Internalizing the simplicial model structure in Homotopy Type Theory

Simon Boulier

A type theory with two equalities

Model structures in mathematics

Internalizing the simplicial model structure in HoTT

Homotopy (co)limits in HoTT

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Martin-Löf Type Theory

- $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \ldots$
- $\Pi x : A.\, B$
- $\Sigma x : A.\, B$
- $x =_A y$
- $A + B\,, \quad \mathbf{0}\,, \quad \mathbf{1}\,, \quad \mathbb{N}\,, \quad \ldots$

Martin-Löf Type Theory

- $\mathcal{U}$
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- Uniqueness of Identity Proofs (UIP) : $p, q : x = y \;\rightarrow\; p = q$
- or Univalence : $A \simeq B \;\rightarrow\; A = B$

Two equalities

$$x \equiv_A y \qquad\qquad x =_A y$$

- $x \equiv_A y \quad \longrightarrow \quad x =_A y$
- $x =_A y \quad \not\longrightarrow \quad x \equiv_A y$

Two equalities

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A new judgment: $\Gamma \vdash A$ Fib
And a new universe: $\mathcal{U}_{\mathsf{F}}$

And we restrict the elimination of $=$:

$$\frac{\Gamma, y : A, e : u = y \vdash P : \mathcal{U} \qquad \Gamma, y, e \vdash P \text{ Fib} \qquad \Gamma \vdash p : u = v \qquad \Gamma \vdash t : P\{y := u, e := \mathsf{refl}_v\}}{\Gamma \vdash J_=(u, P, t, v, p) : P\{y := v, e := p\}}$$

Fibrancy

$$\frac{}{\Gamma \vdash \mathcal{U}_{\mathsf{F}} \ \ \mathsf{Fib}} \qquad \frac{\Gamma \vdash A \ \ \mathsf{Fib} \qquad \Gamma, x : A \vdash B \ \ \mathsf{Fib}}{\Gamma \vdash \Pi x : A.\, B \ \ \mathsf{Fib}}$$

$$\frac{\Gamma \vdash A \ \ \mathsf{Fib} \qquad \Gamma, x : A \vdash B \ \ \mathsf{Fib}}{\Gamma \vdash \Sigma x : A.\, B \ \ \mathsf{Fib}} \qquad \frac{\Gamma \vdash A \ \ \mathsf{Fib}}{\Gamma \vdash x =_A y \ \ \mathsf{Fib}}$$

$$\frac{}{\Gamma \vdash \mathbf{0} \ \ \mathsf{Fib}} \qquad \frac{}{\Gamma \vdash \mathbf{1} \ \ \mathsf{Fib}} \qquad \frac{\Gamma \vdash A \ \ \mathsf{Fib} \qquad \Gamma \vdash B \ \ \mathsf{Fib}}{\Gamma \vdash A + B \ \ \mathsf{Fib}}$$

Summary

- $x \equiv_A y$
  - funext
  - UIP
  - unrestricted elimination
- $x =_A y$
  - funext
  - possibly univalence
  - restricted elimination to fibrant types
- $\mathcal{U}$
- $\mathcal{U}_{\mathsf{F}}$

$\Rightarrow$ this is Homotopy Type System (Voevodsky)

Higher Inductive Types (HITs)

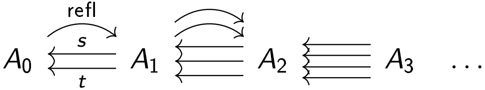
We consider HITs for the path equality $=$.

```
S1 :=
  | base : S1
  | loop : base = base
```

$$\frac{}{\Gamma \vdash \mathbb{S}^1 \ \ \mathsf{Fib}}$$

The model of simplicial sets

- A type A is interpreted as a simplicial set:



- $x = y \; : \; A_1$
- $x \equiv y$: the mathematical equality

The model of simplicial sets

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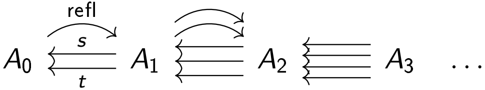

- $x = y \,:\, A_1$
- $x \equiv y$: the mathematical equality

- Fibrant types: Kan complexes.

Which strict equalities?

- the conversion implies $\equiv$
- the provable ones (eg. $n + 0 \equiv 0 + n$)
- UIP: $e, e' : x \equiv y \quad \rightarrow \quad e \;\equiv\; e'$
- funext: $\Pi_{x:A} f(x) \equiv g(x) \quad \rightarrow \quad f \equiv g$

Which strict equalities?

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- ap π_1 $(p, q) \ \equiv \ p$
- ap $(g \circ f)$ $p \ \equiv$ ap g (ap f p)
- ap id $p \ \equiv \ p$
- beta rules for HITs for the path constructors
- eta rules for HITs

HTS in Coq

Hack in Coq: private inductive type and typeclasses.

