Formuler un problème d'ordonnancement juste-à-temps grâce à des inégalités de non-chevauchement

Anne-Elisabeth FALQ

encadrée par Pierre Fouilhoux et Safia Kedad-Sidhoum

18 Novembre 2020, en ligne avec le GSCOP de Grenoble

slides et tapuscrit disponibles sur
http://perso.eleves.ens-rennes.fr/~afalq494/recherche-these.html

Outline

1. Introduction
   Scheduling around a common due date
   Known results about UCDDP and CDDP
   How to encode schedules?

2. A formulation for UCDDP using natural variables

3. How to manage this kind of formulations in practice

4. How to extend this formulation

5. Conclusion

Scheduling around a common due-date on a single machine

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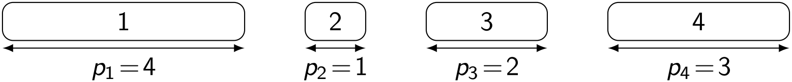

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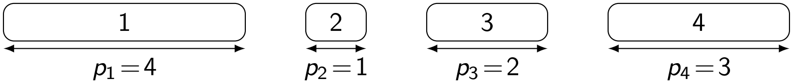

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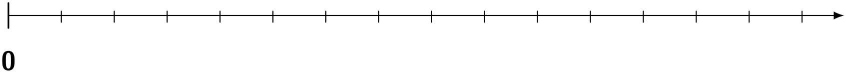

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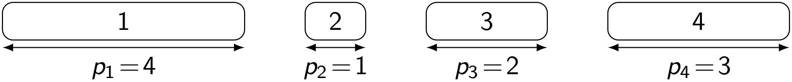

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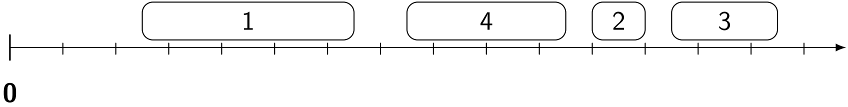

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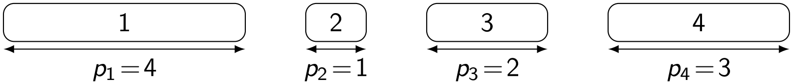

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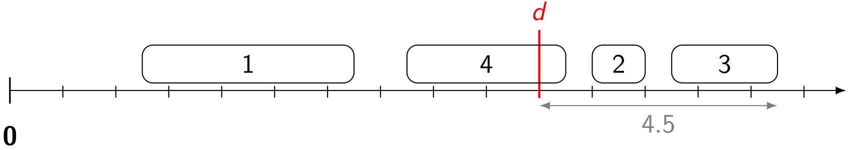

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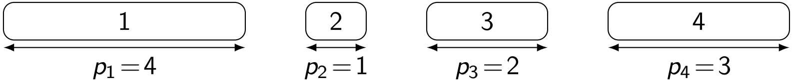

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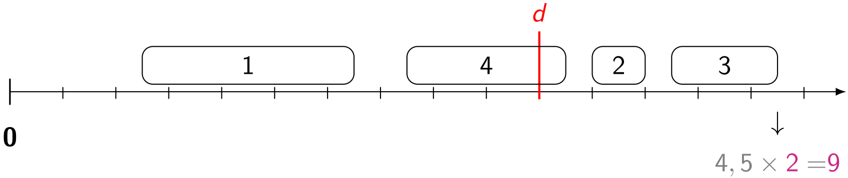

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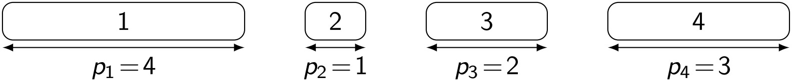

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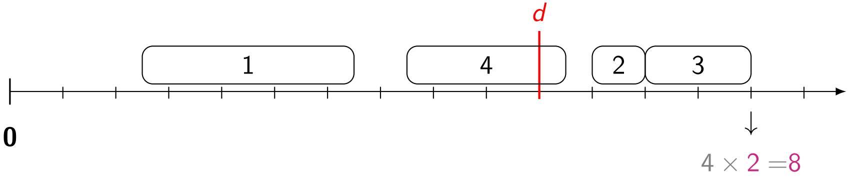

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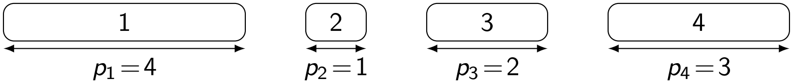

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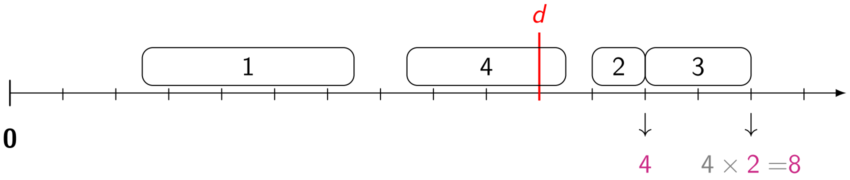

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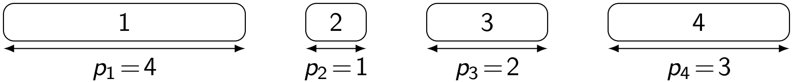

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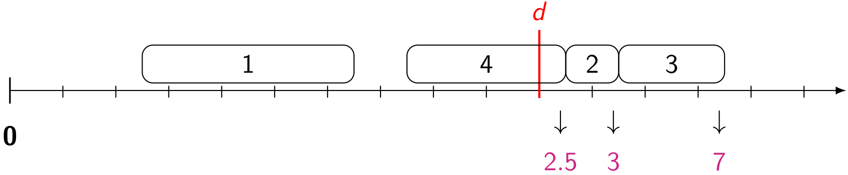

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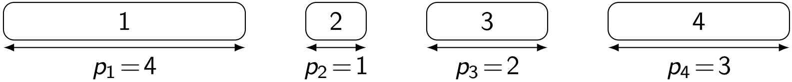

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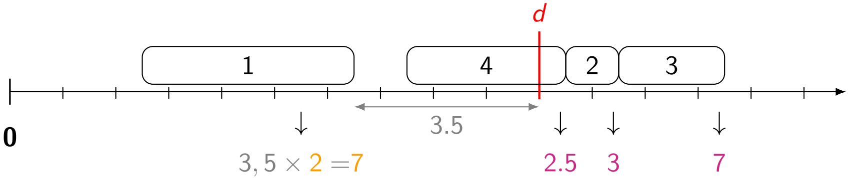

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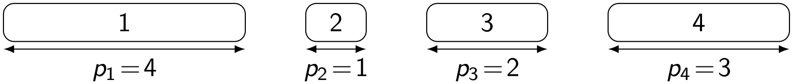

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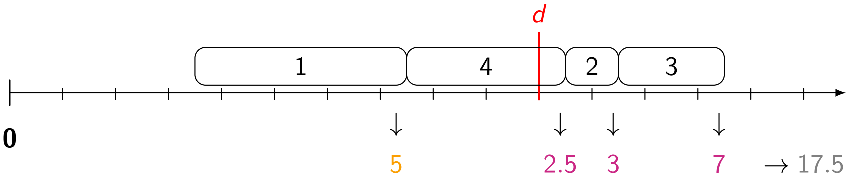

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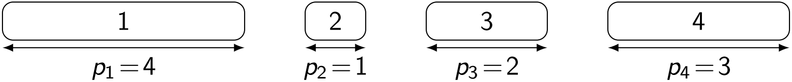

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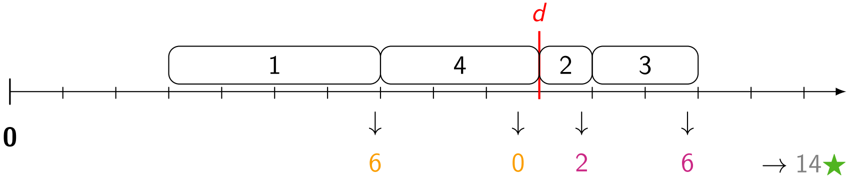

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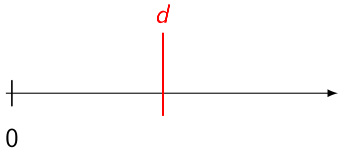

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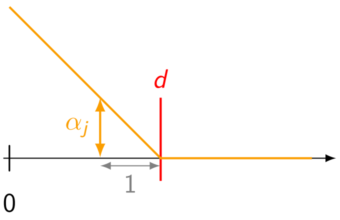

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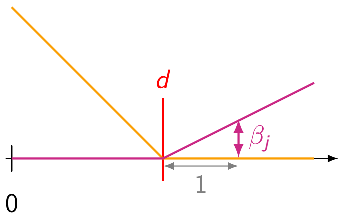

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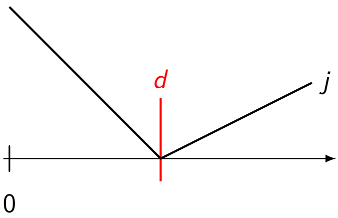

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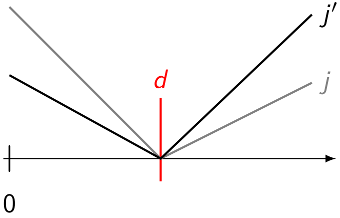

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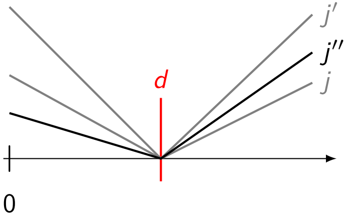

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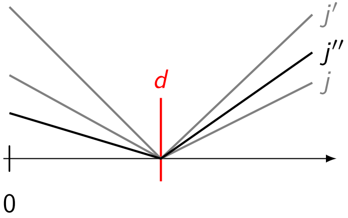

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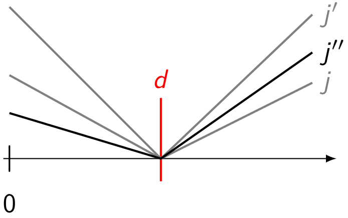

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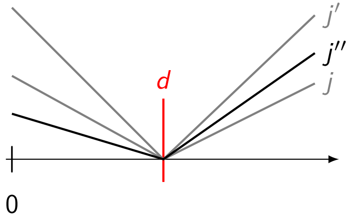

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In both cases,

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Dominance properties and complexity

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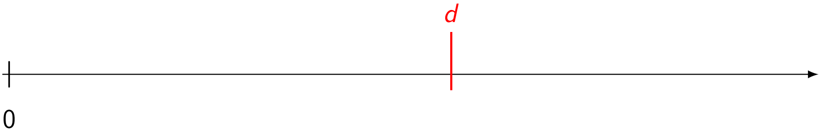
Time-indexed variables



First let us discretize the time horizon .

