Around the Boltzmann-Grad limit

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Abstract

This text presents some general aspects of the kinetic theory of hard-spheres and in particular the mathematical description of the limiting procedure known as the Boltzmann-Grad limit by which Boltzmann's equation and the equations of continuum physics are deduced from a system of N interacting particles when $N \to +\infty$. The first chapter gives the basic tools of kinetic theory we will constantly use throughout this text. Chapter 2 is largely inspired by the article [4] where a rigorous and complete proof of Lanford's theorem is given. The aim of this theorem is to give a mathematical justification of the Boltzmann equation. The idea of the proof is reused in [2] to prove that for a system around the equilibrium, the Brownian motion can be obtained as the limit of a deterministic system of hard-spheres. This is the object of chapter 3.

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Chapter 1

General features of the hard-spheres dynamics

This introductory chapter presents the objects we will constantly work with throughout the next chapters. Most of the results can be found in the the first two parts of [4], in the introduction of [2] or in [3] for a more extensive view on kinetic theory.

1.1 At a microscopic scale : a Newtonian approach

Throughout this text, we will consider a system of N interacting particles, N being of the order of 10^{23}. At a microscopic scale, each particle is modeled by a hard-sphere[1] of diameter ε with position and velocity denoted by $z_i = (x_i, v_i) \in \mathbf{R}^d \times \mathbf{R}^d$. In the following, $Z_N = (X_N, V_N) \in (\mathbf{R}^d)^N \times (\mathbf{R}^d)^N$ will be the vector of all the positions and velocities of the N particles.

In the following we will focus only on the case of monoatomic gases : we assume that all the particles are identical (same mass, same volume...) and interact according to Newton's equations of motion,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \;\; i \in \{1, \dots, N\} \\ \dfrac{dv_i}{dt} = 0 \end{cases} \tag{1.1}$$

on the domain

$$\mathcal{D}_N := \{Z_N \in \mathbf{R}^{2dN} / \; \forall i \neq j, |x_i - x_j| > \varepsilon\}.$$

In this model, we consider that two particles can't overlap[2] : a particle moves in straight line until it encounters an other particle. The two particles then bounce back on each other according to the standard laws of mechanics : a collision is instantaneous and preserves energy and momentum. More precisely, on the boundary of $\mathcal{D}_N$, given two positions x_i and x_j such that $|x_i - x_j| = \varepsilon$, one has

$$\begin{cases} v'_i + v'_j = v_i + v_j \\ |v'_i|^2 + |v'_j|^2 = |v_i|^2 + |v_j|^2 \end{cases}$$

where v'_i and v'_j denote the velocities after the collision. It can be shown that :

$$\begin{aligned} v'_i &= v_i - \nu^{i,j} \cdot (v_i - v_j)\nu^{i,j} \\ v'_j &= v_j + \nu^{i,j} \cdot (v_i - v_j)\nu^{i,j} \end{aligned} \tag{1.2}$$

[1] There exist more accurate models such as the short- and long-range potentials we won't discuss here. A complete study of short-range potentials (similar to this one) is also done in [4]. The case of long-range potentials is a widely open question.

[2] The situation is slightly different for short-range potentials and very different for long-range potentials.

where $\nu^{i,j} := (x_i - x_j)/|x_i - x_j| \in \mathbf{S}^{d-1}$. Here we implicitly assume that v_i and v_j are *pre-collisional* meaning that

$$\nu^{i,j} \cdot (v_i - v_j) < 0.$$

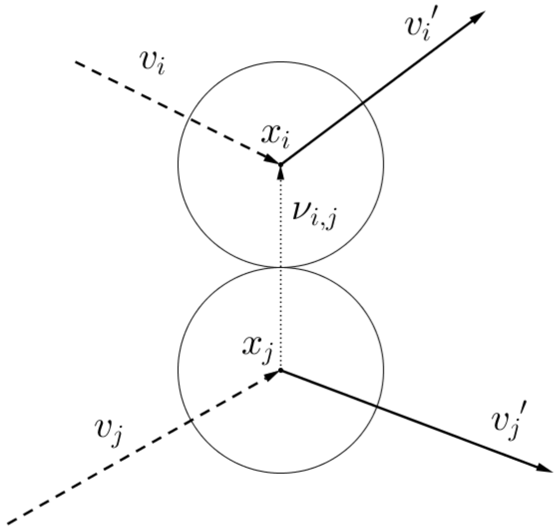

Figure 1.1: v_i and v_j are pre-collisional

When v_i and v_j are *post-collisional*, meaning that $\nu^{i,j} \cdot (v_i - v_j) > 0$, v_i^* and v_j^* will denote the velocities *before* the collision. Note that v_i^* and v_j^* also satisfy (1.2) : since the collision is elastic, the transformation $(v_i, v_j) \mapsto (v_i', v_j')$ is an involution.

$$\begin{aligned} v_i^* &= v_i - \nu^{i,j} \cdot (v_i - v_j)\nu^{i,j} \\ v_j^* &= v_j + \nu^{i,j} \cdot (v_i - v_j)\nu^{i,j} \end{aligned} \tag{1.3}$$

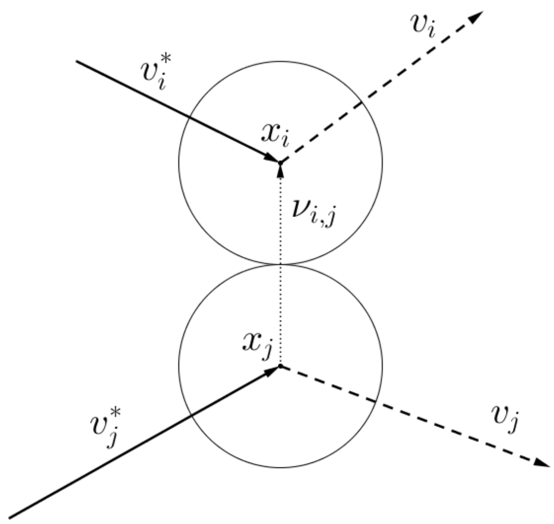

Figure 1.2: v_i and v_j are post-collisional

Note that this boundary condition is a priori not sufficient to define a global dynamics since it is not defined in the following cases : a trajectory is called *pathological* when there is either a collision involving more than two particles or a grazing collision ($\nu^{i,j} \cdot (v_i - v_j) = 0$) or when there are an infinite number of collisions in finite time. Fortunately, the following proposition (see proposition 4.1.1 in [4]) allows us to define a global dynamics for almost all initial data.

Proposition 1.1. *Let N, ε be fixed. The set of initial configurations leading to a pathological trajectory is of measure zero in $\mathbf{R}^{2dN}$.*

Proof. Given $\rho > 0, R > 0$ and $\delta < \varepsilon/2$, let us define :

$$I := \Big\{ Z_N \in B_\rho^N \times B_R^N / \text{ one particle will collide with two others on time interval } [0,\delta] \Big\}$$

Noticing that

$$I \subset \Big\{ Z_N \in B_\rho^N \times B_R^N / \ \exists \{i,j,k\} \text{ distinct}, \ \ |x_i - x_j| \in [\varepsilon, \varepsilon + 2R\delta] \text{ and } |x_i - x_k| \in [\varepsilon, \varepsilon + 2R\delta] \Big\}$$

one has :

$$|I| \leq C(N,\varepsilon,R)\rho^{d(N-2)}\delta^2.$$

We then have constructed a subset $I_0(\delta, R) \subset B_R^N \times B_R^N$ such that any initial configuration belonging to $(B_R^N \times B_R^N) \setminus I_0(\delta, R)$ is well defined up to time δ (since up to removing a zero measure set, each collision is non grazing). It can be shown that the determinant of the Jacobian matrix of the flow of (1.1) is constantly equal to 1, meaning that the measure is preserved by the flow. It is then possible to apply the same procedure starting at time δ, the positions now belonging to $B_{R+R\delta}^N$. Finally, repeating this procedure t/δ times, we construct a subset

$$I_\delta(t,R) := \bigcup_{j=0}^{t/\delta - 1} I_j(\delta, R)$$

of $B_R^N \times B_R^N$, of measure

$$|I_\delta(t,R)| \leq C(N,\varepsilon,R)R^{d(N-2)}\delta^2 \sum_{j=0}^{t/\delta-1} (1+j\delta)^{d(N-2)} \leq C(N,R,t,\varepsilon)\delta$$

such that for any initial configuration belonging to $(B_R^N \times B_R^N) \setminus I_\delta(t,R)$ the flow is well-defined up to time t. The intersection $I(t,R) := \cap_{\delta>0} I_\delta(t,R)$ is of measure zero, and any initial configuration in $B_R^N \times B_R^N$ outside that set generates a well-defined flow until time t. We conclude the proof by taking the countable union of $I(t_n, R_n)$ for $t_n \to +\infty$ and $R_n \to +\infty$. □

1.2 At a mesoscopic scale : the Liouville equation

A typical system in kinetic theory involves way too many particles to perform an explicit computation of the individual trajectories. In the limit $N \to +\infty$ and $\varepsilon \to 0$, we can only hope for a description of the average behaviour of the system. This point of view is mentioned by Hilbert in his sixth problem (formulated after Boltzmann's work) which suggests to mathematically investigate the limiting processes "which lead from the atomistic view to the laws of motion of continua". The intermediary scale between the microscopic and macroscopic scales is often known as the *mesoscopic* scale : at this scale the relevant quantity is the *distribution function of the system* $f_N(t, Z_N)$ which can be interpreted as the probability at time t for the system to be in the state $Z_N \in \mathbf{R}^{2dN}$. In particular, f_N takes its values in $[0,1]$ and satisfies the following relation : given a domain $\Omega \subset \mathbf{R}^{2dN}$, one has

$$\text{Prob}(Z_N(t) \in \Omega) = \int_\Omega f_N(t, Z_N) dZ_N = \int_{\mathbf{Z}_N(0,t,\Omega)} f_N(0, Z_N) dZ_N$$

where $\mathbf{Z}_N(t_1, t_0, Z_N^0)$ denotes the flow of (1.1) at time t_1, initiated at time t_0 in Z_N^0. A change of variables in the last integral leads to :

$$\int_\Omega f_N(t, Z_N) dZ_N = \int_\Omega f_N(0, \mathbf{Z}_N(0, t, Z_N)) \left| \det \frac{D\mathbf{Z}_N(0, t, Z_N)}{DZ_N} \right| dZ_N$$

where $\frac{D\mathbf{Z}_N(0,t,Z_N)}{DZ_N}$ denotes the Jacobian matrix of $Z_N \mapsto \mathbf{Z}_N(0, t, Z_N)$. For the system (1.1), it can be shown that the determinant of this matrix is constantly equal to 1 (with respect to t). Taking the time derivative of the last equation therefore leads to :

$$\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N = 0 \tag{1.4}$$

on $\mathcal{D}_N$ and with the boundary condition $f_N(t, Z_N^*) = f_N(t, Z_N)$ where $Z_N^* = (x_1, v_1^*, \ldots, x_N, v_N^*)$. This equation, which is the starting point of all our study, is called *Liouville's equation*.

In addition, we will assume that all the particles are *indistinguishable*, which means that the distribution function f_N satisfies the following symmetry condition :

$$\forall \sigma \in \mathfrak{S}_N, \quad f_N(t, Z_{\sigma(N)}) = f_N(t, Z_N) \tag{1.5}$$

with $Z_{\sigma(N)} = (z_{\sigma(1)}, \ldots, z_{\sigma(N)})$.

As explained before, our goal is to investigate the behaviour of the system in the limit $N \to +\infty$ and $\varepsilon \to 0$. Of course, since f_N belongs to a functional space that depends on N, it isn't possible take directly the limit, even formally. In the following, we will then focus on the *marginals* of f_N : for $s \in \{1, \ldots, N\}$, the s-th marginal of f_N is defined by

$$f_N^{(s)}(t, Z_s) := \int_{\mathbf{R}^{2d(N-s)}} f_N(t, Z_N) dz_{s+1} \ldots dz_N.$$

Up to defining $f_N^{(s)} = 0$ for $s \in \{N+1, N+2, \ldots\}$, it makes sense to consider for each $s \in \mathbf{N}$ the limit

$$\lim_{N \to +\infty} f_N^{(s)}$$

the exact meaning of the limit being of course still to define.

1.3 The Boltzmann-Grad limit and the Boltzmann equation

An other relevant quantity at a mesoscopic scale is the *mean free path* of the particles which is defined as the average distance traveled by a particle between two collisions. If ℓ denotes this distance, the volume covered by particle between two collisions is then of the order of $\varepsilon^{d-1}\ell$. Interpreting the number N as the number of particles by unit of volume, we have by definition of the mean free path :

$$N\varepsilon^{d-1}\ell \sim 1.$$

Given N and ε, the quantity $N\varepsilon^{d-1}$ can therefore be viewed as the inverse of the mean free path which is also a measure of the number of collisions per unit of time[3]. The aim of this text is to investigate the behaviour of the system in the *Boltzmann-Grad limit* $N \to +\infty$, $\varepsilon \to 0$ and $N\varepsilon^{d-1} = \mathcal{O}(1)$, sometimes known as the *low density limit*. From now and throughout the next chapter, we will consider $N\varepsilon^{d-1} \to 1$. In the last part, we will aim to retrieve a macroscopic behaviour from a deterministic system of particles, which correspond to the case $N\varepsilon^{d-1} \equiv \alpha \to +\infty$ and $N\varepsilon^{d} \to 0$.

The main result we will prove is the Lanford theorem which states that in the Boltzmann-Grad limit, the first marginal $f_N^{(1)}$ of the distribution function will converge (in some sense) to the solution $f(t,x,v)$ of the Boltzmann equation :

$$\partial_t f + v \cdot \nabla_x f = Q(f,f) \tag{1.6}$$

where $Q(f,f)$ is the collision operator defined by :

$$Q(f,f)(v) := \int_{\mathbf{S}_1^{d-1}} \int_{\mathbf{R}^d} (f^* f_1^* - f f_1) b(v - v_1, \omega) dv_1 d\omega. \tag{1.7}$$

with $b(v-v_1,\omega) = ((v-v_1)\cdot\omega)_+$ in the hard-spheres case and where f^* stands for $f(t,x,v^*)$ and f_1 for $f(t,x,v_1)$. We can split $Q = Q_+ - Q_-$ into a gain term

$$Q_+(f,f)(v) := \int_{\mathbf{S}_1^{d-1}} \int_{\mathbf{R}^d} f^* f_1^* b(v - v_1, \omega) dv_1 d\omega$$

and a lose term

$$Q_-(f,f)(v) := \int_{\mathbf{S}_1^{d-1}} \int_{\mathbf{R}^d} f f_1 b(v - v_1, \omega) dv_1 d\omega.$$

The gain term counts the collisions which involve a particle with velocity v^* and produce a particle with velocity v. The lose term counts the collisions during which a particle with velocity v disappears (its velocity becomes v').

The Boltzmann equation (1872) describes the behaviour of a thermodynamic system. In particular, it encodes the notion of *irreversibility* and gives a formal definition of the *entropy* of a system. Physically, the main issue we have to deal with in the limiting procedure is to show that irreversibility can be obtained from a purely deterministic and reversible system governed by Newton's equations.

[3] Up to defining the typical microscopic unit of time as the average time between two collisions, $N\varepsilon^{d-1}$ can be viewed as the number of collisions per unit of time.

1.4 The BBGKY and Boltzmann hierarchies

We are looking for a weak formulation of the system of equations satisfied by the family of marginals $(f_N^{(s)})_{1\leq s\leq N}$. Assuming that f_N decays sufficiently fast in the velocity variable and taking a symmetric test function $\phi(t,Z_s)$, let us perform integrations by parts on :

$$\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}}\Big(\partial_t f_N+\sum_{i=1}^N v_i\cdot\nabla_{x_i}f_N\Big)(t,Z_s)\phi(t,Z_s)\mathbf{1}_{Z_N\in\mathcal{D}_N}dZ_n dt=0.$$

By definition of $f_N^{(s)}$, the integration in time leads to

$$\begin{aligned}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}}\partial_t f_N(t,Z_N)\phi(t,Z_s)\mathbf{1}_{Z_N\in\mathcal{D}_N}dtdZ_N=&-\int_{\mathbf{R}^{2ds}}f_N^{(s)}(0,Z_s)\phi(0,Z_s)dZ_s\\&-\int_{\mathbf{R}_+\times\mathbf{R}^{2ds}}f_N^{(s)}(t,Z_s)\partial_t\phi(t,Z_s)dtdZ_s.\end{aligned}$$

The integration with respect to the space variable is a little bit more tricky since the domain has boundaries $|x_i-x_j|=\varepsilon$ depending on x_i. Using Green's formula, we obtain :

$$\begin{aligned}&\sum_{i=1}^N\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}}v_i\cdot\nabla_{x_i}f_N(t,Z_N)\phi(t,Z_s)\mathbf{1}_{Z_N\in\mathcal{D}_N}dtdZ_N\\&\qquad=-\sum_{i=1}^N\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}}f_N(y,Z_N)v_i\cdot\nabla_{x_i}\phi(t,Z_s)dtdZ_s\\&\qquad+\sum_{i=1}^N\int_{\mathbf{R}_+\times\mathbf{R}^{2dN-d}}\Bigg(\sum_{\substack{j=1\\j\neq i}}^N\int_{\mathbf{S}^{d-1}_{j,\varepsilon}}\nu^{j,i}\cdot v_i f_N(t,Z_N)\phi(t,Z_s)d\sigma^{j,i}(x_i)\Bigg)dt\widehat{dx_i}\end{aligned}$$

where $\nu^{j,i}:=(x_j-x_i)/|x_j-x_i|$, $\mathbf{S}^{d-1}_{j,\varepsilon}$ is the sphere centered in x_j with radius ε and $d\sigma^{j,i}$ is the surface measure on this sphere induced by the Lebesgue measure. We write $\widehat{dx_i}$ for $dx_1dv_1\dots dx_{i-1}dv_{i-1}dv_idx_{i+1}dv_{i+1}\dots dx_Ndv_N$. The boundary term on the right-hand side of the equality is equal to :

$$\sum_{1\leq i\neq j\leq N}\varepsilon^{d-1}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN-d}\times\mathbf{S}^{d-1}}(-\omega)\cdot v_i f_N(t,Z_N)\phi(t,Z_s)dt\widehat{dx_i}d\omega$$

where x_i has to be replaced by $x_j+\varepsilon\omega$ when it appears in the arguments of f_N and ϕ. Now, let us split the sum into four parts :

$$\sum_{1\leq i\neq j\leq N}=\sum_{1\leq i\neq j\leq s}+\sum_{s+1\leq i\neq j\leq N}+\sum_{i=1}^s\sum_{j=s+1}^N+\sum_{i=s+1}^N\sum_{j=1}^s.\tag{1.8}$$

