The couple (A, C) is said to be observable if there exists a constant γ such that for all $x \in L^2(0,T;X), f \in L^2(0,T;X'), x_0 \in H$ verifying*

$$\begin{cases} x_t + Ax = f \text{ in } L^2(0,T;X') \\ x(0) = x_0 \end{cases}$$

we have :

$$|x(T)|_H^2 \leq \gamma \left(\int_0^T (\|f\|_{X'}^2 + \|Cx\|_V^2) dt + |x_0|_H^2 \right)$$

where γ does not depend on T.

Definition 2.2. *The couple (A^*, B^*) is said to be observable if there exists a constant γ such that for all $p \in L^2(0,T;X), f \in L^2(0,T;X'), p_0 \in H$ verifying*

$$\begin{cases} -p_t + A^* p = f \text{ dans } L^2(0,T;X') \\ p(T) = p_0 \end{cases}$$

we have :

$$|p(0)|_H^2 \leq \gamma \left(\int_0^T (\|f\|_{X'}^2 + \|B^* p\|_{U'}^2) dt + |p_0|_H^2 \right)$$

where γ does not depend on T

For the rest of his section we assume :

$$\begin{cases} (A, C) \text{ is observable,} \\ (A^*, B^*) \text{ is observable.} \end{cases} \tag{2.3}$$

We are also interested in the stationnary problem by considering the following functional :

$$J_s(u, x) = \frac{1}{2}(|u|_U^2 + |Cx - z|_V^2)$$

that we try to minimize on the closed set $S = \{(x, u) \in X \times U \mid Ax = Bu\}$.

As in section I, we can obtain stationnary inequatlities thanks to the observability of (A, B). Indeed, let us take $x \in X$, $\hat{x}(t) = tx$ and $f = A\hat{x} + \hat{x}$ in the definition. Then we have :

$$T^2|x|_H^2 \leq \frac{2\gamma T^3}{3}(\|Ax\|_{X'}^2 + \|Cx\|_V^2) + \gamma T\|x\|_{X'}^2$$

Hence, for all T large enough :

$$\|x\|_X^2 \leq \alpha \langle Ax, x\rangle_{X',X} + \beta|x|_H^2.$$

Combining this with the hypothesis (2.2) on A we conclude :

$$\forall x \in X, \quad \|x\|_X^2 \leq \beta(\|Ax\|_{X'}^2 + \|Cx\|_V^2) \tag{2.4}$$

We can also prove that the stationnary problem is well-posed in the sense of the following theorem.

Proposition 2.3. *The fonctunal J_s admits a unique minimum written as* $(\bar{u}, \bar{x})$.

Preuve. According to the stationnary inequlity (2.4), the level sets $\{J_s \leq c\}$ are bounded. But, X and H are Hilbert spaces, then the existence of a minimum is proceeded by weak compacity and especially because of the inequality $|x|_H \leq \liminf_{n\to+\infty} |x_n|_H$ for any sequence $(x_n)_n$ weakly converging to x. The proof of uniqueness is the same than in section I. □

Since $(\bar{u}, \bar{x})$ minimizes J_s on S, its differential is equal to zero :

$$\forall (v, \varphi) \in U \times X \text{ such that } A\varphi = Bv, \quad \langle \bar{u}, v\rangle_U + \langle C\bar{x} - z, C\varphi\rangle_V = 0. \tag{2.5}$$

2.3 Average convergence

Thanks to the expression (2.5) we deduce, taking $v = 0$, that : $C^*(C\bar{x} - z) \in ker(A)^\perp$. But, Im$(A)$ is closed by hypothesis so, Ker$(A)^\perp$ = Im(A^*) (in infinite dimension, this fact is known as the closed range theorem cf [Yos13] p.205), so that we can choose $\bar{p} \in X$ such that

$$A^*\bar{p} = C^* j_V(C\bar{x} - z). \tag{2.6}$$

According to (2.5), for this $\bar{p}$ we have :

$$\forall (v, \varphi) \in U \times X \text{ tel que } A\varphi = Bv, \quad \langle \bar{u}, v\rangle_U + \langle \bar{p}, Bv\rangle_{X,X'} = 0. \tag{2.7}$$

Theorem 2.1. *Under the hypothesis* (2.2) *and* (2.3) *we have :*

$$\frac{1}{T}\min_{u\in L^2} J \underset{T\to+\infty}{\longrightarrow} \min J_s$$

and

$$\frac{1}{T}\int_0^T (|u(t) - \bar{u}|_U^2 + |C(x(t) - \bar{x})|_V^2)dt = \underset{T\to\infty}{\mathcal{O}}(\frac{1}{T}).$$

Preuve. According to [JLL72a] we have the integration formula in $W(0,T)$:

$$\forall f,g \in W(0,T), \quad \langle f(T), g(T)\rangle_H - \langle f(0), g(0)\rangle_H = \int_0^T (\langle f_t, g\rangle_{X',X} + \langle g_t, f\rangle_{X',X})dt. \tag{2.8}$$

Thus, according to this formula :

$$\begin{aligned}\langle x_0, p(0)\rangle_H &= -\int_0^T (\langle x_t, p\rangle_{X',X} + \langle p_t, x\rangle_{X',X})dt \\ &= -\int_0^T (\langle Bu, p\rangle_{X',X} - \langle C^* j_V(Cx-z), x\rangle_{X',X})dt \\ &= \int_0^T (-\langle u, B^*p\rangle_{U,U'} + \langle j_V(Cx-z), Cx\rangle_{V',V})dt\end{aligned}$$

But, $u = -j_U B^* p$ so :

$$\int_0^T (|u|_H^2 + |Cx-z|_V^2)dt = \langle x_0, p(0)\rangle_H - \int_0^T (\langle z, Cx-z\rangle_V)dt. \tag{2.9}$$

Besides, we have by the observability hypothesis

$$|p(0)|_H^2 \leq K\left(\int_0^T (|u|_U^2 + |Cx-z|_V^2)dt\right) \tag{2.10}$$

and :

$$|x(T)|_H^2 \leq K\left(\int_0^T (|u|_U^2 + |Cx|_V^2)dt + |x_0|_H^2\right) \tag{2.11}$$

Combining (2.9) et (2.10) ones gets a constant $c > 0$ which does not depend on T such that :

$$\int_0^T (|u|_U^2 + |Cx|_V^2)dt \leq cT. \tag{2.12}$$

Thus, going back to the observability inequalities (2.10) and (2.11) we get :

$$|p(0)|_H + |x(T)|_H \leq c\sqrt{T}. \tag{2.13}$$

Then, by the Cauchy-Schwarz inequality, we deduce from (2.12) that $\frac{1}{T}\int_0^T u dt$ and $\frac{1}{T}\int_0^T Cx dt$ are bounded in U and in V. But, integrating the evolution equation, we have

$$\frac{1}{T}\int_0^T Ax dt = \frac{1}{T}\int_0^T Bu dt - \frac{x(T) - x_0}{T} \tag{2.14}$$

with the second terme on the right converging to 0 when $T \to +\infty$ thanks to (2.13). So $\frac{1}{T}\int_0^T Ax dt$ also bounded in X'. Thus, applying the stationnary inequality (2.4)

to $\frac{1}{T}\int_0^T xdt$ we deduce that $\frac{1}{T}\int_0^T xdt$ is bounded in X uniformly in T.
By a similar argument, we prove that $\frac{1}{T}\int_0^T pdt$ is bounded in X uniformly in T.