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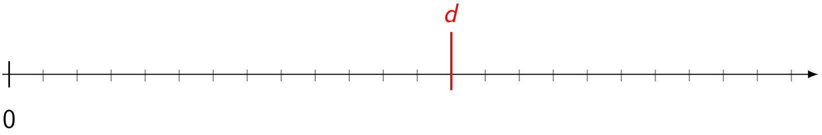

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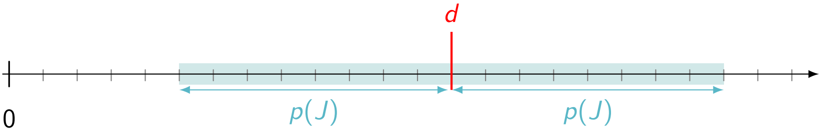

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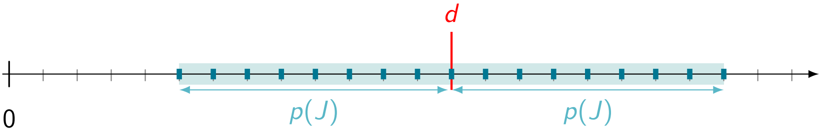

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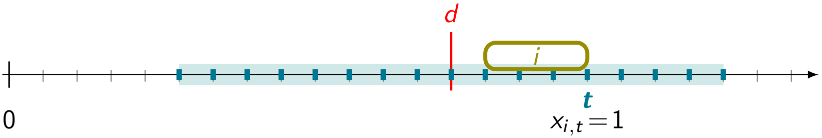

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Variables: $\forall i \in J$, $\forall t \in \mathcal{T}$: $x_{i,t} = \begin{cases} 1 \text{ if } i \text{ completes at } t \\ 0 \text{ otherwise} \end{cases}$

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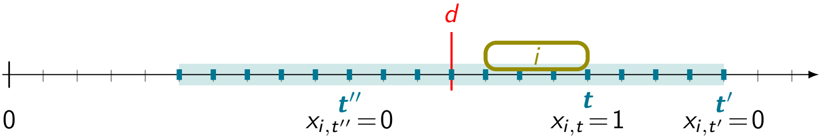

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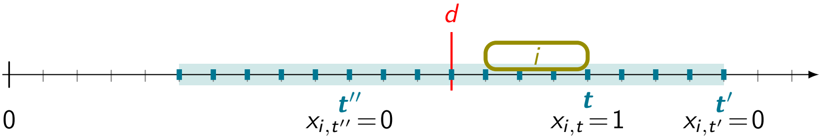
Time-indexed variables



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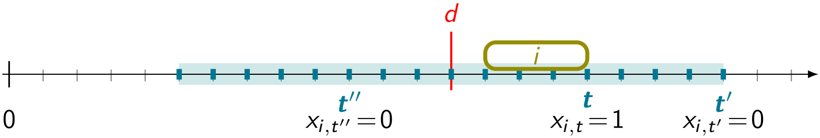

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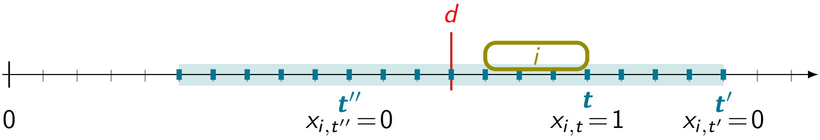
Constraints:

- $\forall i \in J,\ \sum_{t \in \mathcal{T}} x_{i,t} = 1$ task i is placed
- $\forall t \in \mathcal{T},\ \sum_{i \in J} \sum_{\substack{s \in \mathcal{T} \\ s \in [t, t+p_i[}} x_{i,s} \leqslant 1$ at most 1 task is in progress at t
- $\forall i \in J,\ \forall t \in \mathcal{T},\ x_{i,t} \in \mathbb{Z}$ integrity constraint

Time-indexed variables



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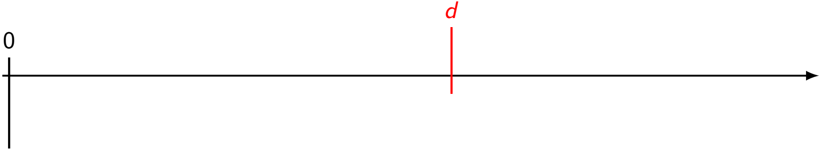
- $+$ easy to formulate as a **MIP**
- $+$ good relaxation value
- $-$ $2n\, p(J)$ binary variables $=$ a pseudo polynomial number
- $-$ $n + n\, p(J)$ inequalities $=$ a pseudo polynomial number

Completion time variables



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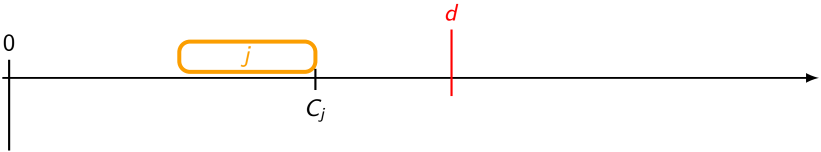

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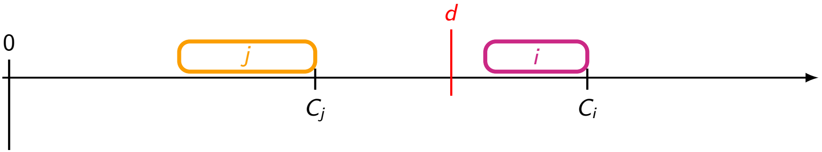

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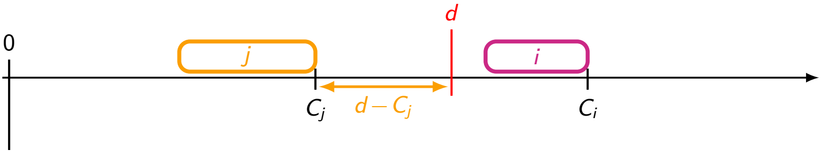

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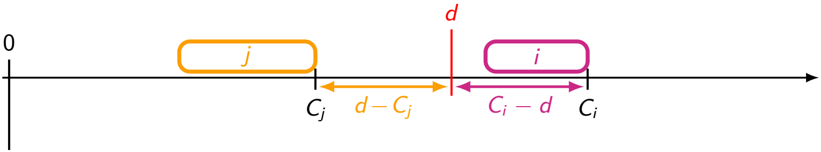

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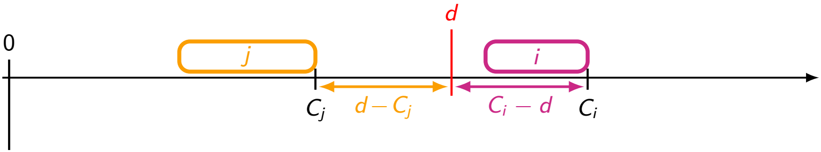

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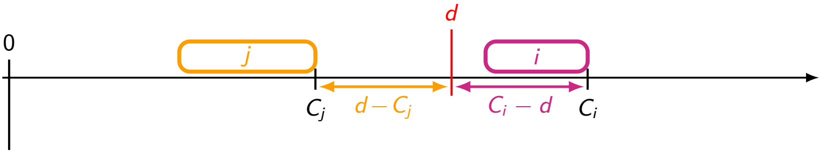

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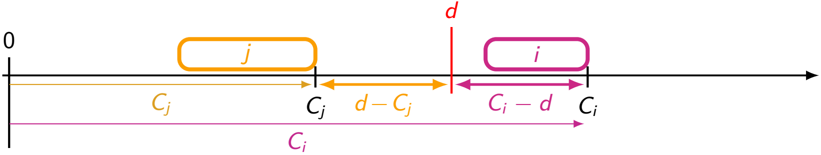

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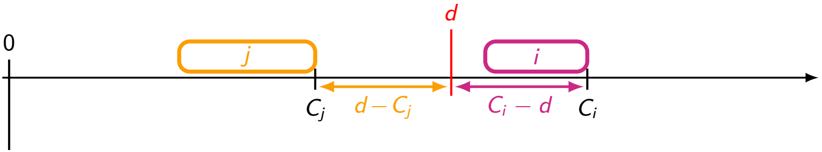

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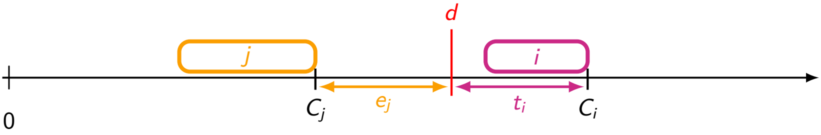
Earliness–Tardiness variables



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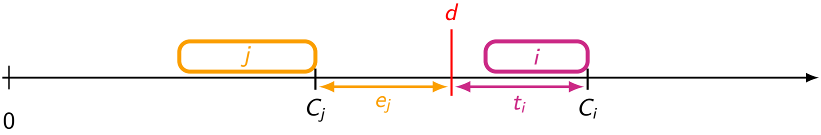