Using the boundary condition, it is possible to show that the first two sums on the right-hand side of the equality are equal to 0. Indeed, thanks to the symmetry condition (1.5), one can see that

$$\begin{aligned}&\sum_{1\leq i\neq j\leq s}\varepsilon^{d-1}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN-d}\times\mathbf{S}^{d-1}}\omega\cdot v_i f_N(t,Z_N)\phi(t,Z_s)dt\widehat{dx_i}d\omega\\&\qquad=\frac12\sum_{1\leq i\neq j\leq N}\varepsilon^{d-1}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN-d}\times\mathbf{S}^{d-1}}\omega\cdot(v_i-v_j)f_N(t,Z_N)\phi(t,Z_s)dt\widehat{dx_i}d\omega\end{aligned}$$

and noticing that

$$\omega \cdot (v_i - v_j) = -\omega \cdot (v_i^* - v_j^*)$$

the boundary condition says that the sum is equal to zero (since it is equal to its opposite). The proof is identical when $s+1 \leq i \neq j \leq N$. For the two last sums in (1.8), this procedure can't be applied since ϕ depends only on the s first variables. However, using the symmetry condition (1.5) and by definition of the $s+1$-th marginal, the last sum is equal to :

$$-(N-s)\varepsilon^{d-1} \sum_{j=1}^{s} \int_{\mathbf{R}_+ \times \mathbf{R}^{ds} \times \mathbf{S}^{d-1} \times \mathbf{R}^d} \omega \cdot v_{s+1} f_N^{(s+1)}(t, Z_s, x_j + \varepsilon\omega, v_{s+1}) \phi(t, Z_s) dt dZ_s d\omega dv_{s+1}$$

and similarly, after a change of variables, the third sum on the right-hand side of (1.8) is equal to

$$(N-s)\varepsilon^{d-1} \sum_{i=1}^{s} \int_{\mathbf{R}_+ \times \mathbf{R}^{ds} \times \mathbf{S}^{d-1} \times \mathbf{R}^d} \omega \cdot v_i f_N^{(s+1)}(t, Z_s, x_i + \varepsilon\omega, v_{s+1}) \phi(t, Z_s) dt dZ_s d\omega dv_{s+1}.$$

Gathering everything, we find that the boundary term induced by Green's formula is equal to :

$$-(N-s)\varepsilon^{d-1} \sum_{i=1}^{s} \int_{\mathbf{R}_+ \times \mathbf{R}^{ds} \times \mathbf{S}^{d-1} \times \mathbf{R}^d} \omega \cdot (v_{s+1} - v_i) f_N^{(s+1)}(t, Z_s, x_i + \varepsilon\omega, v_{s+1}) \phi(t, Z_s) dt dZ_s d\omega dv_{s+1}.$$

Finally, we deduce the weak form of the Liouville equation for the marginals $(f_N^{(s)})_{s\geq 1}$, known as *the BBGKY hierarchy* :

$$\forall s \in \{1, .., N\}, \qquad \partial_t f_N^{(s)} + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i} f_N^{(s)} = \mathcal{C}_{s,s+1} f_N^{(s+1)} \tag{1.9}$$

on $\mathbf{R}_+ \times \mathcal{D}_s$ with the boundary condition $f_N^{(s)}(t, Z_s^*) = f_N^{(s)}(t, Z_s)$ and where $\mathcal{C}_{s,s+1}$ is the collision operator defined by :

$$\mathcal{C}_{s,s+1} g_{s+1} := (N-s)\varepsilon^{d-1} \sum_{i=1}^{s} \int_{\mathbf{S}_1^{d-1} \times \mathbf{R}^d} \omega \cdot (v_{s+1} - v_i) g_{s+1}(t, Z_s, x_i + \varepsilon\omega, v_{s+1}) d\omega dv_{s+1}. \tag{1.10}$$

Denoting by $\mathbf{T}_s(t)$ the backward flow associated to the s-particles system, the Duhamel formula for (1.9) gives the integrated form of (1.9) :

$$f_N^{(s)}(t, Z_s) = \mathbf{T}_s(t) f_N^{(s)}(0, Z_s) + \int_0^t \mathbf{T}_s(t-\tau) \mathcal{C}_{s,s+1} f_N^{(s+1)}(\tau, Z_s) d\tau.$$

Definition 1.2 (Total flow and total collision operator)**.** *We define the operators* $\mathbf{T}$ *and* $\mathbf{C}_N$ *on finite sequences* $G_N = (g_s)_{1\leq s\leq N}$ *by :*

$$\forall s \leq N, \ (\mathbf{T}(t) G_N)_s := \mathbf{T}_s(t) g_s$$

$$\forall s \leq N-1, \ (\mathbf{C}_N G_N) s := \mathcal{C}_{s,s+1} g_{s+1} \ \text{ and } \ (\mathbf{C}_N G_N)_N = 0$$

Definition 1.3 (Mild-solution of the BBGKY hierarchy)**.** $F_N = (f_N^{(s)})_{1\leq s\leq N}$ *solution of :*

$$F_N(t) = \mathbf{T}(t) F_N(0) + \int_0^t \mathbf{T}(t-\tau) \mathbf{C}_N F_N(\tau) d\tau.$$

Now, we can formally derive the *Boltzmann hierarchy* from the BBGKY hierarchy in the Boltzmann-Grad limit. Assuming that the pre-factor of (1.2) is approximatively equal to 1 and neglecting the micro-translation in the argument of g_{s+1}, we define the Boltzmann collision operator :

$$\mathcal{C}^0_{s,s+1}g_{s+1} := \sum_{i=1}^{s} \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} \omega\cdot(v_{s+1}-v_i)g_{s+1}(t,Z_s,x_i,v_{s+1})d\omega dv_{s+1}.$$

The boundary condition and the change of variable $\omega\mapsto-\omega$ lead to

$$\mathcal{C}^0_{s,s+1}f^{(s+1)}(t,Z_s) = \sum_{i=1}^{s} \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} (\omega\cdot(v_{s+1}-v_i))_+ \times \Big[f^{(s+1)}(t,Z_s^*,x_i,v_{s+1}^*) - f^{(s+1)}(t,Z_s,x_i,v_{s+1})\Big] d\omega dv_{s+1}. \quad (1.11)$$

Denoting by $\mathbf{S}_s$ the s-particles free flow we can similarly define a mild-solution to the Boltzmann hierarchy as a family $F=(f^{(s+1)})_{s\geq1}$ satisfying :

$$F(t) = \mathbf{S}(t)F(0) + \int_0^t \mathbf{S}(t-\tau)\mathbf{C}^0F(\tau)d\tau$$

for some initial condition $F(0) = (f_0^{(s)})_{s\geq1}$. Note that if $f^{(s)} = f^{\otimes s}$, then f satisfies the Boltzmann equation with $b(w,\omega)=(\omega\cdot w)_+$. Moreover, we can easily check that if f satisfies the Boltzmann equation with initial data f_0, then $(f^{\otimes s})_{s\geq1}$ is a solution of the Boltzmann hierarchy with initial data $(f_0^{\otimes s})_{s\geq1}$. This property is known as *the propagation of chaos* : it means that the solution of the Boltzmann hierarchy coming from a tensorized initial data is also tensorized. However, these observations are relevant only if we are able to prove the wellposedness of the Boltzmann hierarchy. That is one of the purposes of section 1.6.

1.5 Rigorous formulation of the BBGKY hierarchy

Before going any further, let us point out that it isn't clear that the formulation we gave of the BBGKY hierarchy makes sense : the marginals $f_N^{(s)}$ are defined only almost everywhere (see proposition 1.1) but the collision operator 1.10 is defined by an integral on the sphere $\mathbf{S}^{d-1}$... The idea is to use Fubini's theorem to show that for almost every t, the operator $\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t)$ is well-defined from $L^\infty(\mathcal{D}_{s+1})$ to $L^\infty(\mathcal{D}_s)$ (see [4], section 5.1).

Step 1. *Construct a truncated collision operator.*

Far from the boundary of $\mathcal{D}_{s+1}$, it is possible to see t as the "missing" coordinate[4] on $\partial\mathcal{D}_{s+1}$. This idea is embodied in the definition of the mappings :

$$\Phi^{-,i} : \left| \begin{array}{rcl} \mathcal{D}_s\times[0,\delta]\times\mathbf{S}^{d-1}\times\mathbf{R}^d & \longrightarrow & \mathbf{R}^{2d(s+1)} \\ (Z_s,t,\omega,v_{s+1}) & \longmapsto & Z_{s+1} := (X_s-tV_s, V_s, x_i+\varepsilon\omega-tv_{s+1}, v_{s+1}) \end{array} \right.$$

[4]It means that in the following, $[0,\delta]\times\mathbf{S}^{d-1}$ is viewed as a manifold of dimension d whereas $\mathbf{S}^{d-1}$ is a manifold of dimension $d-1$ only...

and

$$\Phi^{+,i}: \left| \begin{array}{ccc} \mathcal{D}_s \times [0,\delta] \times \mathbf{S}^{d-1} \times \mathbf{R}^d & \longrightarrow & \mathbf{R}^{2d(s+1)} \\ (Z_s, t, \omega, v_{s+1}) & \longmapsto & Z_{s+1} := (X_s - tV_s^*, V_s^*, x_i + \varepsilon\omega - tv_{s+1}^*, v_{s+1}^*) \end{array} \right.$$

for $i \in \{1, \ldots, s\}$. The following lemma is the key argument of the construction : it states that $\Phi^{\pm,i}$ is an actual change of variables. Its proof can be found in appendix A.

Lemma 1.4. *The change of variables $\Phi^{-,i}$ maps the measure*

$$d\mu_i^- := [\omega \cdot (v_{s+1} - v_i)]_- dZ_s dt d\omega dv_{s+1}$$

on the Lebesgue measure dZ_{s+1}. Similarly, the change of variables $\Phi^{+,i}$ maps the measure

$$d\mu_i^+ := [\omega \cdot (v_{s+1} - v_i)]_+ dZ_s dt d\omega dv_{s+1}$$

on the Lebesgue measure dZ_{s+1}.

We now turn to the explicit construction of the truncated collision operator : let us consider a small parameter of time $\delta > 0$ and $R > 0$ such that $Z_{s+1} \in B_R^{2(s+1)}$ (ball in dimension $2d(s+1)$ and of radius R). We define a truncated collision operator that acts on continuous functions on $\mathcal{D}_{s+1}$:

$$(\mathcal{C}_{s,s+1}^{\pm,i,\delta}\varphi_{s+1})(Z_s) := (N-s)\varepsilon^{d-1} \int_{\mathcal{S}_1^{d-1}\times\mathbf{R}^d} (\omega \cdot (v_{s+1} - v_i))_\pm \varphi_{s+1}(Z_s, x_i + \varepsilon\omega, v_{s+1}) \\ \times \prod_{\{k,l\}\neq\{i,s+1\}} \mathbf{1}_{|x_k - x_l| > \varepsilon + 2R\delta} d\omega dv_{s+1} dZ_s dt.$$

Thanks to the change of variables $\Phi^{-,i}$ in the pre-collisional case and $\Phi^{+,i}$ in the post-collisional case, one has :

$$\int_{[0,\delta]\times\mathcal{D}_s} \mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(t)\varphi_{s+1} dt dZ_s = \int_{\Phi^{\pm,i}(\mathcal{D}_s\times[0,\delta]\times\mathbf{S}_1^{d-1}\times\mathbf{R}^d)} \varphi_{s+1}\mathbf{1}_{(\ldots)} dZ_{s+1}$$

In particular the right-hand side of the last equality is well-defined, even if $\varphi_{s+1} \in L^\infty(\mathcal{D}_{s+1}) \subset L^1_{loc}(\mathbf{R}^{2d(s+1)})$. Besides,

$$\int_{[0,\delta]\times\mathcal{D}_s} |\mathcal{C}_{s,s+1}^{\pm,i,\delta} \mathbf{T}_{s+1}(t)\varphi_{s+1}| dt dZ_s \leq C\delta R^{2d(s+1)} \|\varphi\|_{L^\infty(\mathcal{D}_{s+1})}$$

so that $\mathcal{C}_{s,s+1}^{\pm,i,\delta}\mathbf{T}_{s+1}(t)\varphi_{s+1} \in L^1([0,\delta]\times\mathcal{D}_s)$ and thanks to Fubini's theorem, we know that for almost every $t \in [0,\delta]$, it defines a measurable function on $\mathcal{D}_s$. As in the proof of proposition 1.1, it is possible to iterate this procedure to show that for almost every $t \in [0,T]$ (for any given $T > 0$) and every $\varphi \in L^\infty(\mathcal{D}_{s+1})$, $\mathcal{C}_{s,s+1}^{\pm,i,\delta}\mathbf{T}_{s+1}(t)\varphi_{s+1}$ is a measurable function defined on $\mathcal{D}_s$. Moreover, the same proof shows that for any subset $A \subset [0,T]\times\mathcal{D}_s$,

$$\int_A |\mathcal{C}_{s,s+1}^{\pm,i,\delta}\mathbf{T}_{s+1}(t)\varphi_{s+1}|(Z_s) dt dZ_s \leq CR^{2d} \|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})} |A|$$

which implies that for almost every $t \in [0,T]$ and $Z_s \in \mathcal{D}_s$,

$$|\mathcal{C}_{s,s+1}^{\pm,i,\delta}\mathbf{T}_{s+1}(t)\varphi_{s+1}|(Z_s) \leq CR^{2d} \|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}. \tag{1.12}$$

In conclusion, for almost every $t \in [0,T]$ we have defined a bounded collision operator :

$$\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t) : L^\infty(\mathcal{D}_{s+1}) \to L^\infty(\mathcal{D}_s)$$

and we now have to remove the truncation, that is to say to "take the limit" $\delta \to 0$.

Step 2. *Show that for every $\varphi_{s+1} \in L^\infty(\mathcal{D}_{s+1})$, $\left(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\right)_{\delta>0}$ converges in $L^\infty([0,T]\times\mathcal{D}_s)$ when $\delta \to 0$.*

Thanks to the bound (1.12), the family $(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1})_{\delta>0}$ is bounded in $L^\infty([0,T]\times\mathcal{D}_s)$ and it is then possible to extract a weakly-* convergent sub-sequence.[5] Given $g \in L^1([0,T]\times\mathcal{D}_s)$ it is then sufficient to prove that

$$\left(\int_{[0,T]\times\mathcal{D}_s}\left(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\right)(Z_s)g(t,Z_s)dtdZ_s\right)_{\delta>0}$$

is a Cauchy family in $\mathbf{R}$ to prove the weak-* convergence of the whole sequence in $L^\infty([0,T]\times\mathcal{D}_s)$. To do so, let us define for $0<\delta'<\delta$.

$$\mathcal{C}^{\pm,i,\delta',\delta}_{s,s+1} := \mathcal{C}^{\pm,i,\delta'}_{s,s+1} - \mathcal{C}^{\pm,i,\delta}_{s,s+1}$$

which satisfies, by the same arguments as before,

$$\int_{[0,T]\times\mathcal{D}_s}|\mathcal{C}^{\pm,i,\delta',\delta}_{s,s+1}T_{s+1}(t)\varphi_{s+1}|(Z_s)dtdZ_s \le C(R,T)\delta\|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}.$$

Thanks to the Bienaymé-Tchebychev inequality, we know that the set

$$I_{\pm,i,\delta',\delta} := \left\{(t,Z_s)\in[0,T]\times\mathcal{D}_s,\ |\mathcal{C}^{\pm,i,\delta',\delta}_{s,s+1}T_{s+1}(t)\varphi_{s+1}|(Z_s)\ge\sqrt{\delta}\right\}$$

is of measure $\mathcal{O}(\sqrt{\delta})$. This leads to the following bound :

$$\begin{aligned}&\left|\int_{[0,T]\times\mathcal{D}_s}\left(\mathcal{C}^{\pm,i,\delta',\delta}_{s,s+1}T_{s+1}(t)\varphi_{s+1}\right)(Z_s)g(t,Z_s)dtdZ_s\right|\\ &\quad\le \int_{I_{\pm,i,\delta',\delta}}\dots+\int_{{}^cI_{\pm,i,\delta',\delta}}\dots\\ &\quad\le C\sup_{\delta>0}\|\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\|_{L^\infty([0,T]\times\mathcal{D}_s)}\|g\|_{L^1}|I_{\pm,i,\delta',\delta}|+\sqrt{\delta}\|g\|_{L^1}\\ &\quad\to 0.\end{aligned}$$

Step 3. *Dependence with respect to time.*

When φ_{s+1} depends also on the time variable t, it is possible to use the density of piece-wise constant functions in time to extend the previous construction (see [4] for more details).

Note that for the Boltzmann hierarchy, it is possible to require that all functions under study are continuous, since the free transport operator preserves continuity.

[5] It means that there exists a subsequence $\delta_{n_k}\to 0$ such that for all $g\in L^1([0,T]\times\mathcal{D}_s)$,

$$\int_{[0,T]\times\mathcal{D}_s}\left(\mathcal{C}^{\pm,i,\delta_{n_k}}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\right)(Z_s)g(t,Z_s)dtdZ_s\to 0.$$

1.6 Functional spaces and uniform estimates

This last section presents the functional spaces in which the wellposedness of the BBGKY and Boltzmann hierarchies can be proved.

We define the *hamiltonian* of the s-particle system by $E_0(Z_s) = \frac{1}{2}\sum_{i=1}^{s} |v_i|^2$.

Definition 1.5. *Given $\varepsilon > 0$, $\beta > 0$ and $\mu \in \mathbf{R}$,*

– *when $G = (g_s)_{s\geq 1}$, $g_s : \mathcal{D}_s \to \mathbf{R}$ is a sequence of measurable functions,*

$$|g_s|_{\varepsilon,s,\beta} := \operatorname*{supess}_{Z_s\in\mathcal{D}_s}\Big(|g_s(Z_s)|\exp(\beta E_0(Z_s))\Big) \quad \textit{and} \quad \|G\|_{\varepsilon,\beta,\mu} := \sup_{s\geq 1}\Big(|g_s|_{\varepsilon,s,\beta}\exp(\mu s)\Big)$$

– *when $G = (g_s)_{s\geq 1}$, $g_s : \mathbf{R}^{2ds} \to \mathbf{R}$ is a sequence of continuous functions,*

$$|g_s|_{0,s,\beta} := \sup_{Z_s\in\mathbf{R}^{2ds}}\Big(|g_s(Z_s)|\exp(\beta E_0(Z_s))\Big) \quad \textit{and} \quad \|G\|_{0,\beta,\mu} := \sup_{s\geq 1}\Big(|g_s|_{0,s,\beta}\exp(\mu s)\Big)$$

– *$X_{\varepsilon,s,\beta}$ (resp. $X_{0,s,\beta}$) is the Banach space of measurable functions with finite $|\cdot|_{\varepsilon,s,\beta}$ norm (resp. $|\cdot|_{0,s,\beta}$).*

– *$\mathbf{X}_{\varepsilon,s,\beta}$ (resp. $\mathbf{X}_{0,s,\beta}$) is the Banach space of sequences of measurable functions with finite $\|\cdot\|_{\varepsilon,s,\beta}$ norm (resp. $\|\cdot\|_{0,s,\beta}$).*

These functional spaces are rather natural in statistical physics where β and μ are respectively called *the inverse of the temperature* and *the chemical potential*. When β and μ are time dependant functions, we consider the following norms :

Definition 1.6. *Given $T > 0$ and β, μ two real valued functions defined on $[0,T]$ and a sequence $G : t \in [0,T] \mapsto G(t) = (g_s(t))_{s\geq 1}$, we define the norms :*

$$|||G|||_{\varepsilon,\beta,\mu} := \sup_{0\leq t\leq T}\|G(t)\|_{\varepsilon,\beta(t),\mu(t)} \quad \textit{and} \quad |||G|||_{0,\beta,\mu} := \sup_{0\leq t\leq T}\|G(t)\|_{0,\beta(t),\mu(t)}$$

and the respective functional spaces $\mathbb{X}_{\varepsilon,\beta,\mu}$ and $\mathbb{X}_{0,\beta,\mu}$.