Now, let us prove the convergence. We choose $\bar{p}$ verifying (2.6) and (2.7). We have the coupled system :

$$\begin{cases} (x-\bar{x})_t + A(x-\bar{x}) = B(u-\bar{u}) \\ -(p-\bar{p})_t + A^*(p-\bar{p}) = C^* j_V C(x-\bar{x}). \end{cases}$$

By some calculus and the integration formula (2.8) we get :

$$\int_0^T (|u-\bar{u}|_U^2 + |C(x-\bar{x})|_V^2)dt = \langle x(T)-\bar{x},\bar{p}\rangle_H + \langle x_0-\bar{x},p(0)-\bar{p}\rangle_H \tag{2.15}$$

$$-\int_0^T \langle u,\bar{u}+j_U B^*\bar{p}\rangle_U \, dt. \tag{2.16}$$

Thus, according to (2.13) :

$$\frac{1}{T}\int_0^T (|u-\bar{u}|_U^2 + |C(x-\bar{x})|_V^2)dt \le \frac{c}{\sqrt{T}} - \frac{1}{T}\int_0^T \langle u,\bar{u}+j_U B^*\bar{p}\rangle_U \, dt. \tag{2.17}$$

Let us prove that $\frac{1}{T}\int_0^T \langle u,\bar{u}+j_U B^*\bar{p}\rangle_U$ converges to 0 when $T\to+\infty$. Since $\frac{1}{T}\int_0^T udt$ is bounded, this quantity is also bounded. Let us take an eigenvalue, for the sequence of times $(T_n)_n$. By extracting, we can assume that $\frac{1}{T_n}\int_0^{T_n} udt$ weakly converges to a limit $\mu \in X$. Then, taking the limit in (2.14), we know that μ is in $D := \{u \in U \mid Bu \in im(A)\}$. But, according to (2.7), we have $\bar{u}+j_U B^*\bar{p} \in D^\perp$ so :

$$\frac{1}{T_n}\int_0^T \langle u,\bar{u}+j_U B^*\bar{p}\rangle_U \underset{n\to+\infty}{\longrightarrow} \langle \mu,\bar{u}+j_U B^*\bar{p}\rangle = 0.$$

This proves the expected result.

We deduce from (2.17) that :

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T (|u-\bar{u}|_U^2 + |C(x-\bar{x})|_V^2)dt = 0. \tag{2.18}$$

This improved estimation, with respect to (2.12), enables us to identify the limit of the averages $\frac{1}{T}\int_0^T udt$ and $\frac{1}{T}\int_0^T Cxdt$. It converges to $\bar{u}$ in U and $C\bar{x}$ in V. Besides, integrating the equation on $x-\bar{x}$ one has :

$$A\int_0^T (x-\bar{x})dt = \int_0^T B(u-\bar{u})dt + (x_0 - x(T))$$

and taking the norm :

$$\|A\frac{1}{T}\int_0^T (x-\bar{x})dt\|_{X'}^2 \leq \frac{c}{T} + \frac{c}{T}\int_0^T \|u-\bar{u}\|_U^2 dt.$$

Thanks to the stationnary inequality (2.4), we deduce that $\|\frac{1}{T}\int_0^T (x-\bar{x})dt\|_X^2$ is bounded, which proves the convergence of $\frac{1}{T}\int_0^T x dt$ to $\bar{x}$.

As in section I, we can prove that $\frac{1}{T}\int_0^T p dt$ converges in X to a certain $\tilde{p}$ verifying the equations (2.6) and (2.7). Since $u = -j_U B^* p$, we get, taking the average and taking the limit :

$$\bar{u} = -j_U B^* \tilde{p}. \tag{2.19}$$

Now let us go back at the moment where $\bar{p}$ appears in the proof. We had taken arbitrary $\bar{p}$ verifying the equations (2.6) and (2.7). Now we choose $\bar{p} = \tilde{p}$. Since $\tilde{p}$ verify the equation (2.19) we have, using (2.15), (2.16) :

$$\int_0^T (|u-\bar{u}|_U^2 + |C(x-\bar{x})|_V^2)dt = \langle x(T)-x_0, \bar{p}\rangle + \langle x_0-\bar{x}, p(0)-\bar{p}\rangle .$$

By applying the observability hypothesis to $x-\bar{x}$ and $p-\bar{p}$ we get estimations on $x_0-\bar{x}$ and $p(0)-\bar{p}$ of the type (2.10) and (2.11) which imply :

$$\int_0^T (|u-\bar{u}|_U^2 + |C(x-\bar{x})|_V^2)dt \leq c\left(\int_0^T (|u-\bar{u}|_U^2 + |C(x-\bar{x})|_V^2)dt + 1\right)^{\frac{1}{2}}.$$

We conclude :

$$\int_0^T (|u-\bar{u}|_U^2 + |C(x-\bar{x})|_V^2)dt \leq c$$

where c does not depend on T.

Finally, we deduce, using Cauchy-Schwarz inequality :

$$\frac{1}{T}\min_{u\in L^2} J^T \underset{T\to+\infty}{\longrightarrow} \min J_s.$$

$\square$

Remark 2.2. *As a consequence of this result, x and p are bounded uniformly in H in time. Indeed, we have proven that $|x(t)-\bar{x}|_H^2$ and $|p(0)-\bar{p}|_H^2$ are bounded by $\int_0^T (|u-\bar{u}|_U^2 + |C(x-\bar{x})|_V^2)dt$.*

2.4 Pointwise convergence

In order to establish a result of pointwise convergence as in section I, the method is globally the same, establishing a feedback law in the case $z = 0$ thanks to the function $\mathcal{E}$. Then, when $z \neq 0$, we should estimate the difference with the feedback law. However, for infinite dimensionnal problems, the construcion of $\mathcal{E}$ is more delicate.

Firstly, we focus on the construction of $\mathcal{E}$. Let us take a zero target $z = 0$. The evolution equation on x and the equation on p are coupled. We would like to express the optimal control u using x, which will uncouple the system.

For a time horizon T and $x_0 \in H$, let us recall that the following coupled system

$$\begin{cases} x_t + Ax = -Bj_U B^* p \\ -p_t + A^* p = C^* j_V C x \\ x(0) = x_0, \quad p(T) = 0 \end{cases}$$

has a unique soluton. We define :

$$\mathcal{E}(T)x_0 \coloneqq p(0).$$

Proposition 2.4. *The function $\mathcal{E}(T) : H \to H$ is linear, continuous, symmetric and postive definite. Besides, the operator family $(\mathcal{E}(t))_{t\geq 0}$ is uniformly bounded and increasing.*

For the proof of these properties, we refer to the article [AP13] (p4260). It relies on the fact that $\forall x_0 \in H, \quad \langle \mathcal{E}(T)x_0, x_0 \rangle = \langle p(0), x_0 \rangle = \int_0^T (|u(t)|_U^2 + |Cx(t)|_V^2) dt$ $= \min J_0^T$. Here, J_0^T is the functionnal J^T with target $z = 0$. The second equality is obtained by taking $z = 0$ in the equality (2.9) of the theorem 2.1 .

Now, let us consider the equation on the adjoint state : $-p_t + A^* p = -C^* C x$. We write p_1 the solution verifying $p_1(T - t) = 0$, for $t \in [0, T]$ and p the one verifying $p(T) = 0$. Translating the interval $[0, T]$ by t and using the uniqueness, we obtain :

$$\forall t \in [0, T], \quad p(t) = \mathcal{E}(T - t)x(t). \tag{2.20}$$

But, we already knew $u = -j_U B^* p$. So, we have the feedback law when $z = 0$:

$$\forall t \in [0, T], \quad u(t) = -j_U B^* \mathcal{E}(T - t)x(t). \tag{2.21}$$

As in section I the aim is to define an operator $\hat{E} : H \to H$ which is the limit (with exponential convergence) of the operator family $(\mathcal{E}(t))_{t\geq 0}$. To define it the method differs.