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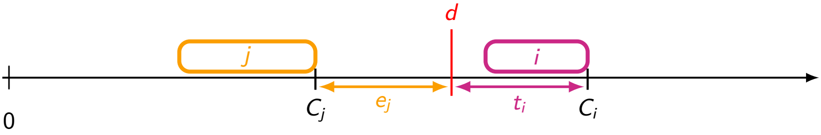

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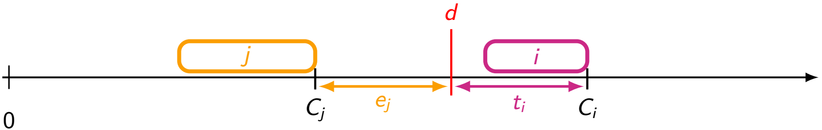

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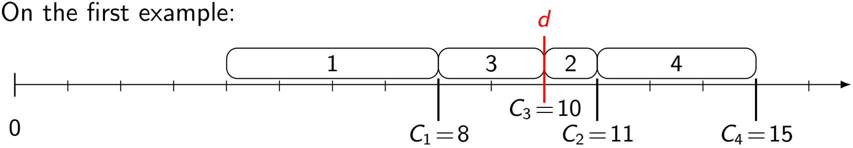

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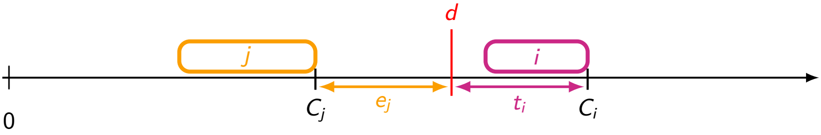

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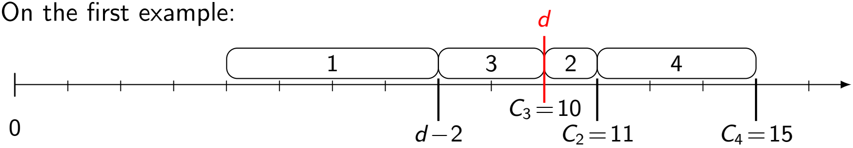

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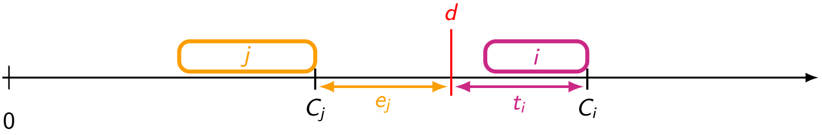

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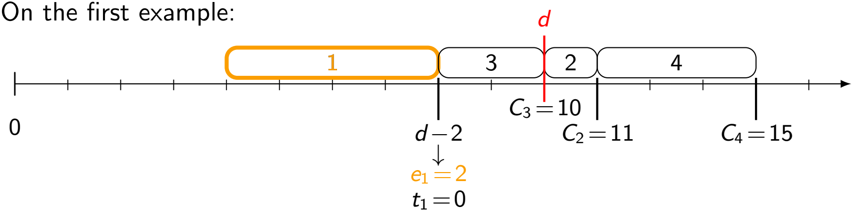

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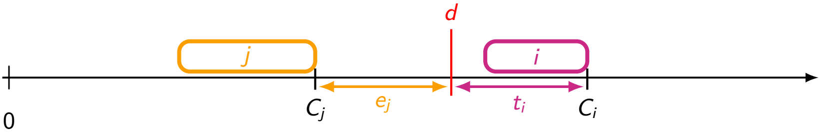

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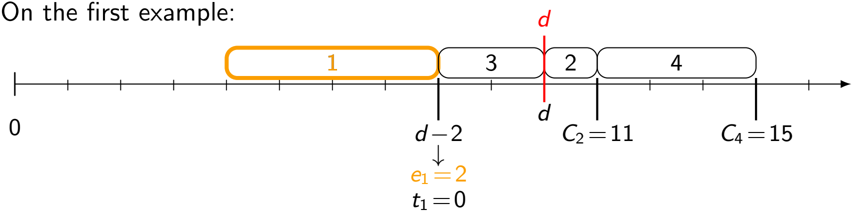

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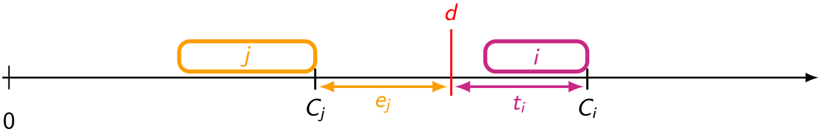

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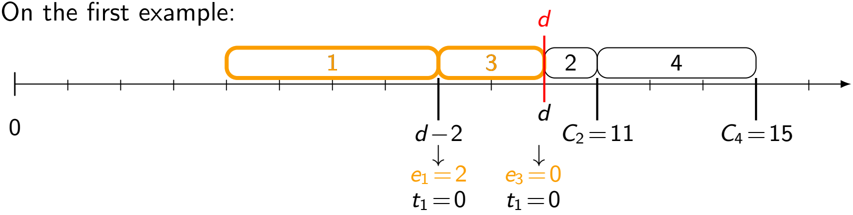

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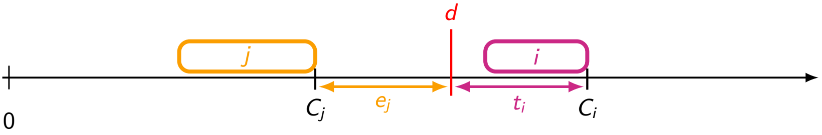

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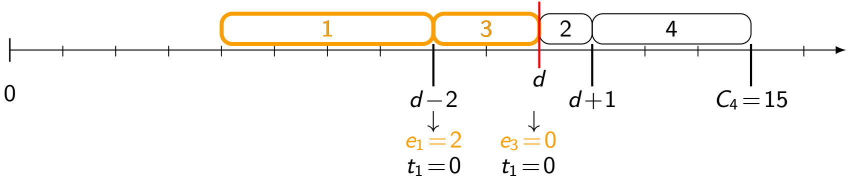

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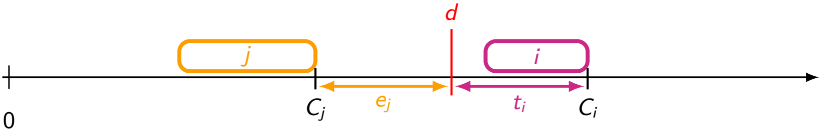

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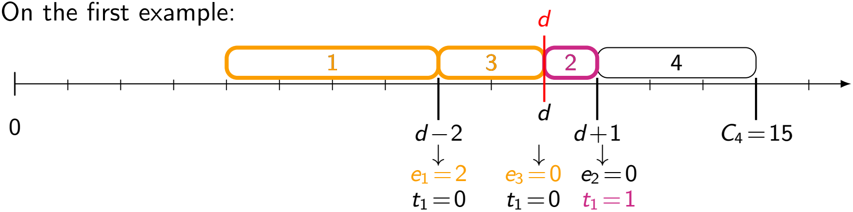

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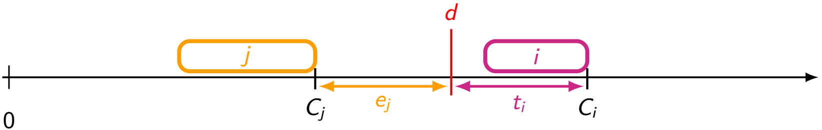

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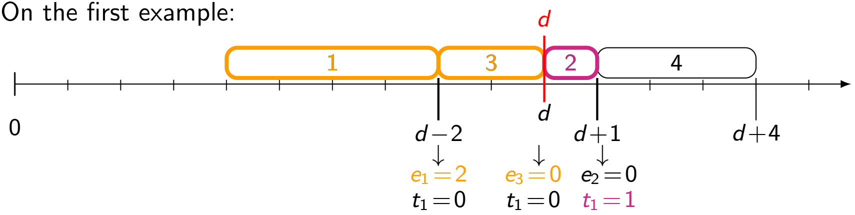

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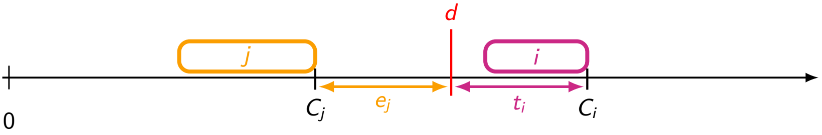

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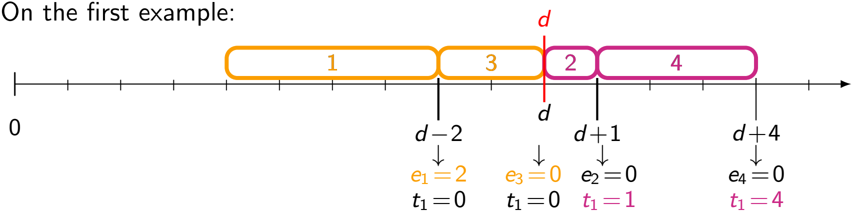

Outline

1. Introduction

2. A formulation for UCDDP using natural variables
   - Describing the solution set for (e, t) variables
   - Non-overlapping inequalities
   - Validity

3. How to manage this kind of formulations in practice

4. How to extend this formulation

5. Conclusion

What is the goal?

We already have: $\boxed{\text{UCDDP} \iff \min\limits_{(e,t)\in\mathscr{S}} g_{\alpha,\beta}(e,t)}$

where : $\rightarrow$ $g_{\alpha,\beta}$ is linear $\left(g_{\alpha,\beta} = (e,t) \mapsto \sum\limits_{j\in J} \alpha_j e_j + \beta_j t_j\right)$

$\rightarrow$ $\mathscr{S}$ is the set of (e,t) vectors that describe a schedule

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We want to describe $\mathscr{S}$ with:

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in order to obtain: $\boxed{\text{UCDDP} \iff \min_{(e,t)\in\mathscr{S}} g_{\alpha,\beta}(e,t)}$

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  if $\mathcal{P} \neq \mathcal{NP}$, it cannot be sufficient (LP-solving $\in \mathcal{P}$ and UCDDP $\in \mathcal{NP}$)
- integrity constraints
- extremality constraints

in order to obtain: $\boxed{\text{UCDDP} \iff \min\limits_{(e,t)\in\text{int}(\text{extr}\, P)} g_{\alpha,\beta}(e,t)}$

What do we need for describing the solution set?