The next proposition gives two continuity estimates for the collision operators we will constantly use throughout the proof. In particular, we will use the second estimate to prove the validity of Lanford's theorem on a short-time interval (see section 2.2.1).

Proposition 1.7 (Continuity estimates)**.** *Given two parameters $0 < \beta' < \beta$ and $g_{s+1} \in X_{\varepsilon,s+1,\beta}$, we have in the Boltzmann-Grad limit :*

$$|\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t)g_{s+1}|(Z_s) \leq C_d\beta^{-d/2}\left(\frac{s}{\sqrt{\beta}} + \sum_{i=1}^{s}|v_i|\right)e^{-\beta E_0(Z_s)}|g_{s+1}|_{\varepsilon,s+1,\beta} \tag{1.13}$$

and

$$|\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t)g_{s+1}|_{\varepsilon,s,\beta'} \leq C_d\beta^{-d/2}\left(\frac{s}{\sqrt{\beta}} + \sqrt{\frac{s}{\beta-\beta'}}\right)|g_{s+1}|_{\varepsilon,s+1,\beta} \tag{1.14}$$

Similar estimates hold for the Boltzmann collision operator $\mathcal{C}^0_{s,s+1}$ with the norms $|\cdot|_{0,s,\beta}$.

Proof. Recalling the definition

$$\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t)g_{s+1} := (N-s)\varepsilon^{d-1}$$
$$\times \sum_{i=1}^{s}\int_{\mathbf{S}^{d-1}\times\mathbf{R}^d}\mathbf{T}_{s+1}(t)g_{s+1}(t,Z_s,x_i+\varepsilon\omega,v_{s+1})\omega\cdot(v_{s+1}-v_i)d\omega dv_{s+1}$$

(1.13) follows from the direct calculation of a gaussian integral (since $\mathbf{T}_{s+1}$ preserves the norm). (1.14) follows from (1.13)$\times e^{\beta' E_0(Z_s)}$ and the Cauchy-Schwarz inequality which implies that $\sum_{i=1}^{s}|v_i| \leq \sqrt{s}\sqrt{E_0(Z_s)}$. The loss in β comes from the fact that the cross-section is unbounded (but with polynomial growth). □

We can now state and prove the following wellposedness results :

Theorem 1.8 (Existence and uniqueness for the BBGKY hierarchy)**.** *Given two parameters $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$, there exists a time $T > 0$ and two non-increasing functions $\beta, \mu : [0,T] \to \mathbf{R}$ with $\beta > 0$, $\beta(0) = \beta_0$ and $\mu(0) = \mu_0$ such that in the Boltzmann-Grad limit, for all bounded initial datum $F_N(0) = (f_N^{(s)}(0))_{1\leq s\leq N} \in \mathbf{X}_{\varepsilon,\beta_0,\mu_0}$, there exists a unique solution $G_N(t) = \left(\mathbf{T}_s(-t)f_N^{(s)}(t)\right)_{1\leq s\leq N} \in \mathbb{X}_{\varepsilon,\beta,\mu}$ to the BBGKY hierarchy, satisfying*

$$|\!|\!| G_N |\!|\!|_{\varepsilon,\beta,\mu} \leq 2\|F_N(0)\|_{\varepsilon,\beta_0,\mu_0} \tag{1.15}$$

Theorem 1.9 (Existence and uniqueness for the Boltzmann hierarchy)**.** *Given two parameters $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$, there exists a time $T > 0$ and two non-increasing functions $\beta, \mu : [0,T] \to \mathbf{R}$ with $\beta > 0$, $\beta(0) = \beta_0$ and $\mu(0) = \mu_0$ such that in the Boltzmann-Grad limit, for all bounded initial datum $F(0) = (f^{(s)}(0))_{1\leq s} \in \mathbf{X}_{0,\beta_0,\mu_0}$, there exists a unique solution $G(t) = \left(\mathbf{S}_s(-t)f^{(s)}(t)\right)_{1\leq s} \in \mathbb{X}_{0,\beta,\mu}$ to the Boltzmann hierarchy, satisfying*

$$|\!|\!| G |\!|\!|_{0,\beta,\mu} \leq 2\|F(0)\|_{0,\beta_0,\mu_0} \tag{1.16}$$

Proof (sketch). Let us first point out that the conservation of energy implies the conservation of the norm :

$$|\mathbf{T}_s(t)g_s|_{\varepsilon,s,\beta} = |g_s|_{\varepsilon,s,\beta} \text{ and } |\mathbf{S}_s(t)g_s|_{0,s,\beta} = |g_s|_{0,s,\beta}$$

The idea of the proof consists in using the continuity estimates to show that in the Boltzmann-Grad limit,

$$\forall 0 < \varepsilon \leq \varepsilon_0, \quad \left|\!\left|\!\left| \int_0^t \mathbf{T}(-\tau)\mathbf{C}_N\mathbf{T}(\tau)G_N(\tau)d\tau \right|\!\right|\!\right|_{\varepsilon,\beta,\mu} \leq \frac{1}{2}|\!|\!| G_N |\!|\!|_{\varepsilon,\beta,\mu} \tag{1.17}$$

for some function β and μ as in the statement of the theorem. This proves that the operator :

$$\mathfrak{L} : G_N \in \mathbb{X}_{\varepsilon,\beta,\mu} \longmapsto \int_0^t \mathbf{T}(-\tau)\mathbf{C}_N\mathbf{T}(\tau)G_N(\tau)d\tau \in \mathbb{X}_{\varepsilon,\beta,\mu}$$

has norm < 1 so that $Id-\mathfrak{L}$ is invertible and $(Id-\mathfrak{L})G_N = F_N(0)$ has a unique solution. The proof of the theorem then relies on the proof of (1.17) : this result is proved in great detail in lemma 5.4.3 of [4]. In particular, they prove that thanks to the continuity estimates, one can choose $\beta(t) = \beta_0 - \lambda t$ and $\mu(t) = \mu_0 - \lambda t$ for an appropriate choice of λ. The proof of the wellposedness of the Boltzmann hierarchy is identical. □

Chapter 2

From Newton to Boltzmann : Lanford's theorem

This chapter is devoted to the statement and the proof of Lanford's theorem which gives a mathematical framework whereby the formal limiting procedure of section 1.4 can be justified. We follow the original outline of the proof given by Lanford in the 1970s. However, the presentation is largely inspired by the rigorous completion of the proof due to I. Gallagher, L. Saint-Raymond and B. Texier in [4].

2.1 Admissible Boltzmann data and Landford's theorem

In this first section, we emphasize the fact that the convergence result only holds for a certain class of initial data called *admissible*. In particular, the definition of such data will show the intrication between the notion of *independence* and the functional spaces previously defined.

We define the set

$$\Omega_s := \{Z_s \in \mathbf{R}^{2ds} / \ \forall i \neq j, x_i \neq x_j\}.$$

2.1.1 Admissible Botzmann data

An admissible Boltzmann datum is defined as the limit of a sequence of distribution functions of a N-particles system when $N \to +\infty$. More precisely :

Definition 2.1 (Admissible Boltzmann datum)**.** *An initial datum $F_0 = (f_0^{(s)})_{s\geq 1}$ satisfying*

1. *For all $s \geq 1$, $f_0^{(s)}$ is continuous over Ω_s, nonnegative and integrable*
2. *For all $s \geq 1$,*

$$\int_{\mathbf{R}^{2d}} f_0^{(s+1)}(Z_s, z_{s+1}) dz_{s+1} = f_0^{(s)} \tag{2.1}$$

is said admissible when there exist BBGKY initial data $F_{N,0} = (f_{N,0}^{(s)})_{1\leq s\leq N} \in \mathbf{X}_{\varepsilon,\beta_0,\mu_0}$ satisfying :

1. $\sup_{N\geq 1} \|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0} < \infty$ *for some β_0 and μ_0 as $N\varepsilon^{d-1} \equiv 1$*
2. *For all $1 \leq s \leq N$,*

$$f_{N,0}^{(s)}(Z_s) = \int_{\mathbf{R}^{2d(N-s)}} \mathbf{1}_{Z_N \in \mathcal{D}_N} f_{N,0}^{(N)}(Z_N) dz_{s+1} \dots dz_N$$

and for all $s \geq 1$, $f_{N,0}^{(s)} \to f_0^{(s)}$ locally uniformly in Ω_s in the Boltzmann-Grad limit.

The following proposition gives a useful characterization of the notion of admissibility.

Proposition 2.2. *Given* $F_0 = (f_0^{(s)})_{s\geq 1}$ *such that for all* $s \geq 1$, $f_0^{(s)}$ *is continuous over* Ω_s, *nonnegative and integrable and satisfies* (2.1)*, the following assertions are equivalent :*

(i) F_0 *is an admissible Boltzmann datum*

(ii) There exist $\beta_0 > 0$ *and* $\mu_0 \in \mathbf{R}$ *such that* $\|F_0\|_{0,\beta_0,\mu_0} < \infty$

(iii) There exist $\beta_0 > 0$, $\mu_0 \in \mathbf{R}$ *and* π *a probability measure over*

$$\mathcal{P} := \left\{ g : \mathbf{R}^{2d} \to \mathbf{R} \text{ measurable},\ g \geq 0,\ \int_{\mathbf{R}^{2d}} g(z)dz = 1 \right\}$$

such that

$$\operatorname{Supp}(\pi) \subset \{ g \in \mathcal{P},\ |g|_{0,1,\beta_0} < e^{-\mu_0} \} \tag{2.2}$$

and

$$\forall s \geq 1,\ \ f_0^{(s)} = \int_{\mathcal{P}} g^{\otimes s} d\pi(g).$$

Remark 2.3. *A bounded initial datum (i.e. satisfying* (*ii*)*) is quasi-independent in the sense that it belongs to the convex hull of tensorized initial data (and for such initial data, the particles are said independant).*

Proof. $(i) \Rightarrow (ii)$ is obvious since the associated BBGKY initial data sequence $(F_{N,0})_N$ is bounded.

$(ii) \Rightarrow (iii)$. The existence of π is a consequence of the Hewitt-Savage theorem B.6 together with remark B.8 about densities of probability. The only thing that remains to check is (2.2) : assume by contradiction that there exists $\alpha > 0$ such that

$$\pi(A_\alpha) = \kappa_\alpha > 0, \text{ where } A_\alpha = \{ g \in \mathcal{P},\ |g|_{0,1,\beta_0} \geq e^{-\mu_0} + \alpha \}.$$

We then have by definition of π, for all $s \geq 1$:

$$f_0^{(s)} = \int_{\mathcal{P}} g^{\otimes s} d\pi(g) \geq \int_{A_\alpha} g^{\otimes s} d\pi(g)$$

hence

$$\|F_0\|_{0,\beta_0,\mu_0} \geq e^{s\mu_0} e^{\beta_0 E_0(Z_s)} f_0^{(s)} \geq \kappa_\alpha (1 + \alpha e^{\mu_0})^s$$

which cannot hold for all $s \geq 1$ since $1 + \alpha e^{\mu_0} > 1$ and $\|F_0\|_{0,\beta_0,\mu_0} < \infty$.

$(iii) \Rightarrow (i)$. Let us first sketch the proof when $\pi = \delta_{f_0}$ for a given f_0 : in this case, we have to prove that the family $(f_0^{\otimes s})_{s\geq 1}$ is admissible (eventually without the assumption of continuity). Let us define for $N \geq 1$:

$$f_{N,0}^{(N)}(Z_N) := \mathcal{Z}_N^{-1} \mathbf{1}_{Z_n \in \mathcal{D}_N} f_0^{\otimes N}(Z_N) \text{ with } \mathcal{Z}_N := \int_{\mathbf{R}^{2dN}} \mathbf{1}_{Z_n \in \mathcal{D}_N} f_0^{\otimes N}(Z_N) dZ_N$$

and its marginals :

$$\forall 1 \leq s \leq N, \quad f_{N,0}^{(s)}(Z_s) = \int_{\mathbf{R}^{2d(N-s)}} f_{N,0}^{(N)}(Z_N) dz_{s+1} \dots dz_N.$$

Then a direct calculation based on the bound

$$\forall s \in \{1, \dots, N\}, \quad 1 \leq \mathcal{Z}_N^{-1} \mathcal{Z}_{N-s} \leq (1 - C\varepsilon |f_0|_{L^\infty(\mathbf{R}^d_x, L^1(\mathbf{R}^d_v))})$$

shows that $F_{N,0} := (f_{N,0}^{(s)})_{1\leq s\leq N}$ satisfies the conditions of definition 2.1 for $F_0 := (f_0^{\otimes})_{s\geq 1}$ (see propositon 6.1.2 in [4] for more details). The generalization of this statement to convex combinations of tensor products follows the same ideas, as explained in proposition 6.1.4 of [4] □

2.1.2 Statement of the convergence result

Before stating the main result of this chapter, we need to define a good notion of convergence.

Definition 2.4 (Convergence)**.** *Given a sequence $(h_N^s)_{1\leq s\leq N}$ of functions $h_N^s \in L^\infty(\mathcal{D}_s, \mathbf{R})$ and a sequence $(h^s)_{s\geq 1}$ of functions $h^s \in C^0(\Omega_s, \mathbf{R})$, we say that $(h_N^s)_{1\leq s\leq N}$ converges in the sense of observables to $(h^s)_{s\geq 1}$, denoted by*

$$(h_N^s)_{1\leq s\leq N} \to (h^s)_{s\geq 1}$$

when for any fixed s, for any test function $\varphi_s \in C_0^0(\mathbf{R}^{ds}, \mathbf{R})$, there holds

$$I_s(h_N^s - h^s)(X_s) := \int_{\mathbf{R}^{ds}} \varphi_s(V_s)(h_N^s - h^s)(Z_s)dV_s \underset{N\to+\infty}{\longrightarrow} 0$$

locally uniformly in Ω_s.

Theorem 2.5 (Lanford)**.** *Given $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$, there exists a time $T > 0$ such that for any admissible Boltzmann datum $F_0 \in \mathbf{X}_{0,\beta_0,\mu_0}$ associated to $F_{N,0} \in \mathbf{X}_{\varepsilon,\beta_0,\mu_0}$, the following convergence in the sense of observables holds :*

$$F_N \to F$$

uniformly in $[0,T]$, where F_N and F are the solutions to the BBGKY and Boltzmann hierarchies.

Remark 2.6. *In particular, if $F_0 = (f_0^{\otimes s})_{s\geq 1}$, then $f_N^{(1)}$ converges to the solution f of the Boltzmann equation (in the sense that the observable $I_1(t, X_s)$ converges locally uniformly to the observable $I_1^0(t, X_s)$).*

Remark 2.7. *If $F_0 = (f_0^{\otimes s})_{s\geq 1}$ with f_0 Lipschitz, we will prove that the convergence holds at a rate $\mathcal{O}(\varepsilon^\alpha)$ for any $\alpha < (d-1)/(d+1)$.*

2.2 Strategy of the convergence proof

For a given $s \geq 1$, let us recall the integrated form of the BBGKY hierarchy (1.9)

$$f_N^{(s)}(t, Z_s) = \mathbf{T}_s(t) f_{N,0}(Z_s) + \int_0^t \mathbf{T}_s(t-\tau)\mathcal{C}_{s,s+1} f_N^{(s+1)}(\tau, Z_s)d\tau.$$

In this formula, the only quantity we know everything about is the initial datum $f_{N,0}^{(s)}$. However, it is still possible to reuse this formula to get rid of $f_N^{(s+1)}$ (we will obtain a formula for $f_N^{(s)}$ that depends on $f_N^{(s+2)}$). Since $f_N^{(k)} = 0$ for $k > N$, we can iterate this procedure a finite number of time (this number going to $+\infty$ with N) and we obtain :

$$f_N^{(s)}(t, Z_s) = \sum_{k=0}^{N-s} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots$$
$$\mathcal{C}_{s+k-1,s+k}\mathbf{T}_{s+k}(t_k) f_{N,0}^{(s+k)} dt_1 \dots dt_k.$$

For a given test function $\varphi_s(V_s)$, the quantity we are interested in is the observable

$$I_s(t, X_s) := \sum_{k=0}^{+\infty} \int dV_s \varphi_s(V_s) \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots$$
$$\mathcal{C}_{s+k-1,s+k}\mathbf{T}_{s+k}(t_k) f_{N,0}^{(s+k)} dt_1 \dots dt_k$$

up to defining $f_N^{(s+k)} = 0$ for $k > N - s$. Similarly for the Boltzmann hierarchy :

$$I_s^0(t, X_s) := \sum_{k=0}^{+\infty} \int dV_s \varphi_s(V_s) \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{S}_s(t-t_1)\mathcal{C}^0_{s,s+1}\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}^0_{s+1,s+2}\dots$$
$$\mathcal{C}^0_{s+k-1,s+k}\mathbf{S}_{s+k}(t_k) f_0^{(s+k)} dt_1 \dots dt_k$$

The strategy is to use the dominated convergence theorem to prove that :

$$\sum_{k=0}^{\infty} I_{s,k}(t, X_s) \xrightarrow[N\varepsilon^{d-1}\equiv 1]{\varepsilon\to 0} \sum_{k=0}^{\infty} I^0_{s,k}(t, X_s).$$

the convergence being uniform in X_s.

