Lecture 2: Combinatorics

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1 Permutations, arrangments, choices

1.1 What we have seen last time

The number of all permutations on n distinct elements is the number $n \cdot (n-1) \cdot (n-2) \cdot ...1 = n!$.

However if we need to permute just k of them (taking from n) we will get the number of arrangements: $n(n-1)(n-2)...(n-k+1) = \frac{n!}{(n-k)!}$

1.2 Pascal's triangle

Pascal's triangle can be obtained line by line by:

- 0th (top) line is "1"
- 1st line is "1 1" on each side of
- on kth line we have k numbers, first and last are 1 and in the middle take sums of neighbors on the $(k-1)$st line. (Figure 1)

Compare the kth line of the triangle and the coefficient of the expansion of $(1+x)^k$. They are the same!

The kth number in the nth line of Pascal's triangle is called a binomial coefficient $\binom{n}{k}$.

This number is actually equal to the number of ways of choosing k elements in the set of n elements.

The big difference with arrangements is that only elements you choose matter and not the order you choose them.

For example if you arrange 3 numbers out of $\{1, 2, ..., 10\}$, then the arrangements $[1, 3, 9]$ and $[3, 9, 1]$ are not the same, as they are not the same lists. But if the two choices $\{1, 3, 9\}$ and $\{3, 9, 1\}$ are the same as sets.

Hence, we have $k!$ arrangements corresponding to one set, therefore:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Figure 1: Pascal's triagle

Problem 1 *1. Show that* $\binom{n}{k} = \binom{n}{n-k}$

2. *Show that* $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ *using Pascal's triangle, polynomes and combinatorially.*
3. *What is the sum of binomial coefficients* $\sum_{k=0}^{n} \binom{n}{k}$*?*
4. *What is the alternate sum of binomial coefficients* $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$*?*
5. *Show that* $\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$ *combinatorially.*

Solution.

1. Observe that $\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k! = \binom{n}{k}}$.

   There is also a combinatorial way to see this identity. The number of ways of choosing k elements is the same as not to choose $n-k$ elements.

2. $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$ in Pascal's triangle correspond to the numbers above left and above right of $\binom{n}{k}$. By definition of Pascal's triangle, the latter one is the sum of the formers.

   Another way to solve this exercise is to observe that $(1+x)^n = (1+x)^{n-1}(1+x)$, which means that:
   $$\sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n-1} \binom{n-1}{k} x^k + \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1}$$
   By associating the the coefficients corresponding to x^k on left and right sides, we obtain the identity.

   You can also see this identity combinatorially. Take the "first" element of a set. When you try to choose k elements from the set, you either choose the first one or not. Depending on the choice for the first element, you choose either k or $k-1$ elements from the rest of the set, which cardinality is $(n-1)$.

3. Take $x = 1$ in $(1+x)^n$. $(1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$ and on the other hand $(1+1)^n = 2^n$, hence the identity $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

   A combinatorial way to solve the problem would be to say that in $\sum_{k=0}^{n} \binom{n}{k}$ you count all possible ways to choose an arbitrary number of elements, or simply said, all possible subsets. When you count subsets, you can simply decide for each element whether you take it or not, therefore the total number of choices is the product $2 \cdot 2 ... 2 = 2^n$.

4. Take $x = -1$ in $(1+x)^n$. $(1-1)^n = \sum_{k=0}^{n} (-1)^n \binom{n}{k} = 0^n$. There are two possibilities: if $n = 0$, $0^n = 1$, otherwise $0^n = 0$. So:
   $$\sum_{k=0}^{n} (-1)^n \binom{n}{k} = \begin{cases} 1, \text{if } n = 0 \\ 0, \text{otherwise} \end{cases}$$

5. Take a set S consisting of n elements. Choose in this set a subset A consisting of k elements and then choose a subset B of A consisting of j elements. The total number of ways to choose the subsets A and B is $\binom{n}{k}\binom{k}{j}$.

   Now, reverse the order. Choose j elements from S to form the set B. What is left for us to choose are the $k-j$ elements from the set $A \setminus B$ from the rest of S. The number of choosing this way is $\binom{n}{j}\binom{n-j}{k-j}$, hence the identity.

2 Inversion formula and disorders

Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be two sequence of numbers, such that:

$$g_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}f_k$$

Problem 2 *Show that:*

$$f_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}g_k$$

Solution. Use what we have seen already!

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k\binom{n}{k}g_k &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}f_j \\
&= \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^k\binom{n}{k}(-1)^j\binom{k}{j}f_j \\
&= \sum_{j=0}^{n}\sum_{k=j}^{n}(-1)^k\binom{n}{k}(-1)^j\binom{k}{j}f_j \\
&= \sum_{j=0}^{n}(-1)^j f_j\sum_{k=j}^{n}(-1)^k\binom{n}{j}\binom{n-j}{k-j} \\
&= \sum_{j=0}^{n}(-1)^j f_j\binom{n}{j}\sum_{k=j}^{n}(-1)^k\binom{n-j}{n-k} \\
&= \sum_{j=0}^{n}(-1)^j f_j\binom{n}{j}\sum_{k=0}^{n-j}(-1)^{k+j}\binom{n-j}{k} \\
&= \sum_{j=0}^{n} f_j\binom{n}{j}\sum_{k=0}^{n-j}(-1)^{k}\binom{n-j}{k}
\end{aligned}$$

As we showed earlier, the sum $\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}$ has 2 different values depending on whether $n-j=0$ or not.

If not, the sum is 0, so the whole term is 0. Otherwise $j=n$, the sum is 1, $\binom{n}{j}=\binom{n}{n}=1$, so the sum is f_n.

Problem 3 *Hat dropping problem n football fans in hats watch a game in a stadium. They root for the same team and this team scores. Everybody throws the hat in the air and the hats fall on the heads of the fans (not necessarily their own). Everybody gets at least one hat on their head in the end. How many ways is there for hats to fall, so that nobody wears his own hat in the end.*

Solution.

Give every fan an indice: 1, 2, 3,..., n. We want to find the number D_n of disorders on these n numbers.

Take a random permutation of the $[1, 2, 3, ..., n]$.

Let F_k be the number of permutations with exactly k fixed points of the permutation.

The $(n-k)$ other fans form a disorder, so $F_k = \binom{n}{k}D_{n-k}$.

If we take the sum of F_k and they give all the permutations:

$$n! = \sum_{k=0}^{n} F_k = \sum_{k=0}^{n} \binom{n}{k} D_{n-k}$$

To solve the problem, we will use the inversion formula on the sequences $((-1)^n D_n)_{n\in\mathbb{N}}$ and $(n!)$:

$$D_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (-1)^k k! = \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!}$$

Therefore:

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!}$$