An instance =

- a set of tasks J
- the processing times of these tasks $(p_j)_{j \in J}$
- an unrestrictive common due-date $d \geqslant \sum p_j$
- a unitary earliness penalty for each task $(\alpha_j)_{j \in J}$
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How to describe the solution set?

To encode a feasible schedule, a vector (e, t) must satisfy :

[**consistancy**] e_j and t_j are not simultaneously strictly positive

[**non-overlapping**] processing intervals are pairwise disjoints

[**positivity**] processing intervals don't begin before time 0

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- Adding disjunctive variables $(\delta_j)_{j\in J}$ such that $\delta_j = \begin{cases} 1 \text{ if } j \text{ is early} \\ 0 \text{ if } j \text{ is tardy} \end{cases}$

$$\begin{aligned} \forall j \in J,\ e_j &\geqslant 0 && (e.0) \\ e_j &\leqslant M\,\delta_j && (e.1) \end{aligned} \qquad \begin{aligned} \forall j \in J,\ t_j &\geqslant 0 && (t.1) \\ t_j &\leqslant M\,(1-\delta_j) && (t.2) \end{aligned} \qquad \text{where } \boldsymbol{M} = \sum_{j\in J} p_j$$

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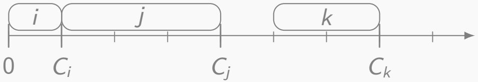
Non-overlapping inequalities for $1 \mid - \mid \min \sum \omega_j C_j$

Queyranne's non-overlapping inequalities



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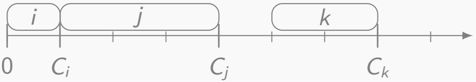

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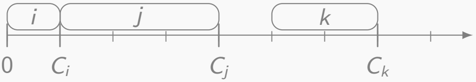

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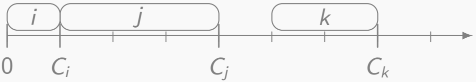

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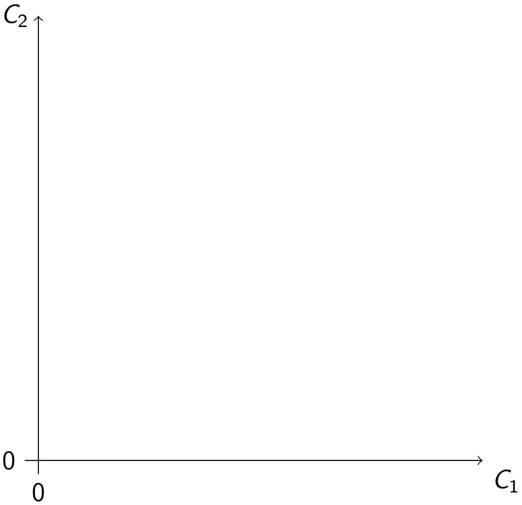
Queyranne, 1993, Mathematical Programming

The set of vectors (C_1, C_2) encoding a 2-task schedule



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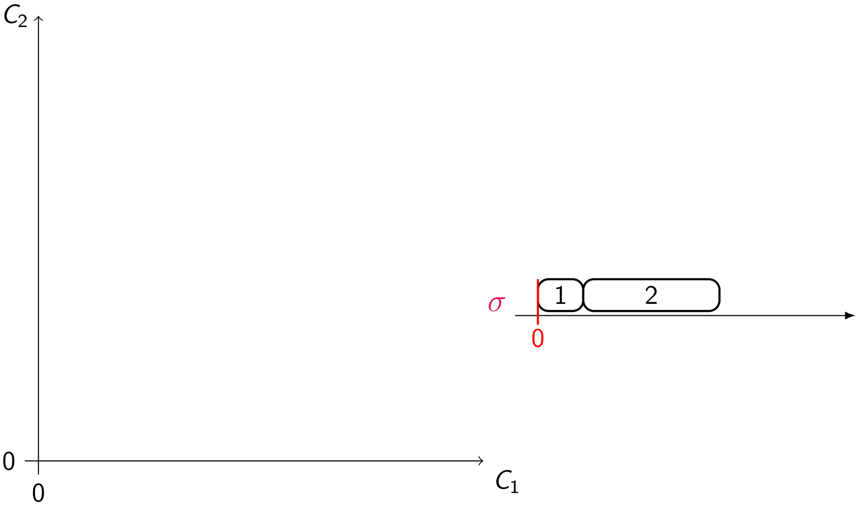

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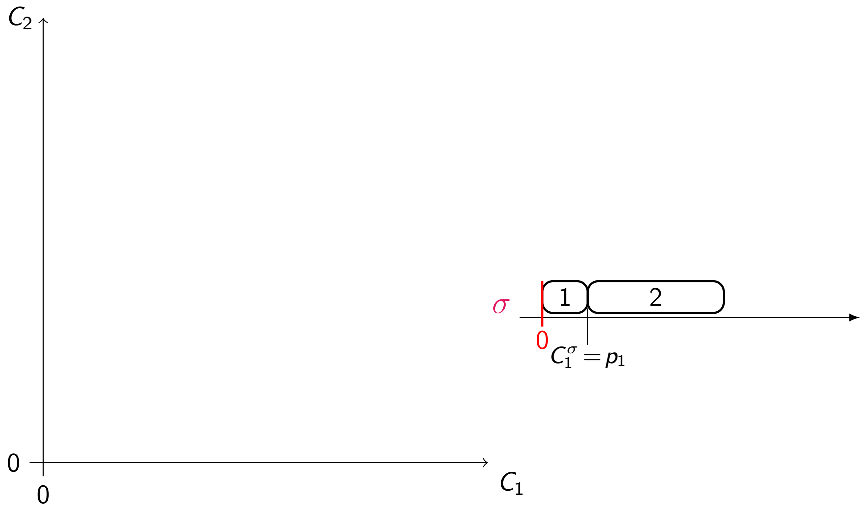

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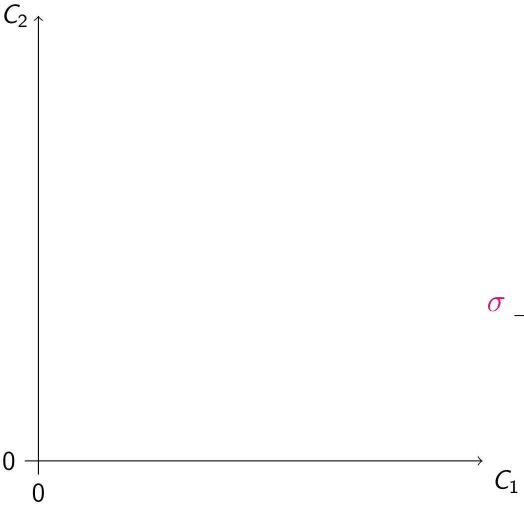

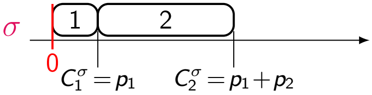

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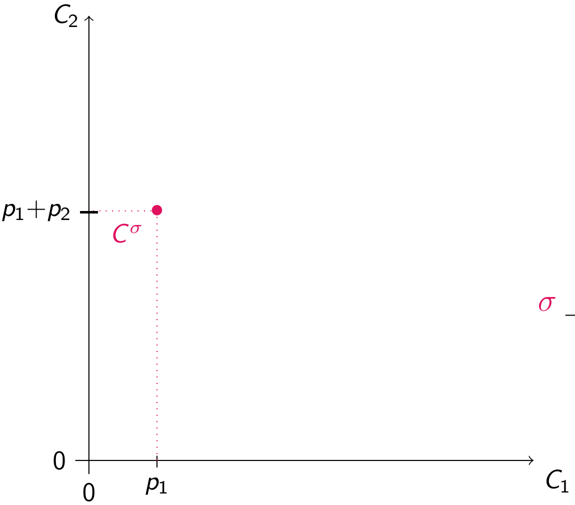

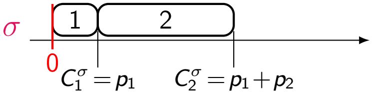

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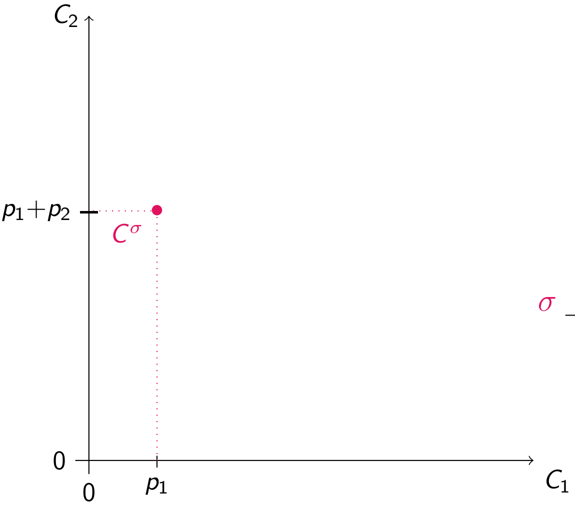

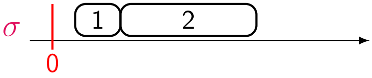

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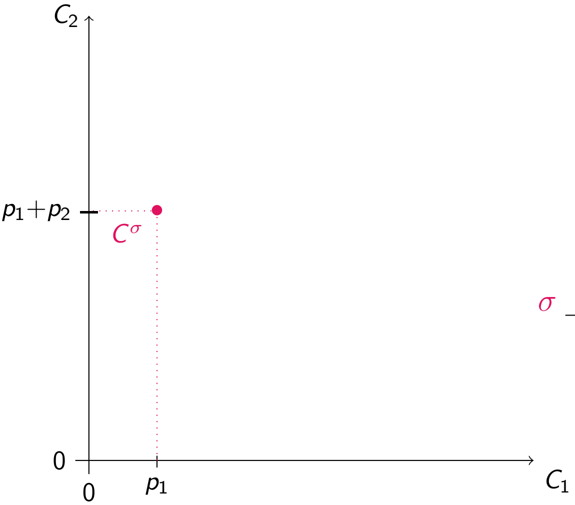