Let us take a sequence of times $(T_n)_n$ which diverges to $+\infty$. We write, x_n and p_n the solutions of the coupled system with time horizon T_n. According to the remark 2.2, $(x_n)_n$ and $(p_n)_n$ are uniformly bounded in H, and bounded in $L^2_{loc}(0,+\infty;X)$. By weak compactness argument, we can assume that $(x_n)_n$ converges weakly to a function $\hat{x} \in L^2_{loc}(0,+\infty;X)$ and $(p_n)_n$ to a function $\hat{p} \in L^2_{loc}(0,+\infty;X)$ verifying :

$$\int_0^{+\infty} (|B^*\hat{p}| + |C\hat{x}|^2)dt \leq c \tag{2.22}$$

and

$$\begin{cases} \hat{x}_t + A\hat{x} = -Bj_U B^*\hat{p} \\ -\hat{p}_t + A^*\hat{p} = C^* j_V C\hat{x} \\ \hat{x}(0) = x_0, \lim\limits_{t\to+\infty} \hat{p}(t) = 0 \end{cases} \tag{2.23}$$

where the last statement is meant in a weak sense.
Besides, we can also assume that $p_n(0)$ converges weakly in H to $\hat{p}(0)$.

Now we define $\hat{E} : H \to H$ by the formula :

$$\hat{E}x_0 = \hat{p}(0).$$

We can prove like we have done before that $\hat{E}$ is in $\mathcal{L}(H,H)$. But, the weak convergence of $(p_n(0))_n$ to $\hat{p}(0)$ means that $\mathcal{E}(T_n)x_0$ weakly converges to $\hat{E}x_0$. Thanks to the proposition 2.2 we already know that $\mathcal{E}(t)x_0$ strongly converges in H. So we have identified the limit :

$$\lim_{t\to+\infty} \mathcal{E}(t)x_0 = \hat{E}x_0.$$

Remark 2.3. *In section I the definition of $\hat{E}$ was exactly this property.*

As for the controlled system with finite time horizon, the operator $\hat{E}$ enables us to uncouple the system (2.23) in the sense of :

$$\forall t \in [0,+\infty[, \quad \hat{p}(t) = \hat{E}\hat{x}(t).$$

As for the law connecting p to x (2.20), the proof of this result consists of a translation of the initial time.

Thus, $\hat{x}$ is the unique solution of the system :

$$\begin{cases} \hat{x}_t + Mx = 0 \\ \hat{x}(0) = x_0. \end{cases} \tag{2.24}$$

where $M := (A + Bj_U B^*\hat{E})$.

The context in which the definition of $\hat{E}$ appears is established. Now we can prove the exponential convergence of $(\mathcal{E}(t))_{t\geq 0}$ to $\hat{E}$.

Proposition 2.5. *The system* (2.25) *is exponentially stable and there exists* $\mu > 0$ *such that :*

$$\forall t > 0, \quad \|\mathcal{E}(t) - \hat{E}\|_{\mathcal{L}(H,H)} \leq ce^{-\mu t}.$$

To prove this proposition we need the following lemma.

Lemma 2.1. *If* $z \in L^\infty(]-\infty, 0[; H)$ *verifies :*

$$\begin{cases} z_t + Az = 0 \text{ in }]-\infty, 0[\\ Cz = 0 \text{ in }]-\infty, 0[\end{cases}$$

then z *is identically zero.*

Proof of the lemma. Let us consider $\tilde{z} = e^{\lambda t}$ verifying the equation $\tilde{z}_t + A\tilde{z} = \lambda\tilde{z}$. Applying the observability hypothesis on $[\tau, t]$, where $t < 0$, and since $C\tilde{z} = 0$, we have :

$$|\tilde{z}(t)|_H^2 \leq c(\lambda^2 \int_0^T \|\tilde{z}\|_{X'}^2 dt + |\tilde{z}(\tau)|_H^2).$$

Since the inclusion $X \subset H$ is continuous, we also have :

$$|\tilde{z}(t)|_H^2 \leq c(\lambda^2 \int_0^T |\tilde{z}|_H^2 dt + |\tilde{z}(\tau)|_H^2).$$

Using Gronwall lemma, we obtain :

$$|\tilde{z}(t)|_H^2 \leq c|\tilde{z}(\tau)|_H^2 e^{\lambda^2 \tilde{c}(t-\tau)}.$$

Then, integrating :

$$\int_\tau^t |\tilde{z}(s)|_H^2 ds \leq \frac{1}{\lambda^2} e^{-\tilde{c}\lambda^2(\tau-t)} |\tilde{z}(\tau)|_H^2 = \frac{1}{\lambda^2} e^{-\tilde{c}\lambda^2(\tau-t)+2\lambda t} |z(\tau)|_H^2.$$

Taking λ larger enough and then taking $\tau \to -\infty$, we finally get $z = 0$ sur $]-\infty, t[$. This being true for every $t < 0$, the proof is ended. $\square$

Proof of the theorem. We define the following quantity for $t \geq 0$:

$$l(t) := \sup_{|x_0|_H \leq 1} \left\langle \hat{E}\hat{x}(t), \hat{x}(t) \right\rangle_H.$$

Let us prove that $\lim\limits_{t\to+\infty} l(t) = 0$. This quantity is bounded because $\hat{E}$ is continuous and $\hat{x}$ is bounded when $|x_0|_H \leq 1$. Besides, $l(t)$ is non-increasing. Indeed, one can prove by calculation that :

$$\frac{d}{dt}\left\langle \hat{E}\hat{x}(t),\hat{x}(t)\right\rangle_H = \frac{d}{dt}\left\langle \hat{x}(t),\hat{p}(t)\right\rangle_H = -|B^*\hat{E}\hat{x}|_H^2 - |C\hat{x}|_H^2 \leq 0.$$

Let us take a sequence of times $(t_n)_n$ such that $\lim_{t\to+\infty} t_n = +\infty$ and some $x_0^n \in H$ such that $|x_0^n|_H \leq 1$ with :

$$\left\langle \hat{E}\hat{x}_n(t_n),\hat{x}_n(t_n)\right\rangle_H \geq l(t_n) - \frac{1}{n}$$

where the $\hat{x}_n$ are the solutions of the system (2.25) with initial data x_0^n. We write $z_n(t) \coloneqq \hat{x}_n(t+t_n)$. It verifies :

$$(z_n)_t + Mz_n = 0 \text{ sur }]-t_n,+\infty[.$$

But, $\hat{x}$ is bounded in H. Indeed, by the observability hypothesis, we have :

$$\begin{aligned}
|\hat{x}(t)|_H^2 &\leq c\left(\int_0^t (|C\hat{x}|_V^2 + \|Bj_U B^*\hat{E}\hat{x}\|_{X'}^2)dt + |x_0|_H^2\right)\\
&\leq \tilde{c}\left(\int_0^t (|C\hat{x}|_V^2 + |Bj_U B^*\hat{E}\hat{x}|_H^2)dt + |x_0|_H^2\right)\\
&= \tilde{c}\left(\int_0^t (|C\hat{x}|_V^2 + \|B^*\hat{p}\|_H^2)dt + |x_0|_H^2\right).
\end{aligned}$$

So, by the inequality (2.23), $\hat{x}$ is bounded in H and the sequence (z_n) is uniformly bounded in time in H.