Categories in HTS

Definition (Category)

A category consists of:

- *a type A of objects,*
- $\forall\, a, b : A$ *, a type* $Hom(a, b)$ *of morphisms*
- $\forall\, a$ *, an identity morphism* $1_a : Hom(a, a)$
- $\forall\, a, b, c$ *, a composition function*
  $\_ \circ \_ : Hom(b, c) \to Hom(a, b) \to Hom(a, c)$
- $\forall\, f$ *, a proof of* $f \circ 1_a \equiv f$ *and* $1_b \circ f \equiv f$
- $\forall\, f, g, h$ *, a proof of* $h \circ (g \circ f) \equiv (h \circ g) \circ f$

$\mathcal{U}$ is a category and $\mathcal{U}_\mathsf{F}$ a full subcategory of $\mathcal{U}$.

Model structure

Definition (Model Structure)

A model structure on a category $\mathcal{C}$ consists of three classes of morphisms:

- W *the weak equivalences*
- F *the fibrations*
- C *the cofibrations*

satisfying ...

- $AF := F \cap W$ are the acyclic (or trivial) fibrations
- $AC := C \cap W$ are the acyclic (or trivial) cofibrations

Lifting property

$\mathcal{C}$ a category.

Definition (Lifting property)

$f : X \to Y$ *and* $g : X' \to Y'$
$f \bowtie g$ *if*

$$\begin{array}{ccc} X & & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & & Y' \end{array}$$

Lifting property

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Definition (Lifting property)

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$f \bowtie g$ if for all squares

$$\begin{array}{ccc} X & \xrightarrow{F} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{G} & Y' \end{array}$$

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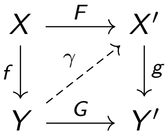


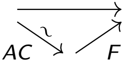
there exists $\gamma : Y \to X'$ filling the diagonal.

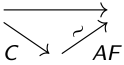
$S \bowtie S'$ if for all $f \in S$ and $g \in S'$, $f \bowtie g$.

Model Structure

Definition (Model Structure)

A model structure on a category $\mathcal{C}$ consists of three classes of morphisms W, F and C satisfying:

- $AC \bowtie F$ *and* $C \bowtie AF$
- *a function can be factorized as* 

- *a function can be factorized as* 

- W *is stable under two out of three*
- W, F *and* C *are stable under retract*

Model category = complete and cocomplete category + model structure.

Model structure on sSet

Classical Model Structure

The classical model structure – or **Quillen model structure** $\mathrm{sSet}_{\mathrm{Quillen}}$ on sSet has the following distinguished classes of morphisms:

Definition 1.

- The **cofibrations** C are simply the monomorphisms $f : X \to Y$ which are precisely the levelwise injections, i.e. the morphisms of simplicial sets such that $f_n : X_n \to Y_n$ is an injection of sets for all $n \in \mathbb{N}$.

- The **weak equivalences** W are **weak homotopy equivalences**, i.e., morphisms whose geometric realization is a weak homotopy equivalence of topological spaces.

- The **fibrations** F are the **Kan fibrations**, i.e., maps $f : X \to Y$ which have the right lifting property with respect to all horn inclusions.

$$\begin{array}{ccc} \Lambda^k[n] & \to & X \\ \downarrow & {}^{\exists}\nearrow & \downarrow^f \\ \Delta[n] & \to & Y \end{array} .$$

- A morphism $f : X \to Y$ of fibrant simplicial sets / Kan complexes is a weak equivalence precisely if it induces an isomorphism on all simplicial homotopy groups.

- All simplicial sets are cofibrant with respect to this model structure.

- The fibrant objects are precisely the Kan complexes.

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Model structure on $\mathcal{U}_\mathsf{F}$

- weak equivalences = HoTT-equivalences
- fibrations = destructors
- cofibrations = constructors

Definition (HoTT-equivalence)

$f : A \to B$ is an equivalence if there exists $g : B \to A$ such that

- *for all x, $g\,(f\,x) = x$*
- *for all y, $f\,(g\,y) = y$*

First factorization system (2008)

THE IDENTITY TYPE WEAK FACTORISATION SYSTEM

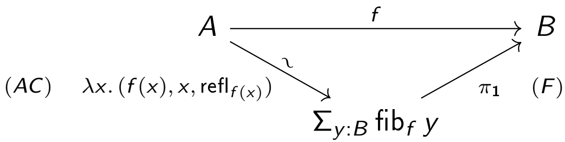
NICOLA GAMBINO AND RICHARD GARNER

Abstract. We show that the classifying category $\mathcal{C}(\mathbb{T})$ of a dependent type theory $\mathbb{T}$ with axioms for identity types admits a non-trivial weak factorisation system. We provide an explicit characterisation of the elements of both the left class and the right class of the weak factorisation system. This characterisation is applied to relate identity types and the homotopy theory of groupoids.