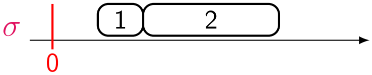

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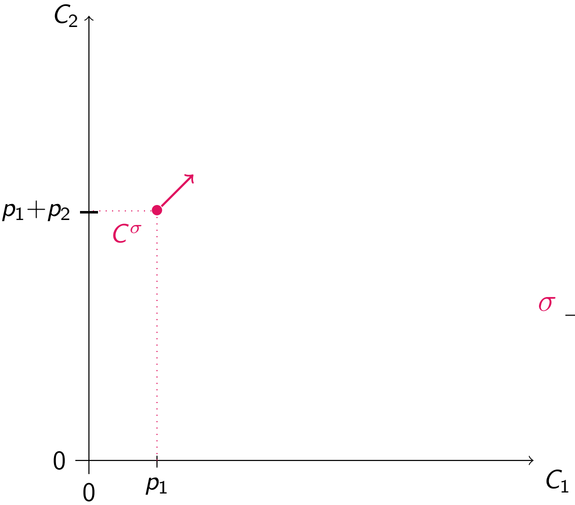

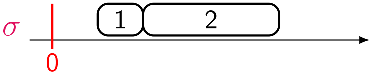

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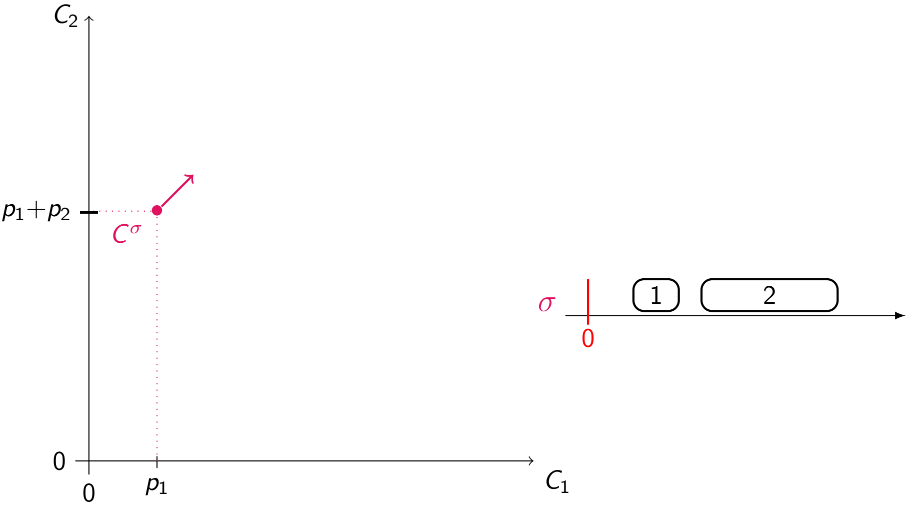

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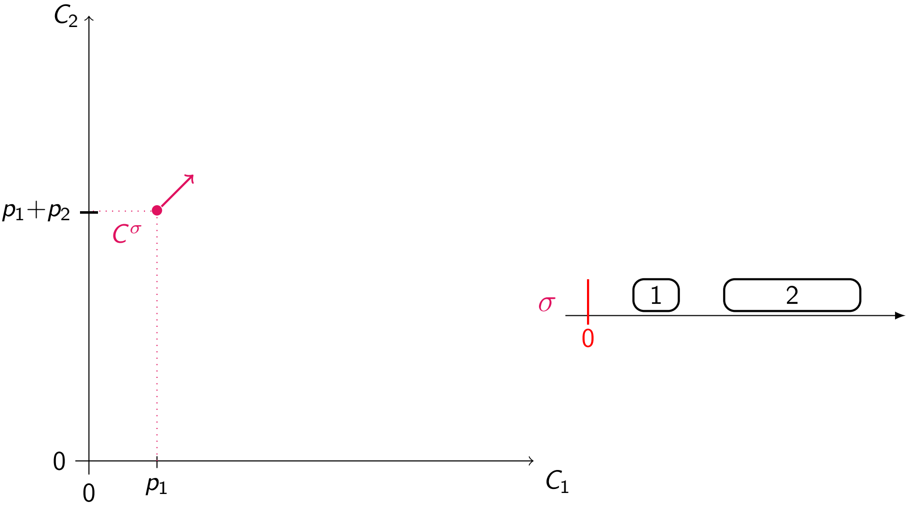

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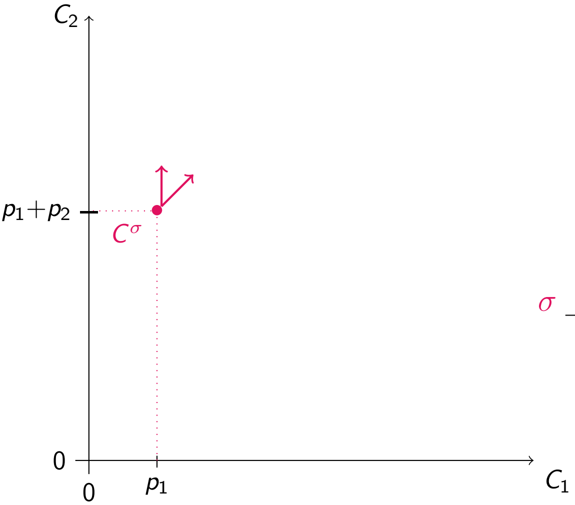

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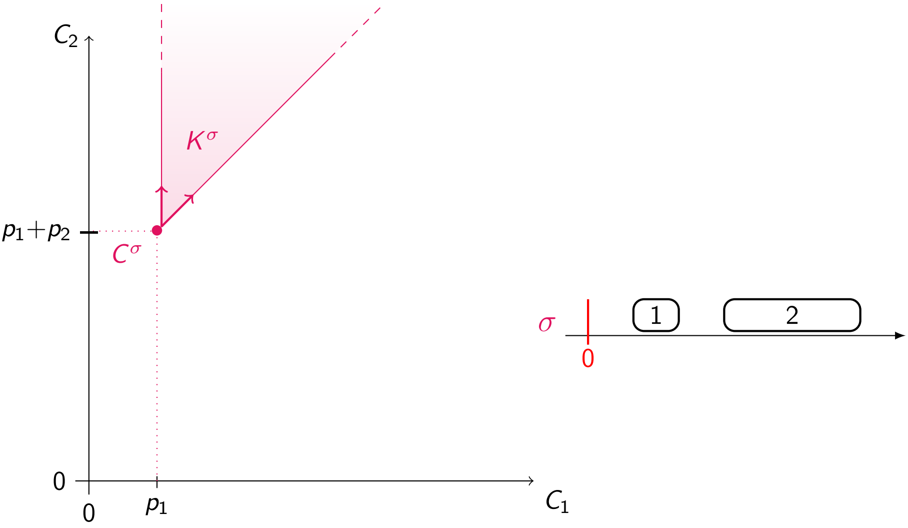

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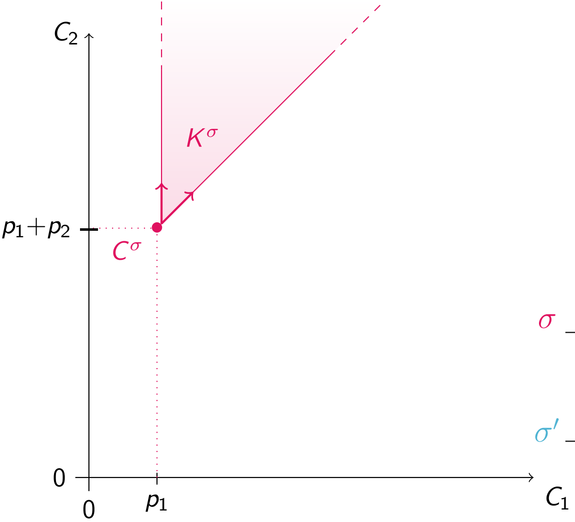

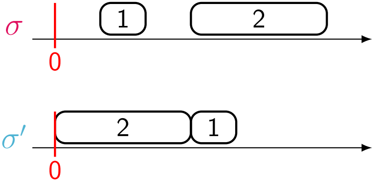

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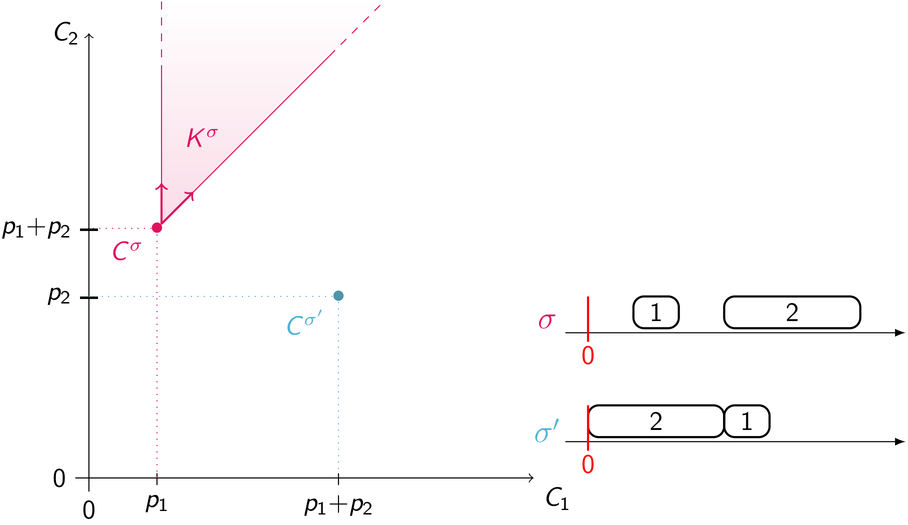