In order to keep lighter notations let us define :

$$\mathcal{T}_k(t) := \{T_k = (t_1, .., t_k)/\ t_i \geq t_{i+1} \text{ and } t_0 = t,\ t_{k+1} = 0\}.$$

2.2.1 The easy part of the proof : domination

Thanks to the continuity estimate (1.13), the domination part is the easiest one (see [9]). let us define a finite arithmetic sequence :

$$\gamma_0 = \frac{\beta_0}{2} < \dots < \gamma_j = \frac{\beta_0}{2} + \frac{j\beta_0}{2k} < \dots < \gamma_k = \beta_0$$

and note that the transport operator preserves the $|\cdot|_{\varepsilon,s,\beta}$ norm. One has :

$$\begin{aligned}
|I_{s,k}(t,X_s)| &\le \|\varphi_s\|_{L^\infty(\mathbf{R}^{ds})}\int dV_s\int_{\mathcal{T}_k(t)} dT_k e^{-\gamma_0 E_0(Z_s)}\left|\mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}\dots f_{N,0}^{(s+k)}\right|_{\varepsilon,s,\gamma_0}\\
&\le C\times\frac{t^k}{k!}\times\prod_{j=0}^{k-1}\left\{C\gamma_{j+1}^{-d/2}\left(\frac{s+j}{\sqrt{\gamma_{j+1}}}+\sqrt{\frac{s+1}{\gamma_{j+1}-\gamma_j}}\right)\right\}\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\\
&\le C^k\times\frac{t^k}{k!}\times\prod_{j=0}^{k-1}\sqrt{s+j}\prod_{j=0}^{k-1}\left\{\sqrt{s+j}+\sqrt{k(s+j)}\right\}\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\\
&\le C^k\times t^k\times\sqrt{k}^k\times\frac{1}{k!}\sqrt{\binom{s+k-1}{s-1}}\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\\
&\le (Ct)^k\times\frac{\sqrt{k}^k}{\sqrt{k!}}\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}
\end{aligned}$$

and thanks to the Stirling's formula : $\frac{\sqrt{k}^k}{\sqrt{k!}}\le C^k$, so that $I_{s,k}(t,X_s)$ is (uniformly) smaller than a geometric term. The domination is therefore proved for t sufficiently small (we can compute the bound explicitly).

2.2.2 Before the proof : technical reductions

To carry out the proof, it will be necessary to work with *truncated* observables. More precisely, we will assume that two collisions are separated in time by at least $\delta>0$, that there is a finite number of collisions $n\ge 1$ and that the energy remains bounded by $R>0$. Of course, at the very end, n and R will go to $+\infty$ and δ will go to 0. This leads to the following definitions :

$$\mathcal{T}_{k,\delta}(t) := \{T_k=(t_1,..,t_k)/\ t_i-t_{i+1}\ge\delta \text{ and } t_0=t,\ t_{k+1}=0\}.$$

and the truncated observable $I_s^{R,\delta}(t)=\sum_{k=0}^n I_{s,k}^{R,\delta}(t)$ where :

$$I_{s,k}^{R,\delta}(t,X_s)=\int\varphi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)}\mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots$$
$$\mathcal{C}_{s+k-1,s+k}\mathbf{T}_{s+k}(t_k)\mathbf{1}_{E_0(Z_{s+k})\le R^2}f_{N,0}^{(s+k)}dT_kdV_s$$

The error term between $I_s(t)$ and $I_s^{R,\delta}(t)$ can be controlled by :

Proposition 2.8. *Given* $s\in\mathbf{N}^*,\ t\in[0,T],\ \exists C,C'>0,$

$$\|I_s(t)-\sum_{k=0}^n I_{s,k}^{R,\delta}(t)\|_{L^\infty(\mathbf{R}^{ds})}\le C\left(2^{-n}+e^{-C'\beta_0R^2}+n^2\frac{\delta}{T}\right)\|\varphi_s\|_{L^\infty(\mathbf{R}^{ds})}\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}$$

Proof (sketch). The error 2^{-n} comes from the estimate (1.17), as well as the error $e^{-C'\beta_0R^2}$. Moreover, it can be proved that the complement of $\mathcal{T}_{k,\delta}(t)$ in $\mathcal{T}_k(t)$ is of measure $\le C\delta k\frac{t^{k-1}}{(k-1)!}$. If $t\le T$, summing for k from 0 to n leads to the error $n^2\delta/T$. □

Of course, we can similarly define a truncated Boltzmann observable $I_{s,k}^{0,R,\delta}$ with the same estimate for the error.

2.2.3 The notion of pseudo-trajectory

This paragraph presents the main idea of the proof, namely the notion of *pseudo-trajectory.* We are going to see that each term $I_{s,k}^{R,\delta}(t)$ and $I_{s,k}^{0,R,\delta}$ has a nice geometric interpretation which will allow us to couple the two hierarchies (see [4] section 7.4 and [2] section 5.1).

Splitting the cross-section into its positive and negative parts, we can write the collision operator (1.10) as :

$$\mathcal{C}_{s,s+1} = \sum_{i=1}^{s} \mathcal{C}_{s,s+1}^{i} = \sum_{i=1}^{s} \mathcal{C}_{s,s+1}^{+,i} - \sum_{i=1}^{s} \mathcal{C}_{s,s+1}^{-,i}.$$

For $J := (j_1, \ldots, j_k) \in \{-,+\}^k$ and $M := (m_1, .., m_k)$ with $m_i \in \{1, \ldots, s+i-1\}$, let us write :

$$I_{s,k}^{R,\delta}(t)(X_s) = \sum_{J,M} \left(\prod_{i=1}^{k} j_i \right) I_{s,k}^{R,\delta}(t, J, M)(X_s)$$

where we define the *elementary truncated observable* :

$$I_{s,k}^{R,\delta}(t, J, M)(X_s) := \int \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}^{j_1,m_1}\mathbf{T}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}^{j_2,m_2}\cdots$$
$$\mathcal{C}_{s+k-1,s+k}^{j_k,m_k}\mathbf{T}_{s+k}(t_k)\mathbf{1}_{E_0(Z_{s+k})\leq R^2} f_{N,0}^{(s+k)} dT_k dV_s \quad (2.3)$$

with a similar formula for the Boltzmann hierarchy :

$$I_{s,k}^{0,R,\delta}(t, J, M)(X_s) := \int \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{S}_s(t-t_1)\mathcal{C}_{s,s+1}^{0,j_1,m_1}\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}^{0,j_2,m_2}\cdots$$
$$\mathcal{C}_{s+k-1,s+k}^{0,j_k,m_k}\mathbf{S}_{s+k}(t_k)\mathbf{1}_{E_0(Z_{s+k})\leq R^2} f_0^{(s+k)} dT_k dV_s. \quad (2.4)$$

In this last expression, the characteristics associated with the operators $\mathbf{T}_{s+i}(t_{i-1}-t_i)$ and $\mathbf{S}_{s+i}(t_{i-1}-t_i)$ are followed backward in time and the collision operators $\mathcal{C}_{i,i+1}$ and $\mathcal{C}_{i,i+1}^0$ are seen as source terms in which "additional particules" are adjoined to the system. This leads to the definition of the two following flows (respectively called *the Boltzmann and BBGKY pseudo-trajectories*) : given an initial configuration Z_s, a k-tuple of collision times T_k, a k-tuple of signs J, a k-tuple of indexes M and a k-tuple of deflection angles and velocities $\Big\{(\nu_{s+i}, v_{s+i}) \in \mathbf{S}^{d-1} \times B_R^d,\ i \in \{1, \ldots, k\}\Big\}$, we can inductively define two flows :

- for $i \in \{1, \ldots, k\}$, $Z_{s+i}^0(\tau)$ will denote the position of the Boltzmann free-flow at time $t_{i+1} < \tau < t_i$, initiated from Z_s and constructed by adjunction at each time t_i of a particle (ν_{s+i}, v_{s+i}) to the particle m_i (the position of the additional particle at time t_i is precisely $x_{m_i}(t_i)$ since the particles are points).
- for $i \in \{1, \ldots, k\}$, $Z_{s+i}^\varepsilon(\tau)$ will denote the position of the $s+i$-particles free-flow at time $t_{i+1} < \tau < t_i$, initiated from Z_s and constructed by adjunction at each time t_i of a particle (ν_{s+i}, v_{s+i}) to the particle m_i (the additional particle is added, according to ν_{s+i}, at a distance ε of $x_{m_i}(t_i)$ since the particles are modeled by hard-spheres).

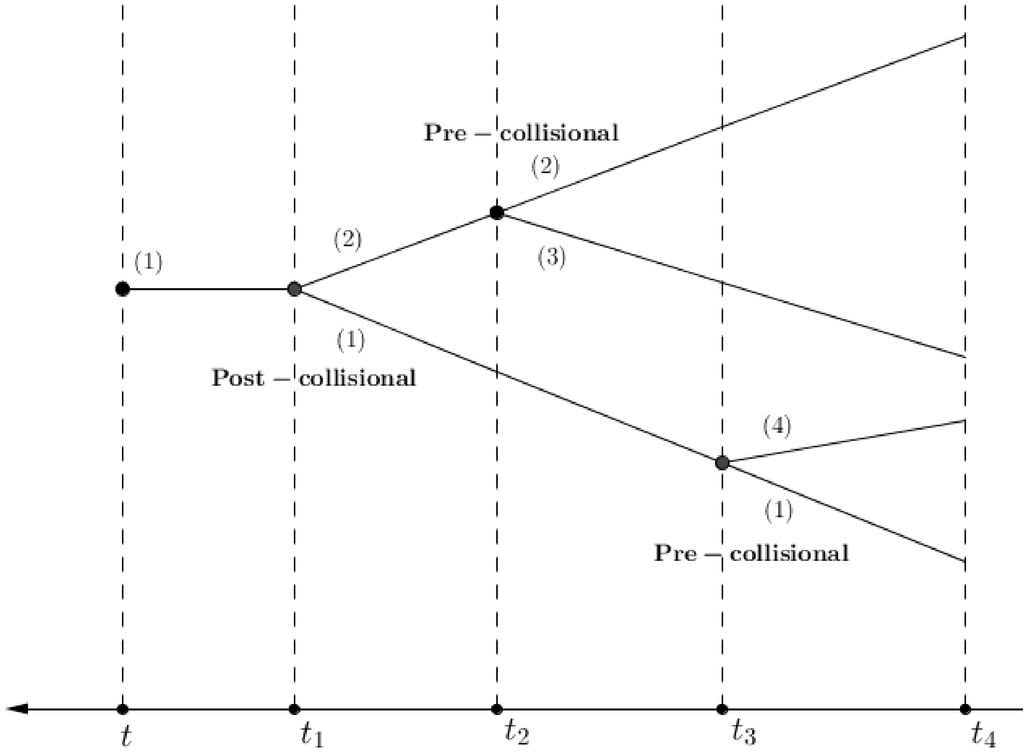

Figure 2.1: A *collision tree.* Each branching point represents the adjunction of a particle. The indexes of the particles is indicated between brackets. Here, we start at time t with one particle. At time t_1, $m_1 = 1$; at time t_2, $m_2 = 2$ and at time t_3, $m_3 = 1$. When a particle is added in a post-collisional way at time t_i, the velocities at time t_i^- are the velocities after scattering 1.3 (here, it happens at time t_1 only).

We will call *recollision* a collision that occurs between two times t_{i+1} and t_i (note that there is no recollision in a Boltzmann pseudo-trajectory). The fundamental idea of Lanford is that if the particles are added in such a way that there is no recollision, then the Boltzmann and BBGKY pseudo-trajectories will remain close and the convergence of the observables can easily be proved since we are able to write them as integral over the pseudo-trajectories. When a recollision occurs, the situation is illustrated below :

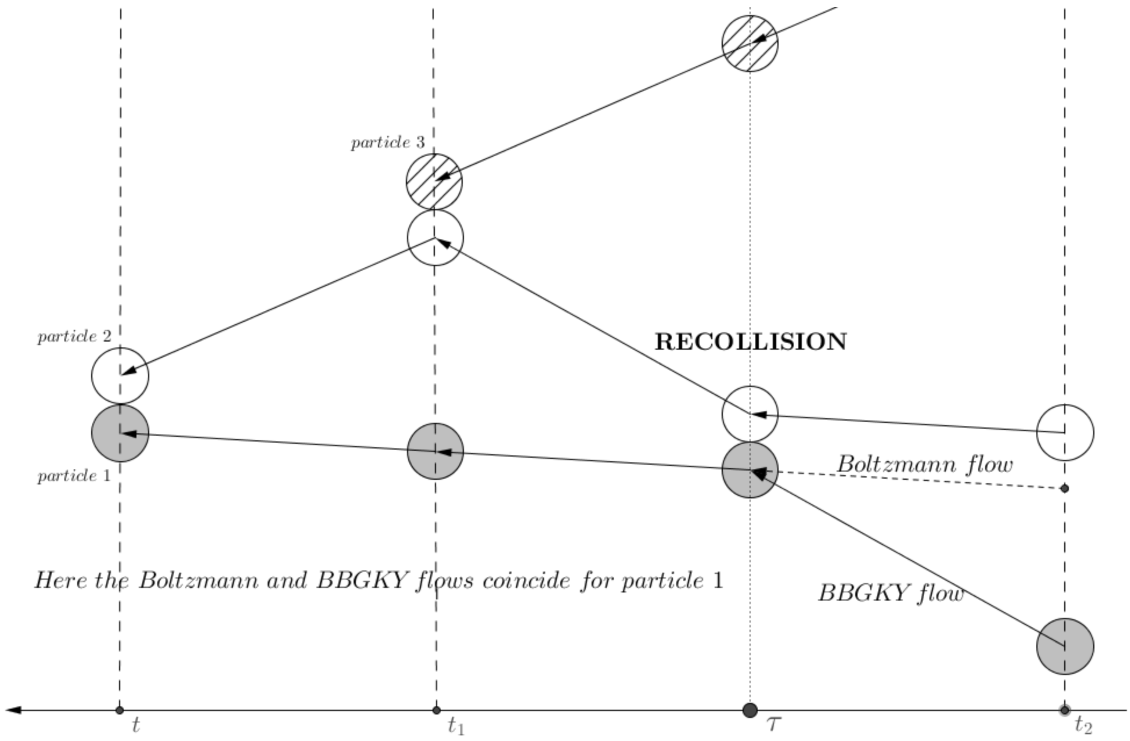

FIGURE 2.2: At time t, particle 2 is added next to particle 1 (in a pre-collisional way). At time t_1, particle 3 is added next to particle 2 (in a post-collisional way). Due to a recollision at time $t_2 < \tau < t_1$, the Boltzmann and BBGKY pseudo-trajectories of particle 1 are no longer close to each other.

2.2.4 Outline of the proof

$$
\begin{array}{ccccccc}
 & & & \overset{\text{Prop. 2.8}}{\downarrow} & & \overset{\text{Section 2.4}}{\downarrow} & \\
I_s(t)(X_s) & = & \sum_{k=0}^{\infty} I_{s,k}(t)(X_s) & \approx & \sum_k I_{s,k}^{R,\delta}(t)(X_s) & \approx & \sum_k J_{s,k}^{R,\delta}(t,J,M)(X_s) \\
 & & & & & & \big\downarrow \text{Section 2.5} \\
I_s^0(t)(X_s) & = & \sum_{k=0}^{\infty} I_{s,k}^0(t)(X_s) & \approx & \sum_k I_{s,k}^{0,R,\delta}(t)(X_s) & \approx & \sum_k J_{s,k}^{0,R,\delta}(t,J,M)(X_s)
\end{array}
$$

In section 2.4 we will slightly modify the truncated elementary observables in order to avoid recollisions. To do so, we will have to remove small[1] bad sets of deflection angles and velocities in the domain of integration of the collisions operators (1.10) so that the adjunction of a particle never leads to a recollision. The exact control of the size of these small bad sets is based on some elementary geometrical considerations presented in section 2.3. It is one of the main contributions of [4] and will be reused in [2] (see chapter 3) to retrieve the macroscopic behaviour of a deterministic system of hard-spheres. Finally,

[1]The Boltzmann-Grad limit being a low density limit, it is heuristically reasonable to think that two particles which already met won't never interact again. This tends to explain why recollisions are rare events.

the term-by-term convergence can easily be proved (see section 2.5) once the bad sets are removed.

2.3 Good configurations

From now, let us consider three new small parameters a, ε_0 (they have the scaling of a position) and η (it has the scaling of a velocity) such that :

$$\varepsilon \ll a \ll \varepsilon_0 \ll \eta\delta. \tag{2.5}$$

We define the sets :

$$\Delta_s(\varepsilon_0) := \{Z_s \in \mathbf{R}^{ds} \times B_R^s / \ \forall l, j,\ |x_l - x_j| \geq \varepsilon_0\}$$

and

$$\Delta_s^X(\varepsilon_0) := \{X_s \in \mathbf{R}^{ds} / \ \forall l, j,\ |x_l - x_j| \geq \varepsilon_0\}.$$

A *good configuration* is a subset of $\Delta_s(\varepsilon_0)$ stable by the free-transport :

$$\mathcal{G}_s(\varepsilon_0) := \{Z_s \in \mathbf{R}^{ds} \times B_R^s / \ \forall \tau \geq 0,\ \forall l, j,\ |x_l - x_j - \tau(v_l - v_j)| \geq \varepsilon_0\}.$$

2.3.1 Two fundamental propositions

Proposition 2.9 (How to add a new particle). *Given $\overline{Z}_k \in \mathcal{G}_k(\varepsilon_0)$, there exists $\mathcal{B}_k(\overline{Z}_k) \subset \mathbf{S}^{d-1} \times B_R$ such that :*

- *$\mathcal{B}_k(\overline{Z}_k)$ is small :*

$$|\mathcal{B}_k(\overline{Z}_k)| \leq Ck \left(R\eta^{d-1} + R^d \left(\frac{a}{\varepsilon_0}\right) + R \left(\frac{\varepsilon_0}{\delta}\right)^{d-1} \right)$$

- *A good configuration close to $\overline{Z}_k$ is stable by adjunction of a particle not belonging to $\mathcal{B}_k(\overline{Z}_k)$: for all $Z_k \in \mathbf{R}^{ds} \times B_R^s$ such that $V_k = \overline{V}_k$ and $|X_k - \overline{X_k}| \leq a$ and for all $(\nu, v) \in {}^c\mathcal{B}_k(\overline{Z}_k)$,*

  - *if $\nu \cdot (v - v_k) < 0$ then*

$$\forall \tau \geq 0, \ \begin{cases} \forall i \neq j \in \{1, \ldots, k\}, & |x_i - \tau\overline{v}_i - (x_j - \tau\overline{v}_j)| \geq \varepsilon \\ \forall j \in \{1, \ldots, k\}, & |x_k + \varepsilon\nu - \tau v - (x_j - \tau\overline{v}_j)| \geq \varepsilon \end{cases}$$

  Moreover, after time δ, the $k+1$ particles are in a good configuration :

$$(X_k - \delta\overline{V}_k, \overline{V}_k, x_k + \varepsilon\nu - \delta v, v) \in \mathcal{G}_{k+1}(\varepsilon_0/2).$$

  - *if $\nu \cdot (v - v_k) > 0$ then*

$$\forall \tau \geq 0, \ \begin{cases} \forall i \neq j \in \{1, \ldots, k\}, & |x_i - \tau\overline{v}_i^* - (x_j - \tau\overline{v}_j^*)| \geq \varepsilon \\ \forall j \in \{1, \ldots, k\}, & |x_k + \varepsilon\nu - \tau v - (x_j - \tau\overline{v}_j^*)| \geq \varepsilon \end{cases}$$

  Moreover, after time δ, the $k+1$ particles are in a good configuration :

$$(X_k - \delta\overline{V}_k^*, \overline{V}_k^*, x_k + \varepsilon\eta - \delta v, v) \in \mathcal{G}_{k+1}(\varepsilon_0/2).$$

Remark 2.10. *In the statement, we have assumed that the additional particle collides with the particle numbered k. Of course, an analogous statement holds if the particle is added close to any other particle m_k in Z_k. In this case, $\mathcal{B}_k^{m_k}(Z_k)$ will denote the bad set of deflection angles and velocities defined in the statement of the proposition.*

Proposition 2.11 (How to prepare the initial configuration)**.** *Given $X_s \in \Delta_s^X(\varepsilon_0)$, there exists a subset of velocities $\mathcal{M}_s(X_s) \subset \mathbf{R}^{ds}$ such that :*

- *$\mathcal{M}_s(X_s)$ is small :*

$$|\mathcal{M}_s(X_s)| \leq CRs^2 \left(\left(R\frac{\varepsilon}{\varepsilon_0} \right)^{d-1} + \left(\frac{\varepsilon_0}{\delta} \right)^{d-1} \right)$$

- *Defining $\mathcal{P}_s := \{Z_s \in \Delta_s(\varepsilon_0) / \, V_s \notin \mathcal{M}_s(X_s)\}$, one has :*

$$\forall \tau \geq 0, \ \mathbf{1}_{\mathcal{P}_s} \circ \mathbf{T}_s(\tau) \equiv \mathbf{1}_{\mathcal{P}_s} \circ \mathbf{S}_s(\tau)$$

and

$$\forall \tau \geq \delta, \ \mathbf{1}_{\mathcal{P}_s} \circ \mathbf{S}_s(\tau) \equiv \mathbf{1}_{\mathcal{P}_s} \circ \mathbf{S}_s(\tau) \circ \mathbf{1}_{\mathcal{G}_s(\varepsilon_0)}.$$

The proof of these two propositions is based on a simple geometrical lemma presented in the next paragraph.