We can also prove that $(z_n)_n$is bounded in $L^2(a,b;X)$, for all $a \leq b$ appropriate. Indeed, A verifies the hypothesis (2.2). Using this, we can easily prove that $M = (A + Bj_U B^*\hat{E})$ also satisfies it. Thus, we have basic energy estimates taking the scalar product with z in the equation $z_t + Mz = 0$ for $z \in \{z_n \mid n \in \mathbb{N}\}$. This enables us to get :

$$\frac{1}{2}(|z(a)|_H^2 - |z(b)|_H^2) \geq \int_a^b (\lambda\|z\|_H^2 - \mu|z|_H^2)dt.$$

As we have proven it, $(z_n)_n$ is uniformly bounded in time in H, which yields the expected result.

Thus, for any $a < b$ appropriate, $(z_n)_n$ is bounded in $L^2(a,b;X)$ and $((z_n)_t)_n$ is bounded in $L^2(a,b;X')$ because $(z_n)_t = -Mz_n$. The Aubin Lemma (cf [Dro01], 951, theorem 2.4.1) enables us to assume, that $(z_n)_n$ converges in $L^2(a,b;H)$ and then in $\mathcal{C}([a,b];H)$, to a function $z \in L^2_{loc}(]-\infty,+\infty[;H)$ verifying :

$$z_t + Mz = 0 \text{ sur }]-\infty,+\infty[.$$

Since l is non-increasing one has :

$$\forall t<0, \quad \left\langle \hat{E}\hat{x}_n(t_n), \hat{x}_n(t_n)\right\rangle_H \leq \left\langle \hat{E}\hat{x}_n(t+t_n), \hat{x}_n(t+t_n)\right\rangle_H \leq l(t+t_n),$$

which yields :

$$\forall t<0, \quad l(t_n)-\frac{1}{n} \leq \left\langle \hat{E}z_n(t), z_n(t)\right\rangle_H \leq l(t+t_n).$$

Because of the convergence of $(z_n)_n$ to z we get, as $t_n \underset{n\to+\infty}{\longrightarrow} +\infty$:

$$\forall t<0, \quad \left\langle \hat{E}z(t), z(t)\right\rangle_H = \lim_{t\to+\infty} l(t).$$

So we have proven $\left\langle \hat{E}z(t), z(t)\right\rangle_H$ is constant on $]-\infty,0[$. But, the derivative of this quantity is $-|B^*\hat{E}z|_H^2-|Cz|_H^2$. Thus :

$$B^*\hat{E}z = Cz = 0 \text{ sur }]-\infty,0[.$$

In brief, z verifies :

$$\begin{cases} z_t + Az = 0 \text{ sur }]-\infty,0[\\ Cz = 0 \\ z \in L^\infty(]-\infty,0[;H). \end{cases}$$

According to the the preliminary lemme, we deduce that $z=0$ and so $\lim_{t\to+\infty} l(t) = 0$, which ends the first part of the proof.

Now, let us prove that

$$\lim_{t\to+\infty} \sup_{|x_0|_H\leq 1} |\hat{x}(t)|_H = 0.$$

Reasoning by contradiction, there would exist $\epsilon>0$, a sequence $(x_0^n)_n \in H^{\mathbb{N}}$, with $|x_0^n|\leq 1$, and a sequence of times $(t_n)_n$, with $t_n \underset{n\to+\infty}{\longrightarrow} +\infty$, such that $|\hat{x}_n(t_n)|_H \geq \epsilon$ where $\hat{x}_n$ is the solution of

$$\begin{cases} \hat{x}_t + M\hat{x} = 0 \\ \hat{x}(0) = x_0^n. \end{cases} \tag{2.25}$$

But, we have $\hat{x}_n(t_n) = z_n(0) \underset{n\to+\infty}{\longrightarrow} z(0)$. Besides, z is continuous at 0 (property of the Sobolev spaces) and z is equal to zero on $]-\infty,0[$ so, $\hat{x}_n(t_n) = z_n(0) \underset{n\to+\infty}{\longrightarrow} 0$, which is absurd. This ends the second part of the proof. This limit result on $\hat{x}$ implies actually the exponential decay $\hat{x}$ calimed in the theorem. Indeed, it is a semi-group result, there exists $\mu>0, c>0$ such that :

$$\forall t \geq 0, \quad |\hat{x}(t)|_H \leq ce^{-\mu t}|x_0|_H.$$

Even if I studied the basis of semi-group theory, I won't get into more details about such a result. The reader of this report can refer to [AB07] (p92, corollary 2.1) .

Let us finally prove the exponential convergence of $(\mathcal{E}(t))_{t\geq 0}$. We consider the optimality systems on (x, p) and $(\hat{x}, \hat{p})$:

$$\begin{cases} x_t + Ax = -Bj_U B^* p \\ -p_t + A^* p = C^* j_V C x \\ x(0) = x_0, p(T) = 0. \end{cases} \qquad \begin{cases} \hat{x}_t + A\hat{x} = -Bj_U B^* \hat{p} \\ -\hat{p}_t + A^* \hat{p} = C^* j_V C \hat{x} \\ \hat{x}(0) = x_0, \lim\limits_{t\to+\infty} \hat{p}(t) = 0 \end{cases}$$

Substracting the equations, taking the scalar product and integrating from 0 to T one obtains :

$$\int_0^T (|B^*(p-\hat{p})|_H^2 + |C(x-\hat{x})|_H^2)dt \leq |\hat{p}(T)|_H |x(T) - \hat{x}(T)|_H. \tag{2.26}$$

Besides, applying the observability assumptions on (A, C) at $x - \hat{x}$, we get :

$$|x(T) - \hat{x}(T)|_H^2 \leq c \int_0^T (|B^*(p-\hat{p})|_H^2 + |C(x-\hat{x})|_H^2)dt.$$

These two inequalities yields :

$$\int_0^T (|B^*(p-\hat{p})|_H^2 + |C(x-\hat{x})|_H^2)dt \leq c|\hat{p}(T)|_H^2$$

Applying the observability asumptions on (A^*, B^*) at $p - \hat{p}$ we get :

$$|p(0) - \hat{p}(0)|_H^2 \leq c(\int_0^T (\|B^*(p-\hat{p})\|_{U'}^2 + \|C(x-\hat{x})\|_V^2)dt + |\hat{p}(T)|_H^2)$$

According to the inequlity (2.26) and the exponential decay of $\hat{x}$ (first part of the proof) we deduce :

$$|p(0) - \hat{p}(0)|_H^2 \leq c|\hat{p}(T)|_H^2 = c|\hat{E}\hat{x}|_H^2 \leq ce^{-2\mu T}|x_0|_H^2.$$

By definition of $\mathcal{E}(T)$ and $\hat{E}$ this means :

$$|\mathcal{E}(T)x_0 - \hat{E}x_0|_H^2 \leq ce^{-2\mu T}|x_0|_H^2.$$

We conclude :

$$\|\mathcal{E}(T) - \hat{E}\|_{\mathcal{L}(H,H)} \leq ce^{-2\mu T}.$$

□

The aim is now to establish the law of the optimal control in the case $z \neq 0$ using the feedback law for a zero target $z = 0$. The distance with the feedback law will then be estimated thanks to the inequality we have just proven.