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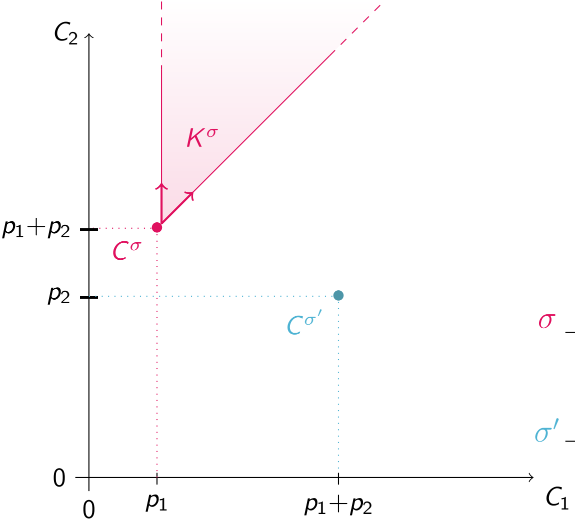

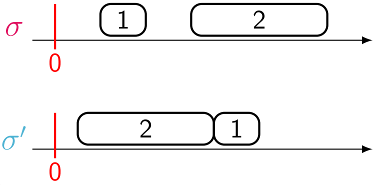

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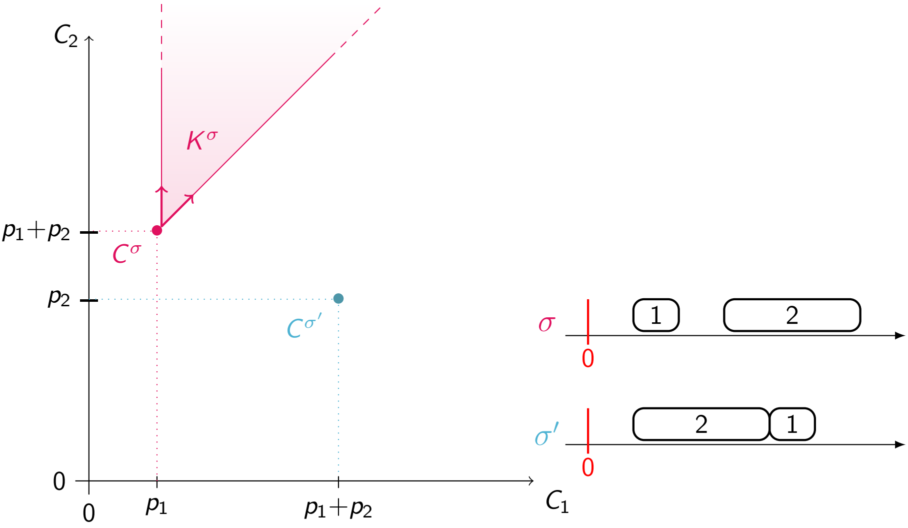

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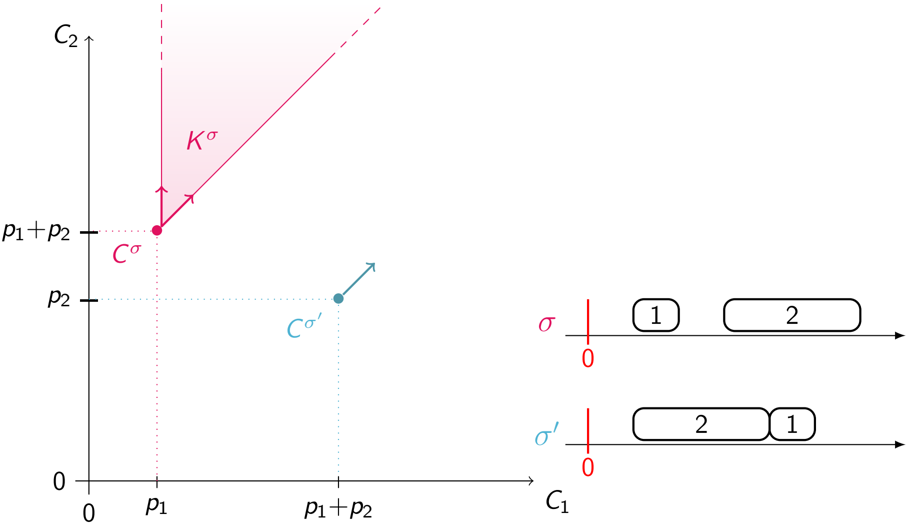

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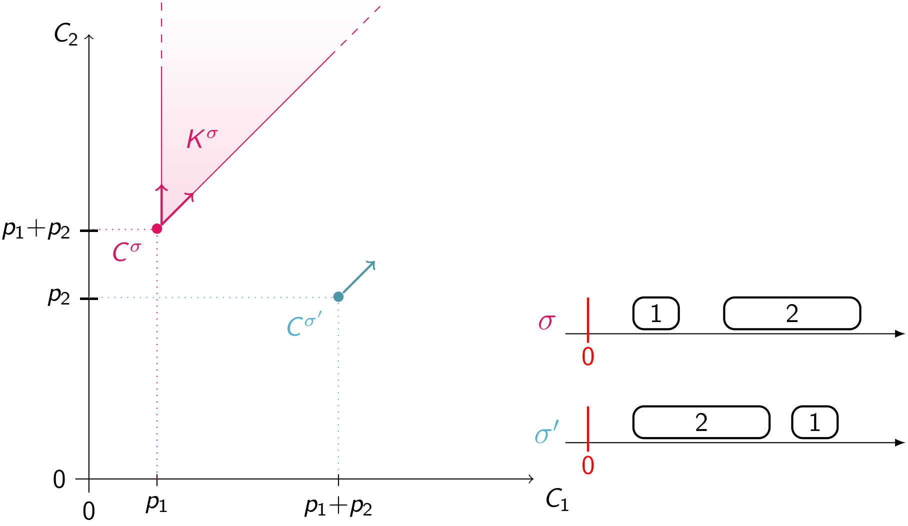

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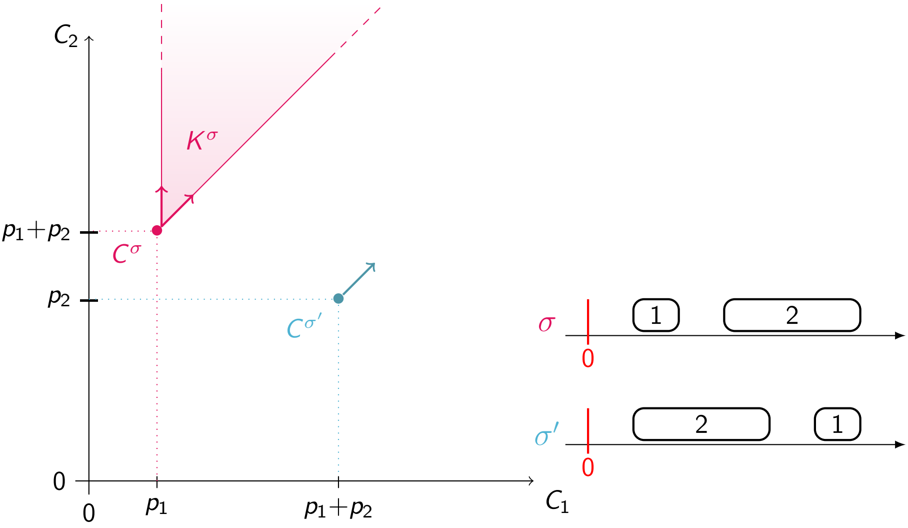

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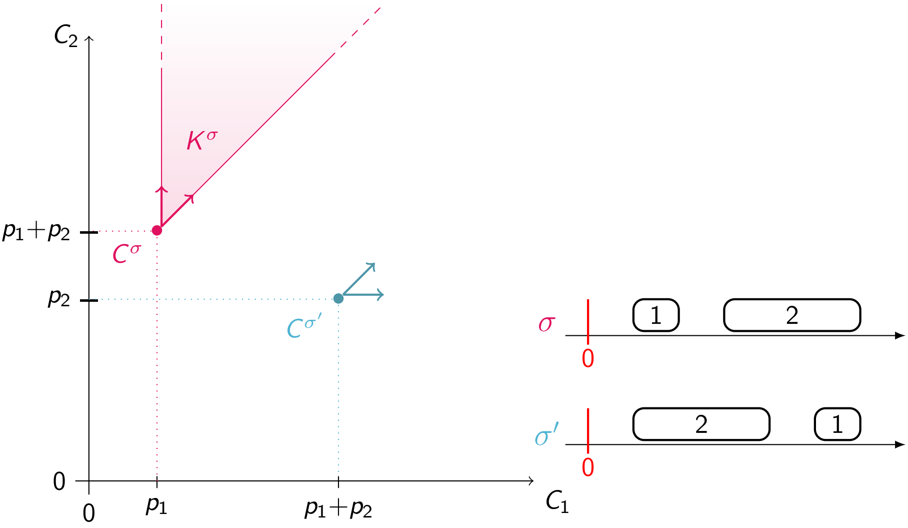