2.3.2 Geometrical considerations

In the following, $K(w, y, \rho)$ will denote a cylinder of origin $w \in \mathbf{R}^d$, of axis $y \in \mathbf{R}^d$ and with radius $\rho > 0$.

Lemma 2.12 (Bad trajectories by free transport)**.** *Consider $\overline{x}_1$ and $\overline{x}_2$ two positions such that $|\overline{x}_1 - \overline{x}_2| \geq \varepsilon_0$ and two velocities v_1 and v_2 in B_R. Then for all $x_1 \in B_a(\overline{x}_1)$ and $x_2 \in B_a(\overline{x}_2)$,*

- *If $v_2 \notin K\left(v_1, \overline{x}_1 - \overline{x}_2, \dfrac{6Ra}{\varepsilon_0}\right)$ then : $\forall \tau \geq 0, \ |(x_1 - v_1\tau) - (x_2 - v_2\tau)| > \varepsilon$.*

- *If $v_2 \notin K\left(v_1, \overline{x}_1 - \overline{x}_2, \dfrac{6\varepsilon_0}{\delta}\right)$ then : $\forall \tau \geq \delta, \ |(x_1 - v_1\tau) - (x_2 - v_2\tau)| > \varepsilon_0$.*

Proof. We want to avoid the velocities such that there exists $\tau \geq 0$ satisfying :

$$|(x_1 - x_2) - \tau(v_1 - v_2)| \leq \varepsilon. \tag{2.6}$$

Provided that ε is small enough, the ball centered at $x_1 - x_2$ and of radius ε is embedded in the ball centered at $\overline{x}_1 - \overline{x}_2$ and of radius $3a$. (2.6) therefore implies that $v_1 - v_2$ belongs to the cone of center O based on the ball $B_{3a}(\overline{x}_1 - \overline{x}_2)$. Recalling that the velocities are bounded by R, the intersection of this cone with the ball of radius $2R$ is embedded in a cylinder of radius smaller than $6Ra/\varepsilon_0$. The first part of the lemma is proved.

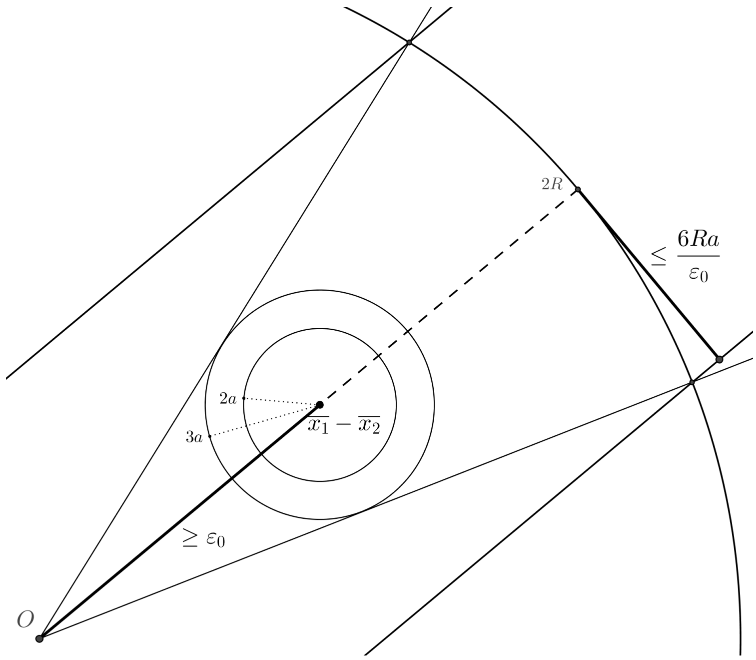

Figure 2.3: The proof.

To prove the second part of the lemma, let us introduce n the unit vector such that $n \cdot (\overline{x}_1 - \overline{x}_2) = 0$. The conclusion follows by writing that

$$\tau |n \cdot (v_1 - v_2)| = |n \cdot ((\overline{x}_1 - \overline{x}_2) - \tau(v_1 - v_2))| \leq 3\varepsilon_0$$

when $\tau \geq \delta$ satisfies $|(x_1 - v_1\tau) - (x_2 - v_2\tau)| \leq \varepsilon_0$ and provided that a is small enough so that

$$|(\overline{x}_1 - v_1\tau) - (\overline{x}_2 - v_2\tau)| \leq 3\varepsilon_0.$$

□

When a particle is added in a post-collisional way, the velocities that will eventually lead to a recollision are the velocities after scattering 1.3. The size of the subset of deflection angles and velocities that lead to a recollision after a "post-collisional adjunction" is controlled by the following lemma (which is based on lemma 2.12).

Lemma 2.13 (Bad trajectories by hard-spheres reflection)**.** *Consider $\rho \ll R$ and $(y, w) \in \mathbf{R}^d \times B_R$. For any $v_1 \in B_R$, define*

$$\mathcal{N}^*(w, y, \rho)(v_1) := \{(\nu, v_2) \in \mathbf{S}^{d-1} \times B_R / \ (v_2 - v_1) \cdot \nu > 0, \ v_1^* \in K(w, y, \rho) \ \text{or} \ v_2^* \in K(w, y, \rho)\}.$$

Then

$$|\mathcal{N}^*(w, y, \rho)(v_1)| \leq C R \rho^{d-1}.$$

Proof (sketch). Noticing that $r = |v_1 - v_2| = |v_1^* - v_2^*|$, the conclusion follows by controlling the measure of the intersection of a sphere of radius r with the cylinder $K(w, y, \rho)$. □

2.3.3 Proof of propositions 2.9 and 2.11

Proof of proposition 2.9. Let us first notice that

$$|x_i - x_j - \tau(\overline{v}_i - \overline{v}_j)| \geq |\overline{x}_i - \overline{x}_j - \tau(\overline{v}_i - \overline{v}_j)| - 2a \geq \frac{\varepsilon_0}{2}$$

so that $Z_k \in \mathcal{G}_k(\varepsilon_0/2)$. In the pre-collisional case, it is clear that for all $\tau \geq 0$,

$$|(x_k + \varepsilon\nu - v_{k+1}\tau) - (x_k - \overline{v}_k\tau)| \geq \varepsilon$$

and if $\tau \geq \delta$, provided that η is chosen according to 2.5, one has

$$|v_{k+1} - \overline{v}_k| > \eta \Rightarrow |(x_k + \varepsilon\nu - v_{k+1}\tau) - (x_k - \overline{v}_k\tau)| \geq \tau|v_k + 1 - \overline{v}_k| - \varepsilon \geq \eta\delta - \varepsilon > \frac{\varepsilon_0}{2}.$$

Moreover, for any $j \in \{1, \ldots, k-1\}$, lemma 2.12 indicates that we have to remove the set

$$B_R \cap K\left(\overline{v}_j, \overline{x}_j - \overline{x}_k, \frac{6Ra}{\varepsilon_0} + \frac{6\varepsilon_0}{\delta}\right)$$

which is of measure smaller than

$$C\left(R^d\left(\frac{a}{\varepsilon_0}\right)^{d-1} + R\left(\frac{\varepsilon_0}{\delta}\right)^{d-1}\right).$$

Finally, in the pre-collisional case, a bad set we should remove is :

$$\mathcal{B}_k^-(\overline{Z}_k) = \mathbf{S}^{d-1} \times \left(B_\eta(\overline{v}_k) \cup \bigcup_{j \leq k-1} \left(B_R \cap K\left(\overline{v}_j, \overline{x}_j - \overline{x}_k, \frac{6Ra}{\varepsilon_0} + \frac{6\varepsilon_0}{\delta}\right)\right)\right)$$

which is of small measure given in the statement of the proposition. In the post-collisional case, the proof is almost identical but we use lemma 2.13 to see that we have to remove the set

$$\mathcal{B}_k^+(\overline{Z}_k) = \mathbf{S}^{d-1} \times B_\eta(\overline{v}_k) \cup \bigcup_{j \leq k-1} \mathcal{N}^*\left(\overline{v}_j, \overline{x}_j - \overline{x}_k, \frac{6Ra}{\varepsilon_0} + \frac{6\varepsilon_0}{\delta}\right)(\overline{v}_k).$$

□

Proof of proposition 2.11 (sketch). The existence of the subset $\mathcal{M}_s(X_s)$ follows by the same arguments as before. Outside this set, the backward nonlinear flow is actually the free-flow and the particles remain at a distance larger than ε for all time and larger than ε_0 after a time δ. The result then follows by definition of $\mathcal{P}_s$. □

2.3.4 A third fundamental proposition

Proposition 2.14 (The pseudo-trajectories remain close)**.** *Assume that the initial configuration Z_s is in $\mathcal{P}_s(\varepsilon_0)$ and that for all $i \in \{1, .., k\}$, $(\nu_{s+i}, v_{s+i}) \in {}^c\mathcal{B}_{s+i}(Z^0_{s+i}(t_i))$, then for ε sufficiently small, for all $i \in \{1, \ldots, k\}$ and for all $m \leq s+i$,*

$$|x^\varepsilon_m(t_{i+1}) - x^0_m(t_{i+1})| \leq \varepsilon i \;\; \textit{and} \;\; v_m(t_{i+1}) = v^0_m(t_{i+1}).$$

Proof (sketch). Gathering the results of the previous section, the result is almost obvious : the particles are initially in a good configuration and the successive additional particles don't perturb this state (except during a small interval of time $[t_i - \delta, t_i]$ following a collision). As a consequence, the only error between the two pseudo-trajectories appears when a particle is added since, in the case of hard-spheres, the center of the additional particle is at a distance ε of the center of the particle next to which it is added. Finally, each time a particle is added, the error grows of at most ε and the conclusion follows. A more explicit proof can easily be carried out by induction since initially $Z_s^\varepsilon(t) = Z_s^0(t)$. □

We are now reaching the heart of proof, that is to say the definition of the Boltzmann and BBGKY approximated functionals (section 2.4) and the term-by-term convergence (section 2.5).

2.4 A matter of good approximations

2.4.1 The Boltzmann system

In the Boltzmann case, the elementary truncated observable can be approximated by :

$$\begin{aligned} J_{s,k}^{0,R,\delta}(t,J,M)(X_s) := & \int_{B_R \setminus \mathcal{M}_s(X_s)} dV_s \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} dT_k \\ & \int_{{}^c\mathcal{B}_s^{m_1}(Z_s^0(t_1))} d\nu_{s+1} dv_{s+1} ((v_{s+1} - v_{m_1}^0(t_1)) \cdot \nu_{s+1})_{j_1} \\ & \dots \int_{{}^c\mathcal{B}_{s+k-1}^{m_k}(Z_{s+k-1}^0(t_k))} d\nu_{s+k} dv_{s+k} ((v_{s+k} - v_{m_k}^0(t_k)) \cdot \nu_{s+k})_{j_k} \\ & \times \mathbf{1}_{E_0(Z_{s+k}^0(0)) \leq R^2} f_0^{(s+k)}(Z_{s+k}^0(0)) \end{aligned} \tag{2.7}$$

as shown in the following proposition, which proof is a direct consequence of propositions 2.9 and 2.11 :

Proposition 2.15. *Given $a, \varepsilon, \eta, \delta$ satisfying* (2.5), *the following estimates holds :*

$$\begin{aligned} & \left| \sum_{k=0}^{n} \sum_{J,M} \left(\prod_{i=1}^{k} j_i \right) \mathbf{1}_{\Delta_s(\varepsilon_0)} \left(I_{s,k}^{0,R,\delta} - J_{s,k}^{0,R,\delta} \right)(t,J,M) \right| \\ & \qquad \leq Cn^2(s+n) \left(R\eta^{d-1} + R^d \left(\frac{a}{\varepsilon_0} \right)^{d-1} + R \left(\frac{\varepsilon_0}{\delta} \right)^{d-1} \right) \|F_0\|_{0,\beta_0,\mu_0} \end{aligned}$$

2.4.2 The BBGKY system

In the BBGKY case, the elementary truncated observable can be approximated by :

$$J^{R,\delta}_{s,k}(t,J,M)(X_s) := \frac{(N-s)!}{(n-s-k)!}\varepsilon^{k(d-1)}\int_{B_R\setminus\mathcal{M}_s(X_s)} dV_s\varphi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)} dT_k$$
$$\int_{{}^c\mathcal{B}^{m_1}_s(Z^0_s(t_1))} d\nu_{s+1}dv_{s+1}((v_{s+1}-v^0_{m_1}(t_1))\cdot\nu_{s+1})_{j_1}$$
$$\dots\int_{{}^c\mathcal{B}^{m_k}_{s+k-1}(Z^0_{s+k-1}(t_k))} d\nu_{s+k}dv_{s+k}((v_{s+k}-v^0_{m_k}(t_k))\cdot\nu_{s+k})_{j_k}$$
$$\times\,\mathbf{1}_{E_0(Z^\varepsilon_{s+k}(0))\le R^2} f^{(s+k)}_{N,0}(Z^\varepsilon_{s+k}(0)) \quad (2.8)$$

as shown in the following proposition, which proof is a direct consequence of propositions 2.9 and 2.11 :

Proposition 2.16. *Given $a,\varepsilon,\eta,\delta$ satisfying* (2.5)*, the following estimates holds :*

$$\left|\sum_{k=0}^{n}\sum_{J,M}\left(\prod_i j_i\right)\mathbf{1}_{\Delta_s(\varepsilon_0)}\left(I^{R,\delta}_{s,k}-J^{R,\delta}_{s,k}\right)(t,J,M)\right|$$
$$\le Cn^2(s+n)\left(R\eta^{d-1}+R^d\left(\frac{a}{\varepsilon_0}\right)^{d-1}+R\left(\frac{\varepsilon_0}{\delta}\right)^{d-1}\right)\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}$$

Note that in the definition of $J^{R,\delta}_{s,k}$, we integrate over ${}^c\mathcal{B}^{m_i}_{s+i}(Z^0_{s+i}(t_i))$ and not over ${}^c\mathcal{B}^{m_i}_{s+i}(Z^\varepsilon_{s+i}(t_i))$ as we could expect. Of course, it will be really useful to prove the convergence $J^{R,\delta}_{s,k}\to J^{0,R,\delta}_{s,k}$. It is made possible by proposition 2.9 which allows us to work with a system *close* to a good configuration : it is the case thanks to proposition 2.14 !

2.5 Term by term convergence

The term-by-term convergence is now easy to carry out. There are three sources of error which lead to the three steps of the proof. We are going to prove that :

$$J^{R,\delta}_{s,k}\approx\widetilde{J}^{R,\delta}_{s,k}\approx\overline{J}^{R,\delta}_{s,k}\to J^{0,R,\delta}_{s,k}$$

these quantities being defined in the following paragraphs.

2.5.1 Error coming from the initial data

Let us define :

$$\widetilde{J}^{R,\delta}_{s,k}(t,J,M)(X_s) := \frac{(N-s)!}{(n-s-k)!}\varepsilon^{k(d-1)}\int_{B_R\setminus\mathcal{M}_s(X_s)} dV_s\varphi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)} dT_k$$
$$\int_{{}^c\mathcal{B}^{m_1}_s(Z^0_s(t_1))} d\nu_{s+1}dv_{s+1}((v_{s+1}-v^0_{m_1}(t_1))\cdot\nu_{s+1})_{j_1}$$
$$\dots\int_{{}^c\mathcal{B}^{m_k}_{s+k-1}(Z^0_{s+k-1}(t_k))} d\nu_{s+k}dv_{s+k}((v_{s+k}-v^0_{m_k}(t_k))\cdot\nu_{s+k})_{j_k}$$
$$\times\,\mathbf{1}_{E_0(Z^\varepsilon_{s+k}(0))\le R^2} f^{(s+k)}_{0}(Z^\varepsilon_{s+k}(0))$$

where $f_{N,0}$ has been changed into f_0 compared to (2.8).