Theorem 2.2. *The optimal control u is given by the following law :*

$$\forall t > 0, \quad u(t) = \bar{u} - j_U B^*[\mathcal{E}(T-t)(x(t) - \bar{x}) + h(t)]$$

where h is the solution of the problem :

$$\begin{cases} -h_t(t) + (A^* + \mathcal{E}(T-t)Bj_UB^*)h(t) = 0 \ \textit{on} \ ]0,T[\\ h(T) = -\bar{p}. \end{cases} \tag{2.27}$$

Then, there exists $\mu > 0$ and $c > 0$ such that :

$$|x(t) - \bar{x}|_H + |u(t) - \bar{u}|_H \leq c(e^{-\mu t} + e^{-\mu(T-t)}).$$

Proof. As in section I, thanks to the exponential convergence (proposition 2.3) and the Gronwall lemma, we can prove that :

$$\forall t \in [0,T], \quad |h(t)|_H \leq ce^{-\mu(T-t)}.$$

Let us prove the affirmation :

$$\forall t \in [0,T], \quad p(t) - \bar{p} = \mathcal{E}(T-t)(x(t) - \bar{x}) - h(t). \tag{2.28}$$

Here, $\mathcal{E}$ is not defined by a differential equation as it was the case in section I. We can not derivate the right term to prove it verifies the same differential equation satisfied by $p - \bar{p}$ and conclude by uniqueness. To get around this, we use a duality argument. The equality (2.298) is equivalent to :

$$\forall \varphi \in H, \quad \langle p(t) - \bar{p}, \varphi \rangle_H = \langle x(t) - \bar{x}, \mathcal{E}(T-t)\varphi \rangle_H + \langle h(t), \varphi \rangle_H$$

because $\mathcal{E}(T-t)$ is symmetric.

Let us prove this equality. Consider $\varphi \in H$, we recall that thanks to (2.20), the adjoint $q(s) = \mathcal{E}(T-s)\varphi$ verifies the coupled system :

$$\begin{cases} -q_t + A^*q = C^*j_VCz \quad \text{in} \quad L^2(t,T;X') \\ z_t + Az = -Bj_UB^*q \quad \text{in} \quad L^2(t,T;X') \\ z(t) = \varphi, q(T) = 0. \end{cases}$$

Thus what we want to prove is :

$$\langle p(t), z(t) \rangle_H - \langle h(t) - h(T), z(t) \rangle_H = \langle x(t) - \bar{x}, q(t) \rangle_H. \tag{2.29}$$

By the integration formula (2.8) we have, for the left term :

$$\begin{aligned}\langle p(t), z(t)\rangle_H - \langle h(t) - h(T), z(t)\rangle_H &= \int_t^T (-\langle p_t, z\rangle_H - \langle p, z_s\rangle_H + \langle h_s, z\rangle + \langle h - h(T), z_s\rangle)ds \\ &= \int_t^T (\langle C^* j_V C x, z\rangle_{X',X} + \langle p, B j_U B^* q\rangle_H + \langle \bar{p}, -B j_U B^* q\rangle_H \\ &- \langle C^* j_V C z + A^* \bar{p}, z\rangle_{X',X})ds.\end{aligned}$$

Similarly, for the right term :

$$\begin{aligned}\langle x(t) - \bar{x}, q(t)\rangle_H &= -\int_t^T (\langle x_s, q\rangle_H + \langle x - \bar{x}, q_s\rangle)ds \\ &= \int_t^T (\langle B j_U B^* p, q\rangle + \langle x, C^* j_V C z\rangle_{X',X} + \langle \bar{x}, A^* q\rangle_H - \langle \bar{x}, C^* j_V C z\rangle_H)ds\end{aligned}$$

But, we have $A\bar{x} = B\bar{u} = -B j_U B^* \bar{p}$ so the term $\langle \bar{x}, A^* q\rangle_H$, which is on the right in (2.30), is on the left in the form $\langle \bar{p}, -B j_U B^* q\rangle_H$. Besides, we have $A^*\bar{p} = C^* j_V (C\bar{x} - z)$ so the term $\langle \bar{x}, C^* j_V C z\rangle_H$, which is on the right, is on the left in the form $-\langle C^* j_V C z + A^* \bar{p}, z\rangle_{X',X}$. This enables us to get the expected equality between the left and right terms.

To end the proof, we apply the same method than in section I : the system (2.24) being asymptotically stable and thanks to the proposition 2.3 we prove that there exists a constant $c > 0$ which does not depend on T such that $\forall t \in [0, T], \quad \|h(t)\|_U \leq c e^{-\mu(T-t)}$. This enables us to prove the inequality and the theorem. □

2.5 Application

As an example of this result, we consdider the Dirichlet problem with internal control and observation.

Let us take a bounded domain Ω of $\mathbb{R}^N$. We consider the following problem :

$$\begin{cases} y_t - \operatorname{div}(M(x)\nabla y) + c(x)y + B(x)\cdot\nabla y = u\chi_\omega \quad \text{in} \quad]0, T[\times\Omega \\ y = 0 \quad \text{in} \quad]0, T[\times\partial\Omega \\ y(0) = y_0 \in L^2(\Omega) \end{cases} \tag{2.30}$$

where $M \in L^\infty(\Omega; \mathbb{R}^N \times \mathbb{R}^N)$ satisfies $\lambda|\xi|^2 \leq \langle M(x)\xi, \xi\rangle \leq \Lambda|\xi|^2$ for $\xi \in \mathbb{R}^N$ and $x \in \Omega$, $c \in L^\infty(\Omega), c(x) \geq 0$ and $B \in L^\infty(\Omega; \mathbb{R}^N)$. The open set ω is included in Ω and χ_ω is the indicator function of ω.

We choose the functional :

$$J(u) = \frac{1}{2}\int_0^T (|u(t)|^2_{L^2(\omega)} + |y(t) - z|^2_{L^2(\omega_0)})dt.$$

where ω_0 is an open set included in Ω.

We actually took in the abstract settings : $X = H_0^1(\Omega)$, $H = L^2(\Omega)$, $X' = H^{-1}(\Omega)$, $U = L^2(\omega)$, $V = L^2(\omega_0)$, $A : y \mapsto -div(M(x)\nabla y) + c(x)y + B(x)\cdot\nabla y$, $B : u \mapsto u$, et $C : y \mapsto y\chi_{\omega_0}$.

We can now easily prove that A verifies the hypothesis (2.2) especially by the Poincaré inequality on $X = H_0^1(\Omega)$. We also may prove the following result.

Lemma 2.2. *The image of the operator $A \in \mathcal{L}(X, X')$ is closed in X'.*

Preuve. Firstly, one can prove that A is one-to-one thanks to the maximum principle.

Now let us show that A verifies :

$$\forall y \in X, \quad \alpha\|y\|_X \leq \|Ay\|_{X'}. \tag{2.31}$$

Reasonning by contradiction, let us assume there exists a sequence $(y_n)_n$ of X verifying $\|y_n\|_X = 1$ and such that $Ay_n \underset{n\to+\infty}{\longrightarrow} 0$ in X'. Since A verifies the assumption (2.2) we have :

$$\forall y \in X, \quad c\|y\|_X \leq \|Ay\|_{X'} + |y|_H.$$

Since $(y_n)_n$ is bounded in X we can assume that $(y_n)_n$ converges in H, thanks to the classical compact injection of $X = H_0^1(\Omega)$ into $H = L^2(\Omega)$. Then, $(y_n)_n$ is a Cauchy sequence in X because of :

$$\forall m, p \in \mathbb{N}, \quad c\|y_m - y_p\|_X \leq \|Ay_m - Ay_p\|_{X'} + |y_m - y_p|_H.$$

Thus, there exists $y \in X$ such that $(y_n)_n$ converges to y in X. By the continuity of the inclusion $X \subset H \subset X'$ and the continuity of A, we deduce that $Ay_n \underset{n\to+\infty}{\longrightarrow} Ay$ in X'. But, we already had $Ay_n \underset{n\to+\infty}{\longrightarrow} 0$ so $Ay = 0$. Since A is one-to-one, $y = 0$. This is absurd because $|y_n|_X = 1$.