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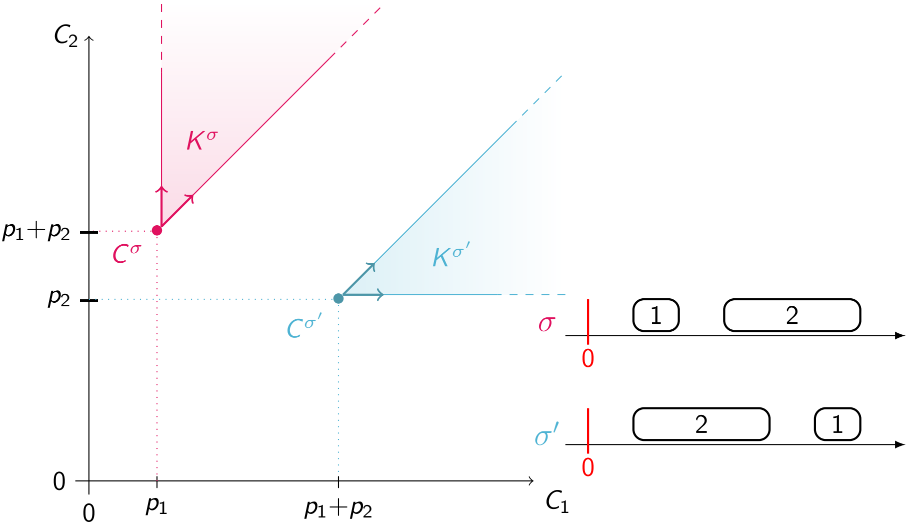

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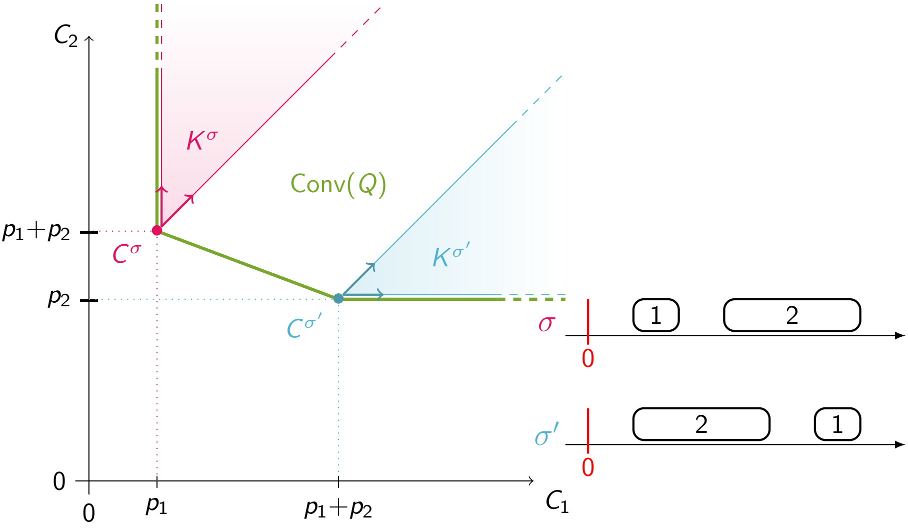
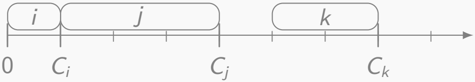


Non-overlapping inequalities for UCDDP

Queyranne's non-overlapping inequalities



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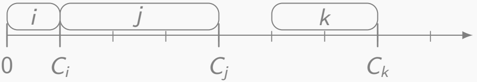

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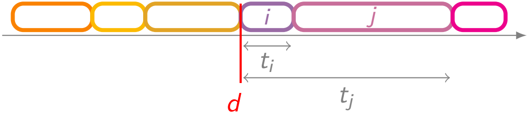

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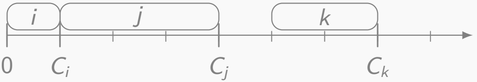

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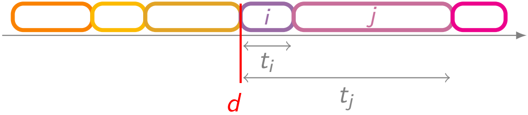

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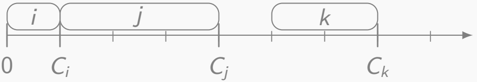

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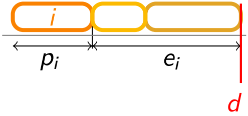

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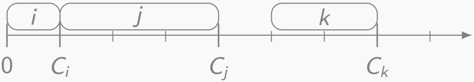

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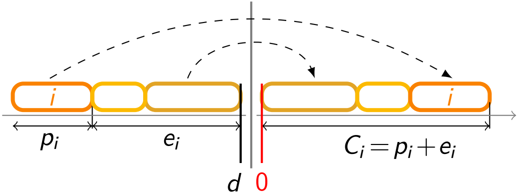

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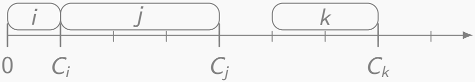

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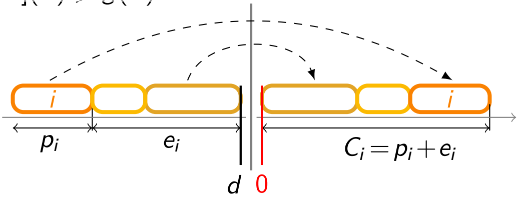

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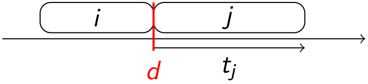

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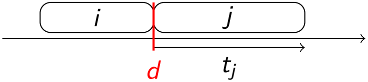

$\hookrightarrow$ So we consider: $\begin{cases} \forall S \subseteq J,\ p * [p+e](S \cap E) \geqslant g(S \cap E) \\ \forall S \subseteq J,\ p * t(S \cap T) \geqslant g(S \cap T) \end{cases}$

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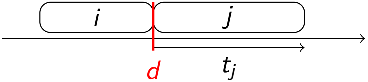

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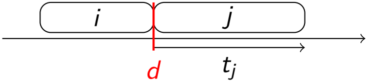

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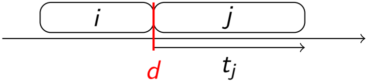

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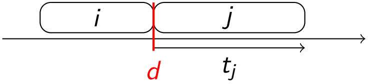

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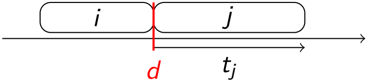

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Falq, Fouilhoux, Kedad-Sidhoum, 2020, Discrete Applied Maths *(to appear)*

Formulation F^3 for UCDDP

$$
\forall (i,j) \in J^<, \quad
\begin{aligned}
X_{i,j} &\geqslant 0 && (x.1)\\
X_{i,j} &\leqslant \delta_i + \delta_j && (x.2)\\
X_{i,j} &\geqslant \delta_i - \delta_j && (x.3)\\
X_{i,j} &\geqslant 2 - \delta_i - \delta_j && (x.4)
\end{aligned}
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$$\forall S\in\mathcal{P}(J),\ \sum_{i\in S} p_i\, e_i \geqslant \sum_{(i,j)\in S^<} p_i\, p_j\, \frac{\delta_i+\delta_j-X_{i,j}}{2} \qquad (S1)$$

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Formulation F^3 for UCDDP

$$
P^3 = \left\{ (e,t,\delta,X) \;\middle|\; \begin{array}{ll}
\forall j \in J, 0 \leqslant \delta_j \leqslant 1 & (\delta) \\
\forall j \in J,\ e_j \geqslant 0 & (e.0) \\
\qquad e_j \leqslant M\,\delta_j & (e.1) \\
\forall j \in J,\ t_j \geqslant 0 & (t.1) \\
\qquad t_j \leqslant M\,(1-\delta_j) & (t.2) \\
\forall (i,j) \in J^<,\ X_{i,j} \geqslant 0 & (x.1) \\
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\end{array} \right\}
$$

Formulation F^3 for UCDDP

$$F^3 : \min \left\{ \sum_{j\in J} \alpha_j\, e_j + \beta_j\, t_j \;\middle|\; (e,t,\delta,X) \in \text{extr}(P^3) \;\text{ and }\; \delta \in \{0,1\}^J \right\}$$

where:

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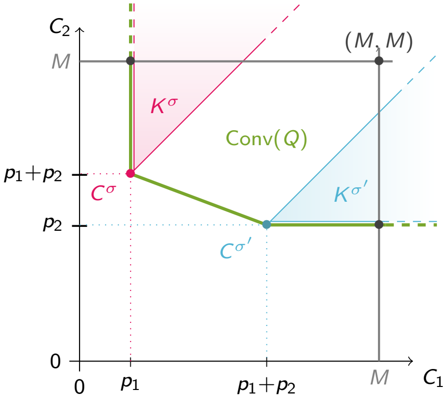
Validity of F^3

- validity proof
  - is not based on a geometrical proof
  - must be compatible with additional inequalities



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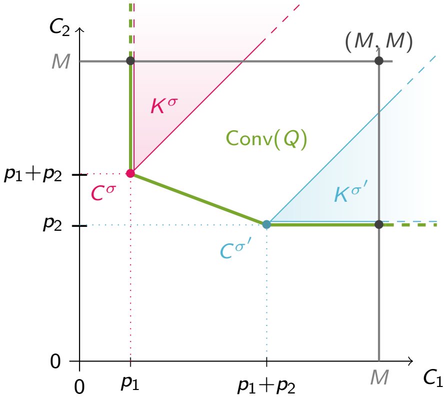
- we provide 2 key lemmas to use non-overlapping inequalities combined with additional inequalities



Two key lemmas for non-overlapping inequalities

Let $y \in \mathbb{R}^J$ satisfying $\forall S \subseteq J,\ \sum_{j \in S} p_j y_j \geqslant g(S)\ \ (Q)$.