Proposition 2.17. *The following estimate of the error holds :*

$$\left| \mathbf{1}_{\Delta_s^X(\varepsilon_0)} \left(J_{s,k}^{R,\delta} - \widetilde{J}_{s,k}^{R,\delta} \right)(t,J,M)(X_s) \right| \leq c_{k,J,M} \| \mathbf{1}_{\Delta_{s+k}(\varepsilon_0/2)} (f_{N,0}^{(s+k)} - f_0^{(s+k)}) \|_{L^\infty(\mathbf{R}^{2d(s+k)})}$$

where $\sum_k \left| \sum_{J,M} (\prod j_i) c_{k,J,M} \right| < +\infty.$

The proof of this result follows from the same calculations as in section 2.2.1. Besides, for tensorized initial data, the explicit estimate that could be found in the proof of proposition 2.2 leads to :

$$\| \mathbf{1}_{Z_{s+k} \in \mathcal{D}_{s+k}} f_0^{\otimes (s+k)} - f_{N,0}^{(s+k)} \|_{L^\infty} \leq C \varepsilon (s+k) \| F_0 \|_{0,\beta_0,\mu_0}.$$

2.5.2 Error coming from the prefactor

Define $\overline{J}_{s,k}^{R,\delta}(t,J,M)(X_s)$ by

$$\widetilde{J}_{s,k}^{R,\delta}(t,J,M)(X_s) = \alpha_{k,\varepsilon} \overline{J}_{s,k}^{R,\delta}(t,J,M)(X_s)$$

where

$$\alpha_{k,\varepsilon} = \frac{(N-s)!}{(N-s-k)!} \varepsilon^{k(d-1)}.$$

In the Boltzmann-Grad limit,

$$\left(\frac{N-s-k+1}{N} \right)^k \leq \alpha_{k,\varepsilon} \leq \left(\frac{N-s}{N} \right)^k$$

so that :

$$|\alpha_{k,\varepsilon} - 1| \leq Ck \frac{s+k-1}{N} \leq C \frac{(s+n)^2}{N}.$$

Proposition 2.18. *The following estimate of the error holds :*

$$\left| \mathbf{1}_{\Delta_s^X(\varepsilon_0)} \sum_{k=0}^n \sum_{J,M} (\prod j_i) \left(\overline{J}_{s,k}^{R,\delta} - \widetilde{J}_{s,k}^{R,\delta} \right)(t,J,M)(X_s) \right| \leq C \frac{(s+n)^2}{N} \| F_0 \|_{0,\beta_0,\mu_0}$$

2.5.3 Error coming from the divergence of trajectories

Here is the conclusion of the proof :

$$\begin{aligned}
(J_{s,k}^{0,R,\delta} - \overline{J}_{s,k}^{R,\delta})(t,J,M)(X_s) \leq & \int_{B_R \setminus \mathcal{M}_s(X_s)} dV_s \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} dT_k \\
& \int_{{}^c\mathcal{B}_s^{m_1}(Z_s^0(t_1))} d\nu_{s+1} dv_{s+1} ((v_{s+1} - v_{m_1}^0(t_1)) \cdot \nu_{s+1})_{j_1} \\
& \dots \int_{{}^c\mathcal{B}_{s+k-1}^{m_k}(Z_{s+k-1}^0(t_k))} d\nu_{s+k} dv_{s+k} ((v_{s+k} - v_{m_k}^0(t_k)) \cdot \nu_{s+k})_{j_k} \\
& \times \mathbf{1}_{E_0(Z_{s+k}^0(0)) \leq R^2} (f_0^{(s+k)}(Z_{s+k}^0(0)) - f_0^{(s+k)}(Z_{s+k}^\varepsilon(0)))
\end{aligned}$$

and since $f_0^{(s+k)}$ is continuous, we conclude thanks to proposition 2.14 :

$$(J_{s,k}^{0,R,\delta} - \overline{J}_{s,k}^{R,\delta})(t,J,M)(X_s) \xrightarrow[N\varepsilon^{d-1}\equiv 1]{\varepsilon\to 0} 0$$

uniformly in X_s and $t < T$. Besides, for tensorized Lipschitz initial data, we know that

$$|f_0^{\otimes(s+k)}(Z_{s+k}^0(0)) - f_0^{\otimes(s+k)}(Z_{s+k}^\varepsilon(0))| \le Cn\varepsilon\|f_0\|_{Lip}.$$

This concludes the proof of theorem 2.5 since the error coming from all the previous approximations can be made as small as we want provided that a good choice of parameters (2.5) is made.

2.5.4 Rate of convergence for tensorized Lipschitz initial data

When the initial data is tensorized and Lipschitz, we can explicitly estimate the rate of convergence. Gathering everything we find :

$$\begin{aligned}
|I_s(t) - I_s^0(t)| \quad \le \quad & C\left(2^{-n} + e^{-C'\beta_0 R^2} + n^2\frac{\delta}{T}\right)\|\varphi_s\|_{L^\infty(\mathbf{R}^{ds})}\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0} \\
& + Cn^2(s+n)\left(R\eta^{d-1} + R^d\left(\frac{a}{\varepsilon_0}\right)^{d-1} + R\left(\frac{\varepsilon_0}{\delta}\right)^{d-1}\right)\|\varphi_s\|_{L^\infty}\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0} \\
& + C\frac{(s+n)^2}{N}\|\varphi_s\|_{L^\infty}\|F_0\|_{0,\beta_0,\mu_0} \\
& + Cn\varepsilon\|f_0\|_{Lip}\|\varphi_s\|_{L^\infty}\|F_0\|_{0,\beta_0,\mu_0}.
\end{aligned}$$

Choosing $n \sim C_1|\log\varepsilon|$, $R^2 \sim C_2|\log\varepsilon|$, $\delta \sim \varepsilon^{(d-1)/(d+1)}$ and $\varepsilon_0 \sim \varepsilon^{d/(d+1)}$, we find that the error is smaller than $C\varepsilon^\alpha$ for all $\alpha < (d-1)/(d+1)$ (choose $a \sim \varepsilon^{(d+\bar{a})/(d+1)}$ and $\eta \sim \varepsilon^{\bar{\eta}/(d+1)}$ with the appropriate $\bar{a}, \bar{\eta} < 1$).

Chapter 3

From hard-spheres to Brownian motion

In [2], T. Bodineau, I. Gallagher and L. Saint-Raymond proved that, following the same pruning procedure presented in [4] (see chapter 2), the Brownian motion can be obtained as the limit of a deterministic system of hard-spheres. The fundamental idea is that if the system is initially around an equilibrium then it is possible to iterate the proof of Lanford's theorem to retrieve the *linear* Boltzmann equation on a time interval diverging with the number of particles. An explicit estimate of the rate of convergence is obtained (theorem 3.1) and with an appropriate macroscopic scaling, it will be the key to prove the diffusion limit (section 3.4) and the convergence of the stochastic process associated to a *tagged* particle to the Brownian motion (section 3.5).

3.1 The linear Boltzmann equation and the main theorem

Let us define the Maxwellian distribution by

$$M_\beta(v) := \left(\frac{\beta}{2\pi}\right)^{d/2} \exp\left(-\frac{\beta}{2}|v|^2\right) \text{ and } M_\beta^{\otimes s}(V_s) := \prod_{i=1}^{s} M_\beta(v_i).$$

We note that M_β is a stationary solution to the Boltzmann equation[1] with α collisions per unit of time :

$$\partial_t f + v \cdot \nabla_x f = \alpha Q(f, f)$$

and that any function of the energy $f_N \equiv F(E_0(Z_N))$ is a stationary solution of the Liouville equation. In particular, an invariant measure for the gas dynamics is given by the Gibbs measure with distribution in $\mathbf{T}^{dN} \times \mathbf{R}^{dN}$:

$$M_{N,\beta}(Z_N) := \frac{1}{\mathcal{Z}_N} \left(\frac{\beta}{2\pi}\right)^{dN/2} \exp\big(\beta E_0(Z_N)\big) \mathbf{1}_{\mathcal{D}_N}(Z_N) = \frac{1}{\mathcal{Z}_N} \mathbf{1}_{\mathcal{D}_N} M_\beta^{\otimes N}(V_N) \tag{3.1}$$

where

$$\mathcal{Z}_N := \int_{\mathbf{T}^{dN} \times \mathbf{R}^{dN}} \mathbf{1}_{\mathcal{D}_N}(Z_N) M_\beta^{\otimes N}(V_N) dZ_N = \int_{\mathbf{T}^{dN}} \prod_{1 \le i \ne j \le N} \mathbf{1}_{|x_i - x_j| > \varepsilon} dX_N.$$

In the following, we consider one tagged particle labeled by 1 with position and velocity $z_1 = (x_1, v_1)$. Initially the system is a perturbation of the density (3.1) with respect to the position x_1 of the tagged particle :

$$f_{N,0}(Z_N) := \rho^0(x_1) M_{N,\beta}(Z_N) \tag{3.2}$$

[1] In fact, it is possible to show that the equation $Q(f, f) = 0$ is only satisfied by the so-called Maxwellian distributions.

where ρ^0 is a continuous density of probability on $\mathbf{T}^d$. The goal of the next sections is to prove that the distribution $f_N^{(1)}(t,x,v)$ of the tagged particle remains close to $M_\beta(v)\varphi_\alpha(t,x,v)$ where φ_α is the solution of the linear Boltzmann equation with hard-spheres cross-section :

$$\begin{aligned} &\partial_t \varphi_\alpha + v\cdot\nabla_x\varphi_\alpha = -\alpha\mathcal{L}\varphi_\alpha \\ &\mathcal{L}\varphi_\alpha(v) := \int_{\mathbf{R}^d\times\mathbf{S}_1^{d-1}} [\varphi_\alpha(v)-\varphi_\alpha(v')]M_\beta(v_1)((v-v_1)\cdot\nu)_+ dv_1 d\nu \end{aligned} \tag{3.3}$$

and with initial data ρ^0. Compared to (1.7), the distribution of the particle with velocity v_1 with which the particle with velocity v collides is given by the Maxwellian M_β. The goal of the next sections is to prove the main theorem of this chapter :

Theorem 3.1. *For all $t>0$ and all $\alpha>1$, in the Boltzmann-Grad limit $N\to+\infty$, $\varepsilon\to 0$, $N\varepsilon^{d-1}\alpha^{-1}=1$, one has :*

$$\|f_n^{(1)}(t,x,v)-M_\beta(v)\varphi_\alpha(t,x,v)\|_{L^\infty(\mathbf{T}^d\times\mathbf{R}^d)} \le C\left[\frac{\alpha t}{(\log\log N)^{\frac{A-1}{A}}}\right]^{\frac{A^2}{A-1}}$$

where $A\ge 2$ can be taken arbitrarily large and C depends on A, β, d and $\|\rho^0\|_{L^\infty}$.

In particular, it is possible to keep a small error even for large concentrations α and large time t, provided that

$$\alpha t \ll (\log\log N)^{\frac{A-1}{A}}.$$

3.2 Setting and a priori estimates

3.2.1 A priori estimates

Let us define the operator :

$$Q_{s,s+n}(t) := \int_0^t\int_0^{t_1}\dots\int_0^{t_{n-1}} \mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\dots\mathbf{T}_{s+n}(t_n)dt_n\dots dt_1.$$

With this notation, the iterated Duhamel formula for the BBGKY hierarchy can be rephrased as :

$$f_N^{(s)}(t) = \sum_{n=0}^{N-s}\alpha^n Q_{s,s+n}(t) f_N^{(s+n)}(0).$$

Similarly, the iterated Duhamel formula for the Boltzmann hierarchy takes the form :

$$g_\alpha^{(s)}(t) = \sum_{n\ge 0}\alpha^n Q^0_{s,s+n}(t) g_\alpha^{(s+n)}(0)$$

where

$$Q^0_{s,s+n}(t) = \int_0^t\int_0^{t_1}\dots\int_0^{t_{n-1}} \mathbf{S}_s(t-t_1)\mathcal{C}^0_{s,s+1}\mathbf{S}_{s+1}(t_1-t_2)\dots\mathbf{S}_{s+n}(t_n)dt_n\dots dt_1.$$

Note that the family $(g_\alpha^{(s)})_{s\ge1}$ defined by

$$g_\alpha^{(s)}(t,Z_s) = \varphi_\alpha(t,z_1)M_\beta^{\otimes s}(V_s)$$

is a solution to the Boltzmann hierarchy with initial data

$$g_0^{(s)}(Z_s) := \rho^0(x_1) M_\beta^{\otimes s}(V_s). \tag{3.4}$$

The following proposition is a consequence of the calculations performed in section 2.2.1.

Proposition 3.2 (Continuity estimates for the operators $Q_{s,s+n}$ and $Q^0_{s,s+n}$)**.** *There exists a constant C depending only on d such that for all $t \geq 0$ and $s, n \in \mathbf{N}^*$, given $f_{s+n} \in X_{\varepsilon,s+n,\lambda}$ and $g_{s+n} \in X_{0,s+n,\lambda}$*

$$\| |Q|_{s,s+n}(t) f_{s+n} \|_{\varepsilon,s,\lambda/2} \leq e^{s-1} \left(\frac{Ct}{\lambda^{(d+1)/2}} \right)^n \| f_{s+n} \|_{\varepsilon,s+n,\lambda}$$

$$\left\| |Q^0|_{s,s+n}(t) g_{s+n} \right\|_{0,s,\lambda/2} \leq e^{s-1} \left(\frac{Ct}{\lambda^{(d+1)/2}} \right)^n \| g_{s+n} \|_{0,s+n,\lambda}$$

where $|Q|_{s,s+n}$ and $|Q^0|_{s,s+n}$ are the operators obtained by summing the absolute values of all elementary contributions.

The maximum principle for the Liouville equation leads to the following proposition :

Proposition 3.3 (Maximum principle)**.** *For any fixed N, the marginal $f_N^{(s)}$ of order s of the solution f_N to the Liouville equation* (1.4) *with initial data* (3.2) *satisfies the following bounds :*

$$\sup_t f_N^{(s)}(t, Z_s) \leq M_{N,\beta}^{(s)}(Z_s) \|\rho^0\|_{L^\infty} \leq C^s M_\beta^{\otimes s}(V_s) \|\rho^0\|_{L^\infty}$$

for some $C > 0$, provided that $\alpha\varepsilon \ll 1$.

Remark 3.4. *Thanks to proposition 3.3 we have for all $t \in \mathbf{R}$:*

$$\|f_N^{(k)}(t)\|_{\varepsilon,k,\beta} \leq C^k \left(\frac{\beta}{2\pi} \right)^{kd/2} \|\rho^0\|_{L^\infty}. \tag{3.5}$$

Similarly, the maximum principle for the linear Boltzmann equation leads to :

$$\|g_\alpha^{(k)}(t)\|_{0,k,\beta} \leq \left(\frac{\beta}{2\pi} \right)^{kd/2} \|\rho^0\|_{L^\infty}. \tag{3.6}$$

These uniform bounds are directly related to the existence of a stationary measure and they will be the key argument to obtain a lifespan diverging with the number of particles.

3.2.2 Collision trees of controlled size

Fixing a small parameter $h > 0$, we will study the dynamics up to time $t := Kh$ for some large integer K by splitting the interval $[0, t]$ into K intervals. Choosing h sufficiently small will allow us to iterate Lanford's argument. We define a tree of controlled size by the condition that it has strictly less than n_k branch points on a time interval $[t - kh, t - (k-1)h]$, where $(n_k)_{k \geq 1}$ is a sequence of integers to be tuned later. The following paragraph shows that we can neglect trees with to many branch points : since we expect the particles to undergo on average α collisions per unit of time, the growth of such pathological trees is typically exponential.

The iterated Duhamel's formula between t and $t-h$ gives for the first marginal :

$$f_N^{(1)}(t) = \sum_{j_1=0}^{n_1-1} \alpha^{j_1} Q_{1,1+j_1}(h) f_N^{(j_1)}(t-h) + R_{1,n_1}(t-h,t)$$

where

$$R_{k,n}(t',t) = \sum_{p=n}^{N-k} \alpha^p Q_{k,k+p}(t-t') f_N^{(k+p)}(t').$$

Iterating this argument K times leads to the following decomposition :

$$f_N^{(1)}(t) = f_N^{(1,K)}(t) + R_N^K(t)$$

where

$$f_N^{(1,K)}(t) := \sum_{j_1=0}^{n_1-1} \dots \sum_{j_K=0}^{n_K-1} \alpha^{J_K-1} Q_{1,J_1}(h) Q_{J_1,J_2}(h) \dots Q_{J_{K-1},J_K}(h) f_{N,0}^{(J_K)}$$

and

$$R_N^K(t) := \sum_{k=0}^{K} \sum_{j_1=0}^{n_1-1} \dots \sum_{j_k=0}^{n_k-1} \alpha^{J_{k-1}-1} Q_{1,J_1}(h) \dots Q_{J_{k-2},J_{k-1}} R_{J_{k-1},n_k}(t-kh, t-(k-1)k)$$

with

$$J_0 := 1 \text{ and } J_k := 1 + j_1 + j_2 + \dots + j_k.$$

From now, we define $n_k = A^k$ for some $A \geq 2$ and we let $h \to 0$.

Proposition 3.5 (Estimate of the remainders)**.** *There exist $c, C, \gamma_0 > 0$ depending on d, A and β such that for any $t > 1$ and any $\gamma \leq \gamma_0$, choosing*

$$h \leq \frac{c\gamma}{\alpha^{A/(A-1)} t^{1/(A-1)}} \quad \textit{and } K = t/h \textit{ integer}$$

we get

$$\|R_N^K(t)\|_{L^\infty(\mathbf{T}^d \times \mathbf{R}^d)} + \|R_\alpha^{0,K}(t)\|_{L^\infty(\mathbf{T}^d \times \mathbf{R}^d)} \leq C\gamma^A \|\rho^0\|_{L^\infty}.$$

Proof. let us estimate

$$\| |Q|_{1,J_{k-1}}((k-1)h) R_{J_{k-1},n_k}(t-kh, t-(k-1)h) \|_{L^\infty(\mathbf{R}^d \times \mathbf{R}^d)}.$$

Thanks to the continuity estimate given in proposition 3.2 we know that this quantity is smaller than

$$\left(\frac{C(k-1)h}{\beta^{(d+1)/2}} \right)^{J_{k-1}-1} \|R_{J_{k-1},n_k}(t-kh, t-(k-1)h)\|_{\varepsilon, J_{k-1}, \beta/2}.$$

And thanks to proposition 3.2 and uniform bound (3.5) :

$$\begin{aligned} \|R_{J_{k-1},n_k}(t-kh, t-(k-1)h)\|_{\varepsilon, J_{k-1}, \beta/2} &\leq \sum_{p=n_k}^{N-J_{k-1}} \left(\frac{C\alpha h}{\beta^{(d+1)/2}} \right)^p \sup_{t \geq 0} \|f_N^{(J_{k-1}+p)}(t)\|_{\varepsilon, J_{k-1}+p, \beta} \\ &\leq \sum_{p=n_k}^{N-J_{k-1}} \left(\frac{C\alpha h}{\beta^{(d+1)/2}} \right)^p C^{J_{k-1}+p} \left(\frac{\beta}{2\pi} \right)^{(J_{k-1}+p)d/2} \|\rho^0\|_{L^\infty} \end{aligned}$$

Finally,

$$\alpha^{J_{k-1}-1}\|\|Q|_{1,J_{k-1}}((k-1)h)R_{J_{k-1},n_k}(t-kh,t-(k-1)h)\|_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d)} \leq \|\rho^0\|_{L^\infty}\beta^{d/2}(\alpha t)^{J_{k-1}-1}\sum_{n_k}^{N-J_{k-1}}\left(\frac{C}{\sqrt{\beta}}\right)^{J_{k-1}+p-1}(\alpha h)^p$$

Assuming from now that

$$\frac{C\alpha h}{\sqrt{\beta}} < \frac{1}{2}$$

we find

$$\alpha^{J_{k-1}-1}\|\|Q|_{1,J_{k-1}}((k-1)h)R_{J_{k-1},n_k}(t-kh,t-(k-1)h)\|_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d)} \leq \|\rho^0\|_{L^\infty}\beta^{d/2}(\alpha t)^{J_{k-1}-1}\left(\frac{C}{\sqrt{\beta}}\right)^{J_{k-1}+n_k-1}(\alpha h)^{n_k}$$

Note that $\mathcal{N}_j := 1+n_1+\ldots+n_j = \frac{A^{j+1}-1}{A-1} \leq \frac{1}{A-1}n_{j+1}$. Then, since $J_{k-1} \leq \mathcal{N}_{k-1}$, we get

$$\alpha^{J_{k-1}-1}\|\|Q|_{1,J_{k-1}}((k-1)h)R_{J_{k-1},n_k}(t-kh,t-(k-1)h)\|_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d)} \leq \|\rho^0\|_{L^\infty}\beta^{d/2}\exp(A^k(\log C+\frac{1}{A-1}\log(\alpha t)+\log(\alpha h)))$$

Therefore, choosing

$$h \leq \frac{C\gamma}{\alpha^{A/(A-1)}t^{1/(A-1)}}$$

for γ small enough so that the previous bound holds, we find

$$\alpha^{J_{k-1}-1}\|\|Q|_{1,J_{k-1}}((k-1)h)R_{J_{k-1},n_k}(t-kh,t-(k-1)h)\|_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d)} \leq \|\rho^0\|_{L^\infty}\beta^{d/2}\exp(A^k\log\gamma)$$

Summing all these contributions leads to the following bound :

$$\begin{aligned}
\|R_N^K\|_{L^\infty(\mathbf{T}^d\times\mathbf{R}^d)} &\leq \beta^{d/2}\sum_{k=1}^{K}\left(\prod_{i=1}^{k}n_i\right)\exp(A^k\log\gamma)\|\rho^0\|_{L^\infty}\\
&\leq \beta^{d/2}\|\rho^0\|_{L^\infty}\sum_{k=1}^{K}\exp(k(k+1)\log A+A^k\log\gamma)\\
&\leq \beta^{d/2}\|\rho^0\|_{L^\infty}\sum_{k=1}^{K}\exp(Ak\log\gamma) \leq C\beta^{d/2}\gamma^A\|\rho^0\|_{L^\infty}
\end{aligned}$$

The argument is identical in the Boltzmann case. □

3.3 Proof of the convergence

The proof of theorem 3.1 is based on the same arguments presented in the last chapter. Some adaptations of the geometrical considerations are needed since we are working on the torus $\mathbf{T}^d$ and not in the whole space $\mathbf{R}^d$.