The inequality (2.31) means that $A^{-1} : \mathrm{Im}(A) \to X$ is continuous. Thanks to the cloed graph theorem, we know that the graph of A^{-1}, that is $\{(z, A^{-1}z) \mid z \in \mathrm{Im}(A)\}$, is closed in $X' \times X$. So we deduce that $\mathrm{Im}(A)$ is closed in X'. $\square$

Besides, by an argument on the eigenvalues of A, we can prove that A and A^* are asymptotically stable (cf [H.B94]). Here we mean that, for A^* by example, there exists $\lambda > 0$ such that every solution of :

$$\begin{cases} x_t + A^*x = 0 \text{ in }]0, T[\\ x(0) = x_0 \end{cases} \tag{2.32}$$

verifies :

$$\forall t \in]0,T[, \quad |x|_X \leq e^{-\lambda t}|x_0|_H.$$

This enables us to prove that the hypothesis (2.3) is verified. Indeed, we use the Duhamel formula for semi-groups (cf [AB07], p130, proposition 3.1). The semi-groups associated to A and A^* being exponentially decreasing, this formula yields the hypothesis (2.3) for any operators B et C.

All hypothesis are know cheched, the theorem 2.2 reads as follow.

Corollary. *There exists $\lambda > 0$ and $K > 0$ such that :*

$$\forall t \in [0,T], \quad |y(t) - \bar{y}|_{L^2(\Omega)} + |u(t) - \bar{u}|_{L^2(\Omega)} \leq K(e^{-\lambda t} + e^{-\lambda(T-t)})$$

where $\bar{u} \in L^2(\Omega)$ and $\bar{x} \in L^2(\Omega)$ are the solution of the stationnary problem associated with (2.30)*.*

3 The case of a boundary contol

3.1 Another way of controlling PDE

In the previous section, we were establishing convergence results for systems controlled with interior control. Let us take the example of the heat equation in a domain Ω :

$$\begin{cases} u_t - \Delta u = g\chi_\omega \text{ dans }]0,T[\times\Omega \\ u(0,x) = u_0(x) \end{cases}$$

where here, g is the control and u the solution.

The aim is to find the oprimal control g, minimizing both the cost and the distance of the solution with the target profile in ω_0.

Another way of controlling this PDE can be on the boundary :

$$\begin{cases} u_t - \Delta u = 0 \text{ in }]0,T[\times\Omega \\ u = g \text{ in }]0,T[\times\partial\Omega \\ u(0,x) = u_0(x) \end{cases}$$

where $g \in L^2(]0,T[\times\partial\Omega)$.

The aim remains to get as close as possible in ω_0 to a target $u_1 : \Omega \to \mathbb{R}$. But now, our action consist of setting the temperature on the boundary of the domain, g, to affect the solution of the heat equation u. The functional we want to minimize is :

$$J^T(g) = \frac{1}{2}\int_0^T |g|^2_{L^2(\partial\Omega)} dt + \frac{1}{2}\int_0^T |u(t,\cdot) - u_1|^2_{L^2(\omega_0)} dt.$$

We can prove there is a unique control minimizing J^T. Thus, we can wonder if the optimal control and trajectory converge to a certain couple $(\bar{g}, \bar{u}) \in L^2(\partial\Omega) \times L^2(\Omega)$ as T goes to $+\infty$.

3.2 The one dimensional heat equation with a boundary control

Let us consider the heat equation with $\Omega :=]0,1[$:

$$\begin{cases} u_t - u_{xx} = 0 \text{ sur }]0,T[\times]0,1[, \\ u(t,0) = 0, \\ u(t,1) = g(t), \\ u(0,x) = u_0(x). \end{cases} \tag{3.1}$$

where, $g \in H^1(0,T;\mathbb{R})$ and $u_0 \in L^2(0,1)$.

We impose $g(0) = 0$, which means that at the beginning, there is not control on the temperature yet. We know that the problem (3.1) admits a unique solution $u \in H^{2,1}(0,T) := L^2(0,T;H^2(0,1)) \cap H^1(0,T;L^2(0,1))$, where here, $H^1(0,T;L^2(0,1))$ is the set of functions $u \in L^2(0,T;L^2(0,1))$ such that $u_t \in L^2(0,T;L^2(0,1))$. This regularity result can be found in [JLL72b] (p28, theorem 4.3).

Let $u_1 \in L^2(0,1)$ be the target profile of temperature. We decide to minimize the derivative of g : we don't want to have to apply brutal variations of temperature. We notice that in $H^1_{00}(0,1) := \{g \in H^1(0,1) \mid g(0) = 0\}$ the norms $|g|_{H^1(0,1)}$ and $|g'|_{L^2(0,1)}$ are equivalent so if the cost concerns g', it will actually also concern the remperature and its variation. So we choose the following functional :

$$J^T(g) = \frac{1}{2}\int_0^T |g'(t)|^2 dt + \frac{1}{2}\int_0^T \int_0^1 |u(t,x) - u_1(x)|^2 dx dt.$$

Let us begin our study with the following result.

Lemma 3.1. *There exists a unique optimal control $g \in H^1_{00}(0,1)$ minimizing the functional J^T.*

Before the proof, we will reduce the problem to an easier one, to a more usual problem at least. Let u be the solution of (3.3) for a control g. The function $v(t,x) = u(t,x) - xg(t)$ verifies :

$$\begin{cases} v_t - v_{xx} = -xg'(t) \text{ in }]0,T[\times]0,1[, \\ v(t,0) = v(t,1) = 0, \\ v(0,x) = u_0(x). \end{cases} \tag{3.2}$$

Thus, the problem (3.3) is equivalent to a problem with control in the domain $]0,1[$. It may be tempting to use the results established in section II taking $Bg = -xg'$ but the functionnal J^T is not adapted to problem (3.2) :

$$J^T(g) = \frac{1}{2}\int_0^T |g'(t)|^2 dt + \frac{1}{2}\int_0^T\int_0^1 |v(t,x) + xg(t) - u_1(x)|^2 dxdt.$$

However, the advantage of this reformulation is that v is equal zero on the boundary of $]0,1[$.

Proof of the lemma. The uniqueness is due to the convexity of the function $g \to u$ where u is the solution of the problem (3.1) associated with g. Since, $g \to |g'|^2_{L^2(0,T)}$ is strictly convex, if g_1 and g_2 are two minima, we obtain $g_1' = g_2'$ so $g_1 = g_2$.