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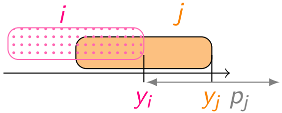

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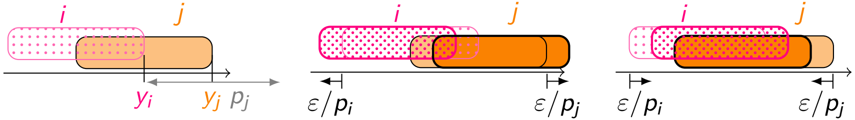

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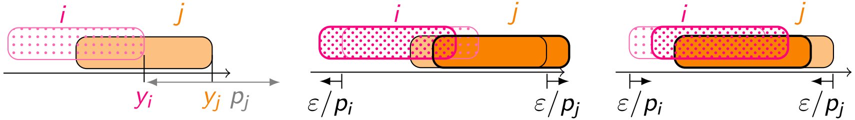

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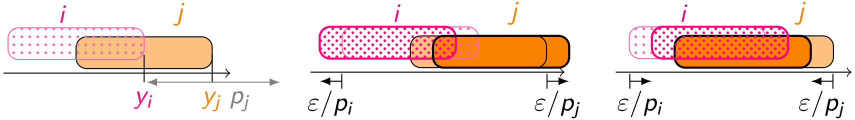

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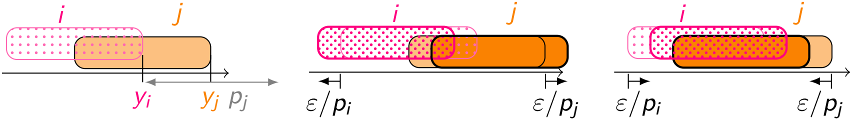

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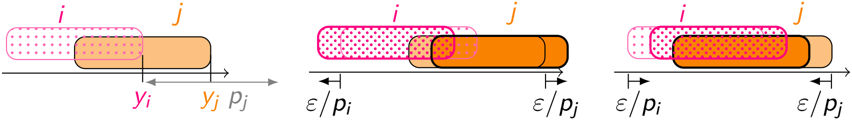

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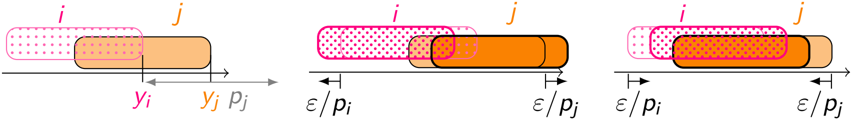

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Outline

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  $\hookrightarrow$ ensuring the solutions extremality in spite of the branching scheme

Inequalities to separate

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$\hookrightarrow$ 2 independent separation problems

but also 2 similar separation problems

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separation of $(S1)$ inequalities $\longrightarrow$ maximization of Γ_1

separation of $(S2)$ inequalities $\longrightarrow$ maximization of Γ_2

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↪ 2 independent separation problems
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separation of $(S1)$ inequalities → maximization of Γ_1 →

separation of $(S2)$ inequalities → maximization of Γ_2 →

$$\max_{S\subset[1..n]} \Big(\sum_{(i,j)\in S^<} \overset{\in \mathbb{R}_+^*}{q_{i,j}} + \sum_{i\in S} \overset{\in \mathbb{R}}{c_i} \Big)$$

Reduction to a min-cut problem

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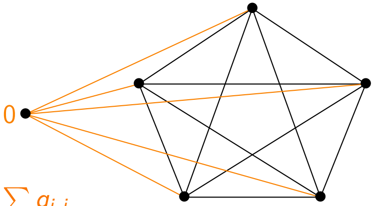

$\forall j\in[1..n]$
$w_{0,j} = 2c_i + \sum_{i\neq j} q_{i,j}$

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Picard et Ratliff, 1975, Networks

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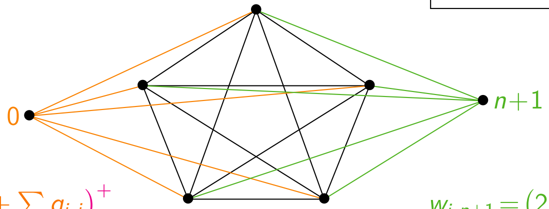

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$w_{0,j} = \left(2c_j + \sum_{i\neq j} q_{i,j}\right)^+$

$\forall i\in[1..n]$

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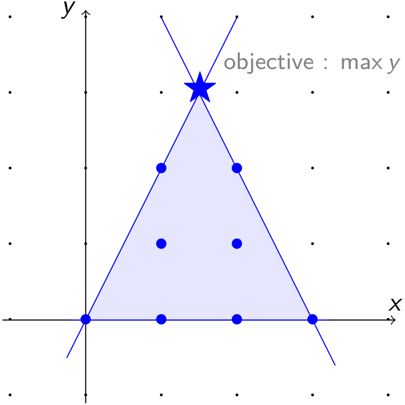

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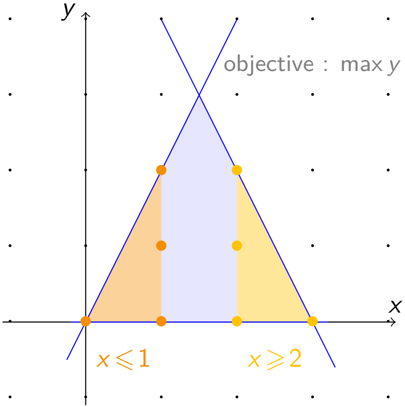

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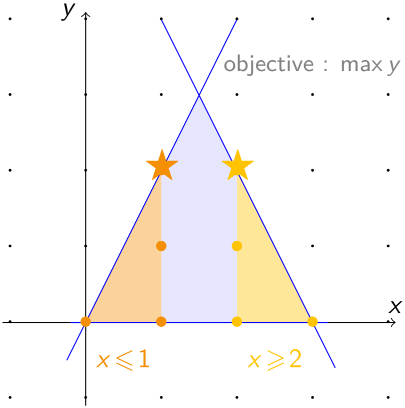

Outline

How to adapt the formulation for the general case ?

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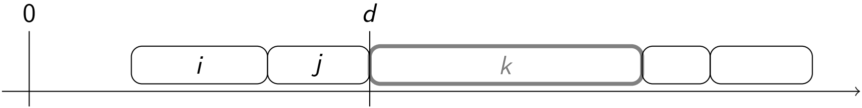

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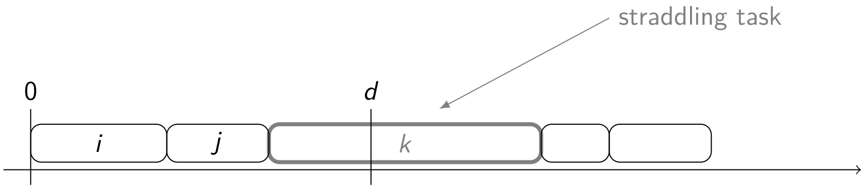

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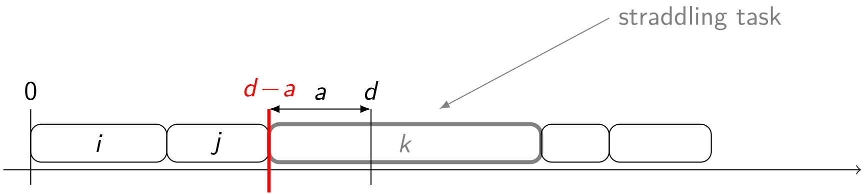

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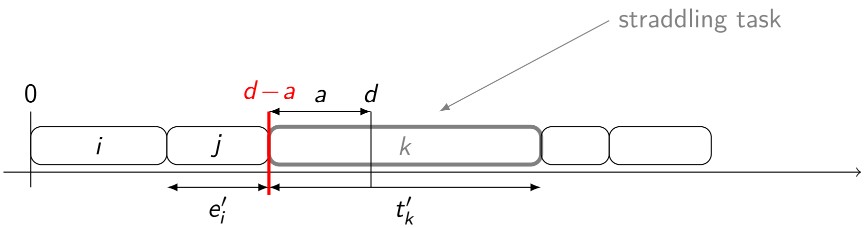

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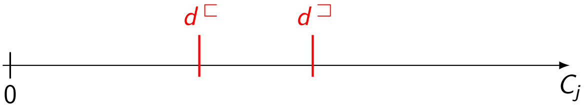
$\rightarrow$ a due window $[d^{\sqsubset}, d^{\sqsupset}]$ instead of a due date d



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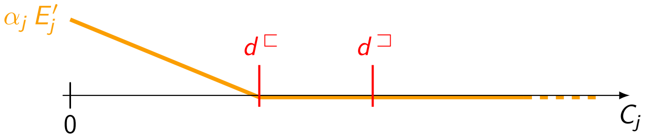

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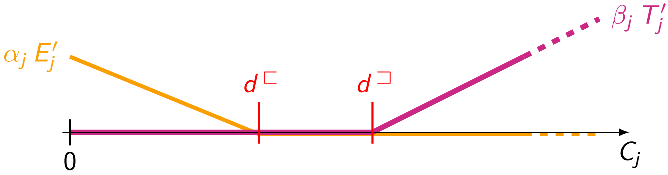

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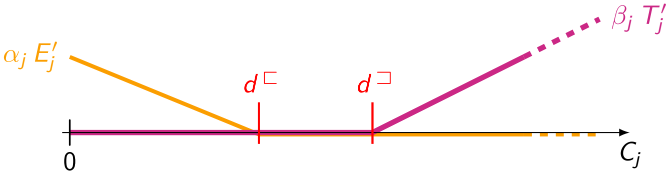

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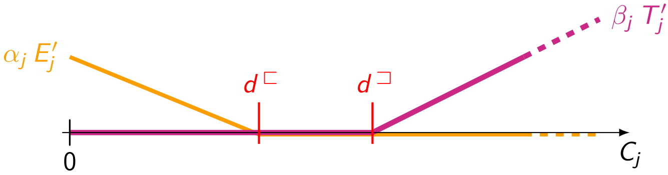

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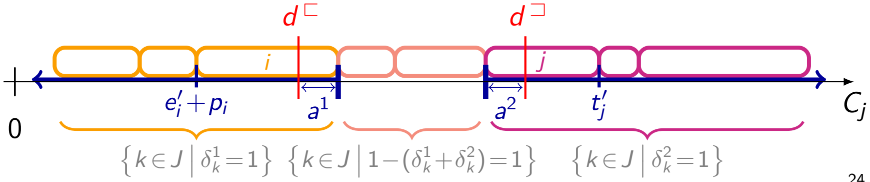

Outline

1. Introduction
2. A formulation for UCDDP using natural variables
3. How to manage this kind of formulations in practice
4. How to extend this formulation
5. Conclusion

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