3.3.1 Reformulation in terms of pseudo-trajectories

$$f_N^{(1,K)}(t) = \sum_{j_1=0}^{n_1-1} \dots \sum_{j_K=0}^{n_K-1} \alpha^{J_K-1} F_N^{(1,K)}(J)(t,z_1), \ \ J=(j_1,\dots,j_K)$$

where

$$\begin{aligned} F_N^{(1,K)}(J) &:= Q_{1,J_1}(h)Q_{J_1,J_2}(h)\dots Q_{J_{K-1},J_K}(h) f_{N,0}^{(J_K)} \\ &= \int_{\mathcal{T}_J(h)} dT_J \mathbf{T}_1(t-t_1)C_{1,2}\mathbf{T}_2(t_2-t_1)\dots \mathbf{T}_{J_K}(t_{J_K-1}) f_{N,0}^{(J_K)} \end{aligned}$$

with

$$\mathcal{T}_J(h) := \big\{ T_J = (t_1,\dots,t_{J_K-1}) /\ t_i < t_{i-1} \text{ and } t_{J_k},\dots,t_{J_{k-1}+1} \in [t-kh, t-(k-1)h] \big\}$$

In terms of pseudo-trajectories and with slightly more compact notations than in the previous chapter :

$$F_N^{(1,K)}(J) = \sum_{m\in\mathcal{M}_J} \left(\frac{\varepsilon^{d-1}}{\alpha}\right)^{J_K-1} \frac{(N-1)!}{(N-J_K)!} F_N^{(1,K)}(J,m)$$

where

$$\mathcal{M}_J := \{ m=(m_1,\dots,m_{J_K-1}),\ 1\le m_i\le i\}$$

and

$$F_N^{(1,K)}(J,m) := \int_{\mathcal{T}_J(h)} dT_J \int_{(\mathbf{S}_1^{d-1}\times\mathbf{R}^d)^{J_K-1}} d\bar\nu d\bar V \prod_{i=1}^{J_K-1} [(v_{i+1}-v_{m_i}(t_i))\cdot\nu_{i+1}] f_{N,0}^{(J_K)}(Z_{J_K}(0))$$

with

$$\bar\nu = (\nu_2,\dots,\nu_{J_K}) \ \text{ and } \ \bar V = (v_2,\dots,v_{J_K}).$$

Here $Z_k(\tau)$ denotes the pseudo-trajectory associated to the BBGKY hierarchy.

3.3.2 Geometrical considerations : summary and adaptation

As in the previous chapter, the heart of the proof consists in neglecting pathological trajectories that lead to recollisions and prevent the two flows to remain close. The same pruning procedure performed in section 2.3 will be used to remove small bad sets of deflection angles and velocities that lead to recollisions. In this paragraph we simply sum up the key arguments developed and proved in the previous chapter. We will need a simple adaptation of lemma 2.12 in the case of the torus (that is the main difference between the two proofs).

Recall that when $\varepsilon_0 \gg \varepsilon$, a *good configuration* with k particles for a time t is the set :

$$\mathcal{G}_k(\varepsilon_0) := \{ Z_k \in \mathbf{T}^{dk}\times\mathbf{R}^{dk} /\ \forall\tau\in[0,t],\ \forall i\neq j,\ d(x_i-\tau v_i, x_j-\tau v_j)\ge\varepsilon_0\}$$

where d denotes the distance on the torus $\mathbf{T}^d$. From now, let us assume that all velocities take value in the ball $B_E := \{v\in\mathbf{R}^d,\ |v|\le E\}$ and let us fix three parameters a, ε_0 and δ such that

$$A^{K+1}\varepsilon \ll a \ll \varepsilon_0 \ll \min(\delta E, 1).$$

The next proposition is identical to proposition 2.9

Proposition 3.6. *Given $\overline{Z}_k \in \mathcal{G}_k(\varepsilon_0)$, there exists $\mathcal{B}_k(\overline{Z}_k) \subset \mathbf{S}^{d-1} \times B_E$ such that :*

- *$\mathcal{B}_k(\overline{Z}_k)$ is small :*

$$|\mathcal{B}_k(\overline{Z}_k)| \leq Ck \left(E^d (Et)^d \varepsilon_0^{d-1} + E^d \left(\frac{a}{\varepsilon_0}\right) + E \left(\frac{\varepsilon_0}{\delta}\right)^{d-1} \right)$$

- *A good configuration close to $\overline{Z}_k$ is stable by adjunction of a particle not belonging to $\mathcal{B}_k(\overline{Z}_k)$: for all $Z_k \in \mathbf{T}^{ds} \times B_E^s$ such that $V_k = \overline{V}_k$ and $|X_k - \overline{X_k}| \leq a$ and for all $(\nu, v) \in {}^c\mathcal{B}_k(\overline{Z}_k)$,*

  - *if $\nu \cdot (v - v_k) < 0$ then*

$$\forall \tau \geq 0, \begin{cases} \forall i \neq j \in \{1, \ldots, k\}, & d(x_i - \tau \overline{v}_i, x_j - \tau \overline{v}_j) \geq \varepsilon \\ \forall j \in \{1, \ldots, k\}, & d(x_k + \varepsilon\nu - \tau v, x_j - \tau \overline{v}_j)| \geq \varepsilon \end{cases}$$

  Moreover, after time δ, the $k+1$ particles are in a good configuration :

$$(X_k - \delta \overline{V}_k, \overline{V}_k, x_k + \varepsilon\nu - \delta v, v) \in \mathcal{G}_{k+1}(\varepsilon_0/2).$$

  - *if $\nu \cdot (v - v_k) > 0$ then*

$$\forall \tau \geq 0, \begin{cases} \forall i \neq j \in \{1, \ldots, k\}, & d(x_i - \tau \overline{v}_i^*, x_j - \tau \overline{v}_j^*) \geq \varepsilon \\ \forall j \in \{1, \ldots, k\}, & d(x_k + \varepsilon\nu - \tau v, x_j - \tau \overline{v}_j^*) \geq \varepsilon \end{cases}$$

  Moreover, after time δ, the $k+1$ particles are in a good configuration :

$$(X_k - \delta \overline{V}_k^*, \overline{V}_k^*, x_k + \varepsilon\eta - \delta v, v) \in \mathcal{G}_{k+1}(\varepsilon_0/2).$$

The proof of this proposition relies on the following lemma which is an adaption of lemma 2.12 :

Lemma 3.7. *Consider $\overline{x}_1$ and $\overline{x}_2$ two positions such that $|\overline{x}_1 - \overline{x}_2| \geq \varepsilon_0$ and two velocities v_1 and v_2 in B_E. Then there exists a subset $K(\overline{x}_1 - \overline{x}_2, \varepsilon_0, a)$ of $\mathbf{R}^d$ with measure bounded by*

$$|K(\overline{x}_1 - \overline{x}_2, \varepsilon_0, a)| \leq CE^d \left(\left(\frac{a}{\varepsilon_0}\right)^{d-1} + (Et)^d a^{d-1} \right)$$

and a subset $K_\delta(\overline{x}_1 - \overline{x}_2, \varepsilon_0, a)$ of $\mathbf{R}^d$ with measure bounded by

$$|K_\delta(\overline{x}_1 - \overline{x}_2, \varepsilon_0, a)| \leq CE^d \left(\left(\frac{\varepsilon_0}{\delta}\right)^{d-1} + (Et)^d E^{d-1} \varepsilon_0^{d-1} \right)$$

such that for all $x_1 \in B_a(\overline{x}_1)$ and $x_2 \in B_a(\overline{x}_2)$,

- *If $v_1 - v_2 \notin K(\overline{x}_1 - \overline{x}_2, \varepsilon_0, a)$ then : $\forall \tau \geq 0,\ d(x_1 - v_1\tau, x_2 - v_2\tau) > \varepsilon$.*
- *If $v_1 - v_2 \notin K_\delta(\overline{x}_1 - \overline{x}_2, \varepsilon_0, a)$ then : $\forall \tau \geq \delta,\ d(x_1 - v_1\tau, x_2 - v_2\tau) > \varepsilon_0$.*

Since we no longer work with observables, we can skip proposition 2.11 and adapt directly proposition 2.14.

Proposition 3.8. *Fix $J = (j_1, \ldots, j_K)$, $m = (m_1, \ldots, m_{J_K - 1}) \in \mathcal{M}_J$ and $T \in \mathcal{T}_{J,\delta}(h)$. Assume that for all $i \in \{1, .., k\}$, $(\nu_{s+i}, v_{s+i}) \in {}^c\mathcal{B}_{s+i}(Z^0_{s+i}(t_i))$, then for ε sufficiently small, for all $i \leq J_K - 1$ and for all $\ell \leq i+1$*

$$|x^\varepsilon_\ell(t_{i+1}) - x^0_\ell(t_{i+1})| \leq \varepsilon i \ \text{ and } \ v_\ell(t_{i+1}) = v^0_\ell(t_{i+1}).$$

3.3.3 Technical truncations

From now, $\sum_J$ will stand for $\sum_{j_1=0}^{n_1-1}\dots\sum_{j_K=0}^{n_K-1}$

Energy truncation. Given $E>0$, define :

$$f_{N,E}^{(1,K)} := \sum_J \alpha^{J_K-1}\sum_{m\in\mathcal{M}_J}\left(\frac{\varepsilon^{d-1}}{\alpha}\right)^{J_K-1}\frac{(N-1)!}{(N-J_K)!}F_{N,E}^{(1,K)}(J,m)$$

where

$$F_{N,E}^{(1,K)}(J,m)(t,z_1) :=$$
$$\int_{\mathcal{T}_J(h)} dT_J\int_{(\mathbf{S}_1^{d-1}\times\mathbf{R}^d)^{J_K-1}} d\bar\nu d\bar V\prod_{i=1}^{J_K-1}[(v_{i+1}-v_{m_i}(t_i))\cdot\nu_{i+1}]\mathbf{1}_{\{E_0(Z_{J_K}(0))\le\frac{E^2}{2}\}}f_{N,0}^{(J_K)}(Z_{J_K}(0)).$$

Similarly :

$$g_{\alpha,E}^{(1,K)} := \sum_J \alpha^{J_K-1}\sum_{m\in\mathcal{M}_J}G_E^{(1,K)}(J,m)$$

where

$$G_E^{(1,K)}(J,m)(t,z_1) :=$$
$$\int_{\mathcal{T}_J(h)} dT_J\int_{(\mathbf{S}_1^{d-1}\times\mathbf{R}^d)^{J_K-1}} d\bar\nu d\bar V\prod_{i=1}^{J_K-1}[(v_{i+1}-v^0_{m_i}(t_i))\cdot\nu_{i+1}]\mathbf{1}_{\{E_0(Z_{J_K}(0))\le\frac{E^2}{2}\}}g_0^{(J_K)}(Z^0_{J_K}(0)).$$

Proposition 3.9. *There is a constant C depending only on β and d such that, as N goes to infinity in the scaling $N\varepsilon^{d-1}\alpha^{-1}=1$, the following bounds hold :*

$$\begin{aligned}\|f_N^{(1,K)}-f_{N,E}^{(1,K)}\|_{L^\infty([0,t]\times\mathbf{T}^d\times\mathbf{R}^d)}+\|g_\alpha^{(1,K)}-g_{\alpha,E}^{(1,K)}\|_{L^\infty([0,t]\times\mathbf{T}^d\times\mathbf{R}^d)}\\ \le A^{K(K+1)}(C\alpha t)^{A^{K+1}}e^{-\frac{\beta}{4}E^2}\|\rho^0\|_{L^\infty}.\end{aligned}$$

Proof. We need to estimate the contribution of pseudo-trajectories such that $\{E_0(Z_{J_K})\ge E^2/2\}$. Since

$$\begin{aligned}\mathbf{1}_{\{E_0(Z_{J_K})\ge E^2/2\}}f_{N,0}^{(J_K)}e^{\frac{\beta}{2}E_0(Z_{J_K})} &= \mathbf{1}_{\{E_0(Z_{J_K})\ge E^2/2\}}f_{N,0}^{(J_K)}e^{\beta E_0(Z_{J_K})}e^{-\frac{\beta}{2}E_0(Z_{J_K})}\\ &\le \mathbf{1}_{\{E_0(Z_{J_K})\ge E^2/2\}}f_{N,0}^{(J_K)}e^{\beta E_0(Z_{J_K})}e^{-\frac{\beta}{4}E^2}\end{aligned}$$

we get thanks to (3.5)

$$\|\mathbf{1}_{\{E_0(Z_{J_K})\ge E^2/2\}}f_{N,0}^{(J_K)}\|_{\varepsilon,J_K,\beta/2}\le\|f_{N,0}^{(J_K)}\|_{\varepsilon,J_K,\beta}e^{-\frac{\beta}{4}E^2}\le C^{J_K}e^{-\frac{\beta}{4}E^2}\|\rho^0\|_{L^\infty}.$$

Now in the limit $N\varepsilon^{d-1}=\alpha$ we have thanks to proposition 3.2

$$\left\|\sum_{m\in\mathcal{M}_J}\left(\frac{\varepsilon^{d-1}}{\alpha}\right)^{J_K-1}\frac{(N-1)!}{(N-J_K)!}(F_N^{(1,K)}-F_{N,E}^{(1,K)})(J,m)\right\|_{L^\infty([0,t]\times\mathbf{T}^d\times\mathbf{R}^d)}$$

$$
\begin{aligned}
&\leq \; \| |Q|_{1,J_K}(t) \mathbf{1}_{E_0(Z_{J_K}) \geq E^2/2} f_{N,0}^{(J_K)} \|_{L^\infty([0,t]\times \mathbf{T}^d \times \mathbf{R}^d)} \\
&\leq \; \left(\frac{Ct}{(\beta/2)^{(d+1)/2}} \right)^{J_K - 1} \| \mathbf{1}_{\{E_0(Z_{J_K}) \geq E^2/2\}} f_{N,0}^{(J_K)} \|_{\varepsilon, J_K, \beta/2} \\
&\leq \; (Ct)^{A^{K+1}} e^{-\frac{\beta}{4} E^2} \| \rho^0 \|_{L^\infty}
\end{aligned}
$$

recalling that $J_K \leq \mathcal{N}_K \leq A^{K+1}$. In conclusion

$$
\begin{aligned}
\| f_N^{(1,K)} - f_{N,E}^{(1,K)} \|_{L^\infty([0,t]\times \mathbf{T}^d \times \mathbf{R}^d)} \; &\leq \; (Ct)^{A^{K+1}} e^{-\frac{\beta}{4} E^2} \| \rho^0 \|_{L^\infty} \sum_J \alpha^{J_K - 1} \\
&\leq \; (Ct)^{A^{K+1}} e^{-\frac{\beta}{4} E^2} \| \rho^0 \|_{L^\infty} \alpha^{A^{K+1}} A^{K(K+1)}
\end{aligned}
$$

A similar estimates holds for the Boltzmann hierarchy so the proposition is proved. □

Time separation. Given a small parameter $\delta > 0$ such that $A^K \delta \ll h$, we define :

$$
f_{N,E,\delta}^{(1,K)} := \sum_J \alpha^{J_K - 1} \sum_{m \in \mathcal{M}_J} \left(\frac{\varepsilon^{d-1}}{\alpha} \right)^{J_K - 1} \frac{(N-1)!}{(N-J_K)!} F_{N,E,\delta}^{(1,K)}(J,m)
$$

where

$$
\begin{aligned}
&F_{N,E}^{(1,K)}(J,m)(t,z_1) := \\
&\quad \int_{\mathcal{T}_{J,\delta}(h)} dT_J \int_{(\mathbf{S}_1^{d-1} \times \mathbf{R}^d)^{J_K - 1}} d\bar{\nu} d\bar{V} \prod_{i=1}^{J_K - 1} [(v_{i+1} - v_{m_i}(t_i)) \cdot \nu_{i+1}] \mathbf{1}_{\{E_0(Z_{J_K}(0)) \leq \frac{E^2}{2}\}} f_{N,0}^{(J_K)}(Z_{J_K}(0)).
\end{aligned}
$$

and

$$
\mathcal{T}_{J,\delta}(h) := \big\{ T_J = (t_1, \dots, t_{J_K - 1}) / \; t_i < t_{i-1} - \delta \text{ and } t_{J_k}, \dots, t_{J_{k-1}+1} \in [t - kh, t-(k-1)h] \big\}
$$