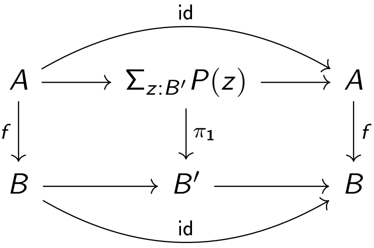
1. Introduction

From the point of view of mathematical logic and theoretical computer science, Martin-Löf's axioms for identity types [25] admit a conceptually clear explanation in terms of the propositions-as-types correspondence [14, 22, 28]. The fundamental idea behind this explanation is that, for any two elements a, b of a type A, we have a new type $\mathrm{Id}_A(a, b)$, whose elements are to be thought of as proofs that a and b are equal. Yet, identity types determine a highly complex structure on each type, which is far from being fully understood. A glimpse of this structure reveals itself as soon as we start applying the construction

First factorization system

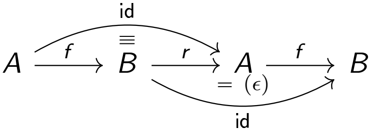


Fibrations:



with $P : B' \to \mathcal{U}_\mathsf{F}$

Injective Equivalences (AC):



$+ \ \epsilon_{f(x)} \equiv \mathsf{refl}_{f(x)}$

Second factorization system (2011)

MODEL STRUCTURES FROM HIGHER INDUCTIVE TYPES

PETER LEFANU LUMSDAINE

Abstract. We show that for any dependent type theory with Martin-Löf identity types and *mapping cylinders* (defined as certain higher-dimensional inductive types), the category of contexts carries a *pre-model-structure*, i.e. a model structure minus the completeness conditions. The (trivial cofibrations,fibrations) are the Gambino-Garner weak factorisation system of [GG08], while the weak equivalences are equivalences in the sense of Voevodsky [Voe].

It follows that any categorical model of this type theory carries a pre-model-structure, and so, if it is additionally complete and co-complete, is a model category.

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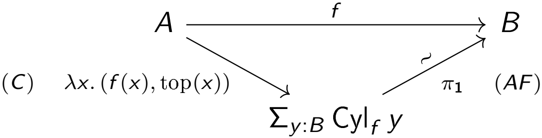
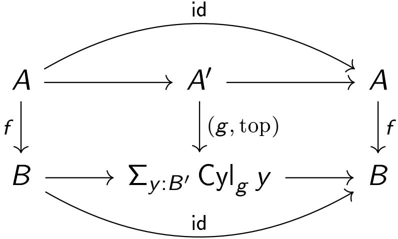
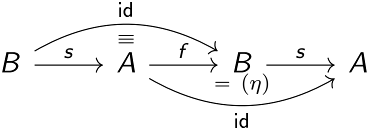
Second factorization system



Cofibrations:



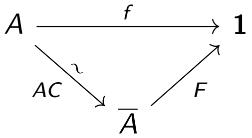
Surjective Equivalences (AF):



$$+\ \mathsf{ap}\ \ f\ \ \eta_x\ \equiv\ \mathsf{refl}_{f(x)}$$

Toward an extension to $\mathcal{U}$

The fibrant replacement:



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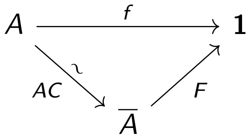


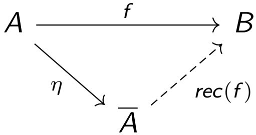
$$\mathrm{sSet} \underset{\iota}{\overset{\overline{X}}{\rightleftarrows}} \mathrm{Kan} \qquad\qquad \mathcal{U} \underset{\iota}{\overset{\overline{A}}{\rightleftarrows}} \mathcal{U}_{\mathsf{F}}$$

Toward an extension to $\mathcal{U}$

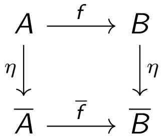
Fibrant replacement

A strict modality:

- $\eta : A \to \overline{A}$
- $\mathrm{Fib}\, \overline{A}$
- if $\mathrm{Fib}\, B$:



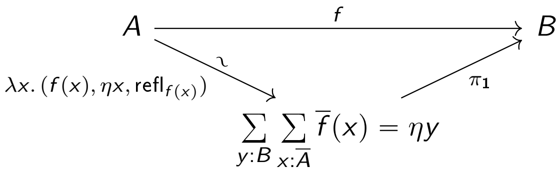
- $rec(f) \circ \eta \equiv f$ and $rec(f \circ \eta) \equiv f$



Toward an extension to $\mathcal{U}$

New weak equivalences: $f \in W$ if $\overline{f}$ is an HoTT-equivalence.

Slight modification of the first factorization system :



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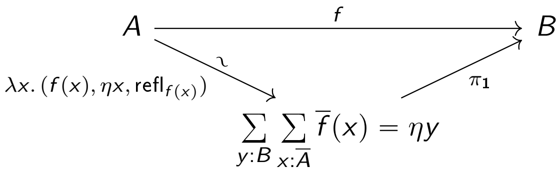

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The example of the equalizer

$$A \overset{f}{\underset{g}{\rightrightarrows}} B$$

Strict limit: $\Sigma_x f(x) \equiv g(x)$

(Presumed) homotopy limit: $\Sigma_x f(x) = g(x)$

limit and **holimit** are two functors : $\{\text{diag of equalizer}\} \to \mathcal{U}$.

holimit is indeed an homotopy limit if it is a right deformation retract of **limit**.

The example of the equalizer

The fibrant replacement of the diagram:

$$\sum_{y:B\times B}\sum_{x:A}(f\,x, g\,x) = y \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} B$$

Conclusion