For the existence, let us consider a minimizing sequence $(g_n, u_n)_n$ for the functional J^T. Since $(g_n')_n$ is bounded in $L^2(0,T)$, we deduce that $(g_n)_n$ is bounded in $H^1_{00}(0,T)$ and we can assume it weakly converges to a function $g \in H^1_{00}(0,T)$. Since $v_n = u_n - xg_n$, the sequence $(v_n)_n$ is bounded in $L^2(]0,T[\times]0,1[)$ and we can assume it weakly converges to a function $v \in L^2(]0,T[\times]0,1[)$. In particular, $v_n(0,\cdot)$ weakly converges to $v(0,\cdot)$ so that $v(0,\cdot) = u_0$. By the weakly converge, there is convergence of the distribution so : $v_t - v_{xx} = -xg'$ as a distribution equality. Thus, v is solution of the simplified problem (3.2) for the control g. Besides, the sequence $(v_n + xg_n - u_1)_n)$ weakly converges to $v + xg - u_1$ so that :

$$\begin{aligned} &\frac{1}{2}\int_0^T |g'|^2_{L^2(\partial\Omega)} dt + \frac{1}{2}\int_0^T\int_0^1 |v(t,x) + xg(t) - u_1(x)|^2 dxdt \\ \le &\liminf_{n\to+\infty} \left(\frac{1}{2}\int_0^T |g_n'|^2_{L^2(\partial\Omega)} dt + \frac{1}{2}\int_0^T\int_0^1 |v_n(t,x) + xg_n(t) - u_1(x)|^2 dxdt\right) \\ = &\inf J^T. \end{aligned}$$

$\square$

In what follows, we refer to (g,u) and (g,v) as the optimal solutions of the initial and simplified problem, (3.1) and (3.2).

As in the previous sections, the optimal couple (g,u) is asociated with an adjoint system. Such a system can be obtained by trial and error or imitating the Pontryagin maximum principle.

Lemma 3.2. *Let us take p solution of the following system :*

$$\begin{cases} -p_t - p_{xx} = v + xg - u_1 \ \text{in }]0,T[\times]0,1[, \\ p(t,0) = p(t,1) = 0, \\ p(T\cdot) = 0. \end{cases} \tag{3.3}$$

Then we have

$$\forall h \in H^1_{00}(0,T), \quad \int_0^T g'(t)h'(t)dt = \int_0^T p_x(t,1)h(t)dt.$$

As for the system (3.1), the problem (3.3) admits a unique solution $p \in H^{2,1}(0,T)$.

Firstly, we give a necessary condition for g to be the minimum of J^T. Let $\delta g \in H^1_{00}$ be a perturbation of g. We write δu the solution of the initial problem with $u_0 = 0$ and $\delta v = \delta u - x\delta g$. We have :

$$\begin{aligned} J^T(g+\delta g) =& J^T(g) + \langle g', (\delta g)' \rangle + \langle u - u_1, \delta u \rangle + o(\delta g) \\ &= J^T(g) + \langle g', (\delta g)' \rangle + \langle v + xg - u_1, \delta v + x\delta g \rangle + o(\delta g). \end{aligned}$$

But the fonction $\delta g \mapsto \langle g', (\delta g)' \rangle + \langle u - u_1, \delta u \rangle$ is linear so that we obtain the condition on g : $\forall \delta g \in L^2(0,T), \langle g', (\delta g)' \rangle + \langle u - u_1, \delta u \rangle = 0$, which means :

$$\forall \delta g \in H^1_{00}(0,T), \quad \langle g', (\delta g)' \rangle + \langle v + xg - u_1, \delta v + x\delta g \rangle = 0.$$

Taking $\delta g = h - g$ where $h \in H^1_{00}(0,T)$ and writing v_h, v_g the solutions associated with h and g one obtains :

$$\forall h \in H^1_{00}(0,T), \quad \langle g', h' - g' \rangle + \langle v_g + xg - u_1, v_h - v_g + x(h-g) \rangle = 0. \tag{3.4}$$

Proof of the lemma. We define $\phi \coloneqq v_h - v_g$. Let p be the solution of the system (3.3). By the integration formula in Sobolev spaces (2.8) we have :

$$\begin{aligned} \langle v_g + xg - u_1, \phi \rangle &= -\langle p_t + p_{xx}, \phi \rangle \\ &= \langle p, \phi_t - \phi_{xx} \rangle + [p(t,1)\phi(t,1)]_{t=0}^{t=T} \\ &- \int_0^T \phi(t,1)p_x(t,1)dt + \int_0^T \phi_x(t,1)p(t,1)dt \\ &= -\langle p, x(h' - g') \rangle . \end{aligned}$$

Besides :

$$
\begin{aligned}
\langle v_g + xg - u_1, x(h-g)\rangle &= \langle -p_t - p_{xx}, x(h-g)\rangle \\
&= \langle p, x(h'-g')\rangle + \langle p_x, h-g\rangle - \langle h-g, p_x(\cdot,1)\rangle .
\end{aligned}
$$

So the condition (3.4) is equivalent to :

$$
\forall h \in H^1_{00}(0,T), \quad \langle g', h'-g'\rangle + \langle p_x, h-g\rangle - \langle p_x(t,1), h-g\rangle = 0.
$$

Since $p(t,1) = p(t,0)$, we have $\langle p_x, h-g\rangle = 0$ and we obtain the expected formula. $\square$

The aim is now to define the stationnary states to which the control and the solution should converge. By trial and error we choose it as following.

Definition 3.1. *We define $(\bar{g}, \bar{u})$ as the unique solution of the control problem :*

$$
\begin{cases}
\bar{u}_{xx} = 0 \ \textit{sur} \ ]0,1[, \\
\bar{u}(t,0) = 0, \\
\bar{u}(t,1) = \bar{g}.
\end{cases}
$$

with the following functional to minimize :

$$
J_s(\bar{g}) = \frac{1}{2} |u_{\bar{g}} - u_1|^2_{L^2(0,1)}.
$$

Besides, we define $\bar{v} = \bar{u} - \bar{g}x$ and we take $\bar{p}$ solution of :

$$
\begin{cases}
-\bar{p}_{xx} = x\bar{g} - u_1 \ \textit{sur} \ ]0,1[\\
\bar{p}(1) = \bar{p}(0) = 0.
\end{cases}
\tag{3.5}
$$

It is easy to compute that :

$$
\bar{g} = 3\int_0^1 x u_1(x)dx,
$$

$$
\bar{u}(x) = \bar{g}x,
$$

$$
\bar{v} = 0,
$$

$$
\bar{p}(x) = -\frac{\bar{g}}{6}x^3 + \left(\frac{\bar{g}}{2} - \int_0^1 u_1(y)dy\right)x + \int_0^x (x-\eta)u_1(\eta)d\eta.
$$

So we have the coupled system :

$$
\begin{cases}
v_t - v_{xx} = -x(g-\bar{g})' \text{ in }]0,T[\times]0,1[\\
-(p-\bar{p})_t - (p-\bar{p})_{xx} = v + x(g-\bar{g}) \text{ in }]0,T[\times]0,1[.
\end{cases}
\tag{3.6}
$$

Theorem 3.1. *For the one dimensionnal heat equation with boundary control, there is convergence in average :*

$$\frac{1}{T}\int_0^T |g'(t)|^2 dt + \frac{1}{T}\int_0^T\int_0^1 |u(t,x)-\bar{u}(x)|dxdt = \underset{T\to\infty}{\mathcal{O}}(\frac{1}{T}).$$

In particular :

$$\frac{1}{T}\int_0^T u(t,\cdot)dt \underset{T\to+\infty}{\longrightarrow} \bar{u} \text{ dans } L^1(0,1),$$

$$\frac{1}{T}\int_0^T g(t)dt \underset{T\to+\infty}{\longrightarrow} \bar{g}.$$

Proof. As it has been all along this report, we work on the coupled system (3.5) by taking the following scalar product and developing by the integration formula :

$$\begin{aligned}\langle (p-\bar{p})_t, v\rangle &= -\langle p-\bar{p}, v_t\rangle + \int_0^1 \left[(p-\bar{p})(T,x)v(T,x) - (p-\bar{p})(0,x)v(0,x)\right]dx \\ &= -\langle p-\bar{p}, v_t\rangle + \langle -\bar{p}, v(T,\cdot)\rangle - \langle (p-\bar{p})(0,\cdot), u_0\rangle\end{aligned}$$

We write α the term $\langle -\bar{p}, v(T,\cdot)\rangle - \langle (p-\bar{p})(0,\cdot), u_0\rangle$, which we will easily estimate in what follows.
By the equation on v we have then :

$$\begin{aligned}\langle (p-\bar{p})_t, v\rangle &= -\langle p-\bar{p}, v_{xx} - x(g-\bar{g})'\rangle + \alpha \\ &= \langle p-\bar{p}, x(g-\bar{g})'\rangle - \langle p-\bar{p}, v_{xx}\rangle + \alpha.\end{aligned}$$

But,

$$\begin{aligned}\langle p-\bar{p}, v_{xx}\rangle &= \langle (p-\bar{p})_{xx}, v\rangle + \int_0^T ((p-\bar{p})v_x(t,1) - (p-\bar{p})v_x(t,0))dt \\ &= \langle (p-\bar{p})_{xx}, v\rangle\,,\end{aligned}$$

So :

$$\langle (p-\bar{p})_t, v\rangle = \langle p-\bar{p}, x(g-\bar{g})'\rangle - \langle (p-\bar{p})_{xx}, v\rangle + \alpha. \tag{3.7}$$

Because of the equation (3.6) on $(p-\bar{p})$ we get :

$$\langle (p-\bar{p})_t, v\rangle = -\langle v, v\rangle - \langle x(g-\bar{g}), v\rangle - \langle (p-\bar{p})_{xx}, v\rangle\,. \tag{3.8}$$

Combining (3.6) and (3.7) we get :

$$-|v|^2 - \langle x(g-\bar{g}), v\rangle = \langle p-\bar{p}, x(g-\bar{g})'\rangle + \alpha.$$

Let us develop the right term :

$$\begin{aligned}\langle p-\bar{p}, x(g-\bar{g})'\rangle = & \int_0^1 \left[(p-\bar{p})(T,x)x(g-\bar{g})(T) - (p-\bar{p})(0,x)x(g(0)-\bar{g})\right] dx \\ & - \langle (p-\bar{p})_t, x(g-\bar{g})\rangle\end{aligned}$$

We write β the term $\int_0^1 \left[(p-\bar{p})(T,x)x(g-\bar{g})(T) - (p-\bar{p})(0,x)x(g(0)-\bar{g})\right] dx$, then we have :

$$\langle p-\bar{p}, x(g-\bar{g})'\rangle = \langle (p-\bar{p})_{xx} + v + x(g-\bar{g}), x(g-\bar{g})\rangle + \beta.$$

By going back to (3.8) one finally gets :

$$|v|^2 + 2\langle v, x(g-\bar{g})\rangle + |x(g-\bar{g})|^2 = -\langle (p-\bar{p})_{xx}, x(g-\bar{g})\rangle - \alpha - \beta.$$

But, $u = v + xg$ so $u - \bar{u}$ appears as follows :

$$|u-\bar{u}|^2 = -\langle (p-\bar{p})_{xx}, x(g-\bar{g})\rangle - \alpha - \beta.$$

Let us develop the term $\langle (p-\bar{p})_{xx}, x(g-\bar{g})\rangle$. For $t \in]0,T[$, we define $f(x) = (p-\bar{p})(t,x)$. We know that $p \in L^2(0,T;H^2(0,1))$ so $f \in H^2(0,1)$ and we can prove that $\int_0^1 xf''(x)dx = f'(1) - f(1) + f(0)$. This yields to :

$$\begin{aligned}\langle (p-\bar{p})_{xx}, x(g-\bar{g})\rangle &= \int_0^T (g(t)-\bar{g}) \left(\int_0^1 x(p-\bar{p})_{xx}(t,x)dx\right) dt \\ &= \int_0^T (g(t)-\bar{g})(p-\bar{p})_x(t,1) \\ &= |g'|^2 - \bar{g}\int_0^T p_x(t,1)dt.\end{aligned}$$

For the last time, let us define $\gamma := -\bar{g}\int_0^T p_x(t,1)dt$.
We finally established the equation :

$$|u-\bar{u}|^2 + |g'|^2 = -\alpha - \beta - \gamma. \tag{3.9}$$

The terms α, β et γ are boundary terms. In order to estimate them, we use the well-posedness of the equation on p, on $p-\bar{p}$ and on v, there exists a constant $c > 0$ independant on T such that :

$$|p|_{L^2} + |p_t|_{L^2} \le c|u - u_1|_{L^2(Q)}$$

$$|p - \bar{p}|_{L^2} + |(p - \bar{p})_t|_{L^2} \le c(|v|_{L^2(Q)} + |g - \bar{g}|_{L^2(0,T)})$$

$$|v|_{L^2} + |v_t|_{L^2} \le c|g'|_{L^2(0,T)}. \tag{3.10}$$

We write ϕ the quantity $|u - \bar{u}|^2 + |g'|^2$. The aim is to estimate it. Thanks to the previous inequalities, one can prove :

$$|\alpha| \le c\phi, \quad |\beta| \le c(1 + \sqrt{T\phi}) \text{ et } |\gamma| \le c\sqrt{T}|u - u_1|_{L^2(Q)}.$$

Indeed, let us prove the estimation on γ for example. According to the Cauchy-Schwartz inequality, to the continuity of the trace and to the inequality (3.10) we have :

$$|\gamma| \le cT^{\frac{1}{2}} \left(\int_0^T |p_x(t,1)|^2 dt \right)^{\frac{1}{2}} \le cT^{\frac{1}{2}} \left(\int_0^T |p(t,\cdot)|_{H^2(0,1)} dt \right)^{\frac{1}{2}} \le c\sqrt{T}|u - u_1|_{L^2(Q)}.$$

Combining these three estimations and going back to (3.9) one gets :

$$\forall T > 0, \quad \phi^2 \le c(1 + \sqrt{T}\phi + \sqrt{T}|u - u_1|_{L^2(Q)}) \tag{3.11}$$

where the constant c doesn't depend on T.
However, we know that the solution $\tilde{u}$ of the heat equation (3.1) without control exponentially converges to 0. Therefore, there exists a constant $c > 0$ such that for every $T > 0$, we have : $\min J^T \le J(\tilde{u}) \le c$. So, because of the minimum, we also have for u : $|u - u_1|_{L^2} \le \min J^T \le c$. Thus, thanks to (3.11), we obtain :

$$\forall T > 0, \quad \phi^2 \le c(1 + \sqrt{T} + \sqrt{T}\phi) \tag{3.12}$$

By elementary reasonning one deduces from (3.12) that :

$$\lim_{T \to +\infty} \frac{1}{T}\phi^2 = 0,$$

which yields the first convergence claimed in the theorem. The second consequence is an immediate consequence.

For the last convergence we should just notice that :

$$\frac{1}{T}|g - \bar{g}|_{L^1(0,T)} \le \frac{1}{\sqrt{T}}|g - \bar{g}|_{L^2(0,1)} = \frac{3}{\sqrt{T}}|xg - x\bar{g}|_{L^2(Q)} = \frac{3}{\sqrt{T}}|u - \bar{u} - v|_{L^2(Q)} \le c(\frac{1}{T}\phi^2)^{\frac{1}{2}}.$$

$\square$

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