Similarly :

$$
g_{\alpha,E,\delta}^{(1,K)} := \sum_J \alpha^{J_K - 1} \sum_{m \in \mathcal{M}_J} G_{E,\delta}^{(1,K)}(J,m)
$$

where

$$
\begin{aligned}
&G_{E,\delta}^{(1,K)}(J,m)(t,z_1) := \\
&\quad \int_{\mathcal{T}_{J,\delta}(h)} dT_J \int_{(\mathbf{S}_1^{d-1} \times \mathbf{R}^d)^{J_K - 1}} d\bar{\nu} d\bar{V} \prod_{i=1}^{J_K - 1} [(v_{i+1} - v^0_{m_i}(t_i)) \cdot \nu_{i+1}] \mathbf{1}_{\{E_0(Z_{J_K}(0)) \leq \frac{E^2}{2}\}} g_0^{(J_K)}(Z^0_{J_K}(0)).
\end{aligned}
$$

Proposition 3.10. *There is a constant C depending only on β and d such that, as N goes to infinity in the scaling $N\varepsilon^{d-1}\alpha^{-1} = 1$, the following bounds hold :*

$$
\begin{aligned}
&\| f_{N,E}^{(1,K)} - f_{N,E,\delta}^{(1,K)} \|_{L^\infty([0,t]\times \mathbf{T}^d \times \mathbf{R}^d)} + \| g_{\alpha,E}^{(1,K)} - g_{\alpha,E,\delta}^{(1,K)} \|_{L^\infty([0,t]\times \mathbf{T}^d \times \mathbf{R}^d)} \\
&\qquad\qquad \leq A^{(K+1)(K+2)} (C\alpha t)^{A^{K+1}} \frac{\delta}{t} \| \rho^0 \|_{L^\infty}.
\end{aligned}
$$

PROOF. Given J, as in proposition 2.8, the error term comes from the integration over two consecutive times such that $|t_{i+1} - t_i| \leq \delta$ which has a contribution $\frac{\delta}{t} J_K$ with $J_K \leq A^{K+1}$. The conclusion of the proof follows by the same arguments as in the proof of proposition 3.9 : the sum over $\mathcal{M}_J$ is smaller than $(Ct)^{A^{K+1}} \frac{A^{K+1}\delta}{t} \|\rho^0\|_{L^\infty}$, the prefactor $\alpha^{J_{K-1}-1}$ is smaller than $\alpha^{A^{K+1}}$ and summing over all possible choices of j_k leads to an extra factor $A^{K(K+1)}$. □

3.3.4 Conclusion of the proof

Neglecting pathological pseudo-trajectories. As in the previous chapter, let us define :

$$\tilde{f}^{(1,K)}_{N,E,\delta} := \sum_J \alpha^{J_K - 1} \sum_{m \in \mathcal{M}_J} \left(\frac{\varepsilon^{d-1}}{\alpha} \right)^{J_K - 1} \frac{(N-1)!}{(N-J_K)!} \tilde{F}^{(1,K)}_{N,E,\delta}(J,m)$$

where

$$\tilde{F}^{(1,K)}_{N,E}(J,m)(t,z_1) :=$$
$$\int_{\mathcal{T}_{J,\delta}(h)} dT_J \int_{\mathcal{B}(J,T,m)^c} d\bar{v} d\bar{V} \prod_{i=1}^{J_K - 1} [(v_{i+1} - v_{m_i}(t_i)) \cdot \nu_{i+1}] \mathbf{1}_{\{E_0(Z_{J_K}(0)) \leq \frac{E^2}{2}\}} f^{(J_K)}_{N,0}(Z_{J_K}(0)).$$

Similarly :

$$\tilde{g}^{(1,K)}_{\alpha,E,\delta} := \sum_J \alpha^{J_K - 1} \sum_{m \in \mathcal{M}_J} \tilde{G}^{(1,K)}_{E,\delta}(J,m)$$

where

$$\tilde{G}^{(1,K)}_{E,\delta}(J,m)(t,z_1) :=$$
$$\int_{\mathcal{T}_{J,\delta}(h)} dT_J \int_{\mathcal{B}(J,T,m)^c} d\bar{v} d\bar{V} \prod_{i=1}^{J_K - 1} [(v_{i+1} - v^0_{m_i}(t_i)) \cdot \nu_{i+1}] \mathbf{1}_{\{E_0(Z_{J_K}(0)) \leq \frac{E^2}{2}\}} g^{(J_K)}_0(Z^0_{J_K}(0)).$$

The following proposition is a direct consequence of the results proved in section 3.3.2

Proposition 3.11. *There is a constant C depending only on β and d such that, as N goes to infinity in the scaling $N\varepsilon^{d-1}\alpha^{-1} = 1$, the following bounds hold :*

$$\|f^{(1,K)}_{N,E,\delta} - \tilde{f}^{(1,K)}_{N,E,\delta}\|_{L^\infty([0,t]\times\mathbf{T}^d\times\mathbf{R}^d)} + \|g^{(1,K)}_{\alpha,E,\delta} - \tilde{g}^{(1,K)}_{\alpha,E,\delta}\|_{L^\infty([0,t]\times\mathbf{T}^d\times\mathbf{R}^d)}$$
$$\leq A^{(K+1)(K+2)} (C\alpha t)^{A^{K+1}} \left(E^d \left(\frac{a}{\varepsilon_0} \right)^{d-1} + E^d (Et)^d \varepsilon^{d-1} + E \left(\frac{\varepsilon_0}{\delta} \right)^{d-1} \right).$$

Discrepency between $\tilde{f}^{(1,K)}_{N,E,\delta}$ and $\tilde{g}^{(1,K)}_{N,E,\delta}$. The following proposition is a simpler version of what is done in section 2.5.

Proposition 3.12. *There is a constant C depending only on β and d such that, as N goes to infinity in the scaling $N\varepsilon^{d-1}\alpha^{-1} = 1$, the following bounds hold :*

$$\|\tilde{f}^{(1,K)}_{N,E,\delta} - \tilde{g}^{(1,K)}_{N,E,\delta}\|_{L^\infty([0,t]\times\mathbf{T}^d\times\mathbf{R}^d)} \leq A^{K(K+1)} (C\alpha t)^{A^{K+1}} \left(\frac{A^{2(K+1)}}{N} + \alpha\varepsilon \right) \|\rho^0\|_{L^\infty}.$$

Proof. For a fixed J_K and in the limit $N\varepsilon^{d-1}\alpha^{-1} = 1$, the error due to the prefactors in the collision operator can be bounded by :

$$\left(1 - \frac{(N-1)\dots(N-J_K+1)}{N^{J_K+1}}\right) \leq C\frac{J_K^2}{N}.$$

Summing all contributions and following the same lines as before, it leads to an error of the form :

$$A^{K(K+1)}(C\alpha t)^{A^{K+1}}\frac{A^{2(K+1)}}{N}.$$

Besides, note that :

$$g_0^{(J_K)}(Z_{J_K}) = g_0^{(J_K)}(Z_{J_K}^0)$$

since by construction, both pseudo-trajectories have the same velocities and $x_1 = x_1^0$. And since $Z_{J_K}(0) \in \mathcal{G}_{J_K}(\varepsilon_0/2)$, proposition **??** gives :

$$\|\mathbf{1}_{\mathcal{G}_{J_K}(\varepsilon_0/2)}(f_{N,0}^{(J_K)} - g_0^{(J_K)})\|_{0,J_K,\beta} \leq \|\rho^0\|_{L^\infty} C^{J_K}\alpha\varepsilon.$$

The result follows by the continuity estimates 3.2 □

Estimate of the main term. Gathering all the previous estimates, we get

Proposition 3.13. *In the scaling :*

$$\alpha t \ll (\log\log N)^{\frac{A-1}{A}} \ \text{ and } \ K \leq \frac{\log\log N}{2\log A}$$

then, as N goes to infinity

$$\|f_N^{(!,K)} - g_\alpha^{(1,K)}\|_{L^\infty([0,t]\times\mathbf{T}^d\times\mathbf{R}^d)} \leq \|\rho^0\|_{L^\infty}\varepsilon^{\frac{d-1}{d+1}}\exp(C(\log N)^{1/2}\log\log N). \tag{3.7}$$

Proof.

$$\begin{aligned}
&\|f_N^{(1,K)} - g_\alpha^{(1,K)}\|_{L^\infty([0,t]\times\mathbf{T}^d\times\mathbf{R}^d)} \\
&\quad \leq A^{K(K+1)}(C\alpha t)^{A^{K+1}}e^{-\frac{\beta}{4}E^2}\|\rho^0\|_{L^\infty} + A^{(K+1)(K+2)}(C\alpha t)^{A^{K+1}}\frac{\delta}{t}\|\rho^0\|_{L^\infty} \\
&\qquad + A^{(K+1)(K+2)}(C\alpha t)^{A^{K+1}}\left(E^d\left(\frac{a}{\varepsilon_0}\right)^{d-1} + E^d(Et)^d\varepsilon^{d-1} + E\left(\frac{\varepsilon_0}{\delta}\right)^{d-1}\right) \\
&\qquad + A^{K(K+1)}(C\alpha t)^{A^{K+1}}\left(\frac{A^{2(K+1)}}{N} + \alpha\varepsilon\right)\|\rho^0\|_{L^\infty}
\end{aligned}$$

Choosing

$$\delta \sim \varepsilon^{\frac{d-1}{d+1}},\ \ \varepsilon_0 \sim \varepsilon^{\frac{d}{d+1}},\ \ E \sim \sqrt{|\log\varepsilon|},\ \ a = A^{K+1}\varepsilon$$

the conclusion follows since $A^K \leq \sqrt{\log N}$. □

We can now prove theorem 3.1

PROOF OF THEOREM 3.1. Gathering the results of propositions 3.5 and 3.13, we get

$$\begin{aligned}\|f_N^{(1)} - g_\alpha\|_{L^\infty([0,t]\times\mathbf{T}^d\times\mathbf{R}^d)} &\leq C(\gamma^A + C_0\varepsilon^{\frac{d-1}{d+1}}\exp(C(\log N)^{1/2}\log\log N))\|\rho^0\|_{L^\infty} \\ &\leq C\left(\frac{(\alpha t)^{A/(A-1)}}{\log\log N}\right)^A \|\rho^0\|_{L^\infty}\end{aligned}$$

where we used the relation

$$\gamma = \frac{(\alpha t)^{A/(A-1)}}{CK}$$

with the choice

$$K = \left\lfloor \frac{\log\log N}{2\log A}\right\rfloor.$$

□

3.4 Diffusion limit

In the macroscopic limit, the trajectory of the tagged particle is defined by

$$\Xi(\tau) = x_1(\alpha\tau)$$

where τ is the typical macroscopic time. In this section we show that the distribution of $\Xi(\tau)$ given by $f_N^{(1)}(\alpha\tau, x, v)$ can be approximated by the diffusion in the following sense.

Theorem 3.14. *Given N hard spheres on the space $\mathbf{T}^d\times\mathbf{R}^d$ initially distributed according to $f_{N,0}$ defined in (3.2) with ρ^0 a continuous function on $\mathbf{T}^d$. Then the distribution $f_N^{(1)}(\alpha\tau, x, v)$ satisfies :*

$$\|f_N^{(1)}(\alpha\tau, x, v) - \rho(\tau, x)M_\beta(v)\|_{L^\infty([0,T]\times\mathbf{T}^d\times\mathbf{R}^d)} \to 0$$

in the limit $N\to\infty$, with $\alpha = N\varepsilon^{d-1}$ going to infinity much slower than $\sqrt{\log\log N}$ and where $\rho(\tau, x)$ is the solution of the linear heat equation :

$$\partial_\tau\rho - \kappa_\beta\Delta\rho = 0 \ \text{ in } \mathbf{T}^d, \quad \rho_{|\tau=0} = \rho^0.$$

Thanks to theorem 3.1 it is sufficient to prove that $\varphi_\alpha(\alpha\tau, x, v)$ can be approximated by a diffusion. Indeed, we proved that for any $\tau > 0$ and any $\alpha > 1$, then in the Boltzmann-Grad limit :

$$\|f_n^{(1)}(\alpha\tau, x, v) - M_\beta(v)\varphi_\alpha(\alpha\tau, x, v)\|_{L^\infty(\mathbf{T}^d\times\mathbf{R}^d)} \leq C\left[\frac{\alpha^2\tau}{(\log\log N)^{\frac{A-1}{A}}}\right]^{\frac{A^2}{A-1}}$$

where $A\geq 2$ can be taken arbitrarily large. The limit $\alpha\to\infty$ can therefore be taken, provided that $\alpha \ll (\log\log N)^{\frac{A-1}{2A}}$. let us define

$$\tilde\varphi_\alpha(\tau, x, v) = \varphi_\alpha(\alpha\tau, x, v),$$

which satisfies

$$\partial_\tau\tilde\varphi_\alpha + \alpha v\cdot\nabla_x\tilde\varphi_\alpha + \alpha^2\mathcal{L}\tilde\varphi_\alpha = 0. \tag{3.8}$$

Theorem 3.14 then follows directly from the following result :

$$\sup_{\tau\in[0,T]} \sup_{(x,v)\in\mathbf{T}^2\times\mathbf{R}^d} |M_\beta(v)(\tilde\varphi_\alpha(\tau,x,v)-\rho(\tau,x,v))| \to 0. \tag{3.9}$$

The proof of this estimates is classical and relies on some properties of the operator $\mathcal{L}$ which can be proved by standard functional analysis (see for example [1]). The starting argument consists in writing $\tilde\varphi_\alpha$ as a formal power series in terms of α^{-1} (called *Hilbert's expansion*) :

$$\tilde\varphi_\alpha(\tau,x,v) = \tilde\rho_0(\tau,x,v) + \frac{1}{\alpha}\tilde\rho_1(\tau,x,v) + \frac{1}{\alpha^2}\tilde\rho_2(\tau,x,v) + \dots$$

Then, replacing this expansion in equation (3.8) leads to the following set of equations for the first three terms :

$$\begin{aligned} \mathcal{L}\tilde\rho_0 &= 0.\\ v\cdot\nabla_x\tilde\rho_0 + \mathcal{L}\tilde\rho_1 &= 0.\\ \partial_\tau\tilde\rho_0 + v\cdot\nabla_x\tilde\rho_1 + \mathcal{L}\tilde\rho_2 &= 0. \end{aligned} \tag{3.10}$$

In the following we will prove that $\tilde\rho_0$ is the solution to the heat equation. To do so, we need to investigate more carefully the algebraic properties of the operator $\mathcal{L}$. The proof of the following proposition can be found in a more general setting in [1].

Proposition 3.15. *For any $\beta > 0$, the operator $\mathcal{L}$ is self-adjoint on $L^2(\mathbf{R}^d, M_\beta dv)$ with domain $\mathscr{D}(\mathcal{L}) = L^2(\mathbf{R}^d,(1+|v|^2)M_\beta dv)$.* Ker $\mathcal{L}$ *reduces to a.e. constant functions and $\mathcal{L}$ is invertible on the set*

$$\left\{ g\in L^2(\mathbf{R}^d, M_\beta dv),\ \int_{\mathbf{R}^d} g(v)M_\beta(v)dv = 0\right\}.$$

PROOF. let us write

$$\mathcal{L} = a_\beta(v)Id - \mathcal{K}$$

where

$$a_\beta(v) := \int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} M_\beta(v_1)((v-v_1)\cdot\nu)_+ d\nu dv_1$$

and

$$\mathcal{K}\phi(v) := \int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} \phi(v')M_\beta(v_1)((v-v_1)\cdot\nu)_+ d\nu dv_1.$$

Step 1. *Prove that $\mathcal{K} : L^2(\mathbf{R}^d,(1+|v|^2)M_\beta dv) \subset L^2(\mathbf{R}^d, M_\beta dv) \to L^2(\mathbf{R}^d, M_\beta dv)$ is well-defined.*

It can easily be shown using Cauchy-Schwarz inequality and the fact that $|v|^2+|v_1|^2 = |v'|^2+|v_1'|^2$. Note also that the change of variables $(v,v_1)\mapsto(v',v_1')$ has unit jacobian.

Step 2. *Prove the following inequality :*

$$\forall\phi\in L^2(\mathbf{R}^d,(1+|v|^2)M_\beta dv),\ \int_{\mathbf{R}^d}\phi(v)\mathcal{L}\phi(v)M_\beta(v)dv \geq 0 \tag{3.11}$$

with equality if and only if ϕ is constant a.e.

Using the change of variables $(v, v_1, \eta) \mapsto (v', v_1', -\nu)$ and the equalities :

$$|v|^2 + |v_1|^2 = |v'|^2 + |v_1'|^2 \text{ and } (v - v_1)\cdot\nu = -(v' - v_1')\cdot\nu,$$

one has :

$$\begin{aligned}
\int_{\mathbf{R}^d} \phi(v)^2 a_\beta(v) M_\beta(v)dv &= \iint_{\mathbf{S}^{d-1}\times\mathbf{R}^d\times\mathbf{R}^d} \phi(v)^2 M_\beta(v)M_\beta(v_1)((v-v_1)\cdot\nu)_+ dvdv_1d\nu \\
&= \iiint_{\mathbf{S}^{d-1}\times\mathbf{R}^d\times\mathbf{R}^d} \phi(v')^2 M_\beta(v')M_\beta(v_1')((v'-v_1')\cdot\nu)_+ dv'dv_1'd\nu \\
&= \iiint_{\mathbf{S}^{d-1}\times\mathbf{R}^d\times\mathbf{R}^d} \phi(v)^2 M_\beta(v)M_\beta(v_1)((v'-v_1)\cdot\nu)_+ dvdv_1d\nu \\
&= \int_{\mathbf{R}^d} \phi(v)^2 a_\beta(v) M_\beta(v)dv.
\end{aligned}$$

This shows that

$$\begin{aligned}
\int_{\mathbf{R}^d} \phi(v)\mathcal{L}\phi(v)M_\beta(v)dv &= \frac{1}{2}\iiint_{\mathbf{S}^{d-1}\times\mathbf{R}^d\times\mathbf{R}^d} (\phi(v)-\phi(v'))^2 M_\beta(v)M_\beta(v_1)((v-v_1)\cdot\nu)_+ dvdv_1d\nu \\
&\geq 0
\end{aligned}$$

with equality if and only if $\phi(v) = \phi(v')$ a.e. with respect to v, v_1, ν, *i.e.* ϕ is constant a.e.

Step 3. *Prove that $\mathcal{K}$ is self-adjoint for the scalar product inherited from $L^2(\mathbf{R}^d; M_\beta dv)$*

Let ϕ_1 and ϕ_2 be two elements of $L^2(\mathbf{R}^d; (1+|v|^2)M_\beta dv)$. The same change of variables used in step 2 shows that :

$$\begin{aligned}
\langle \mathcal{K}\phi_1, \phi_2\rangle_{M_\beta} &= \int_{\mathbf{R}^d} \mathcal{K}\phi_1(v)\phi_2(v)M_\beta(v)dv \\
&= \iiint_{\mathbf{S}^{d-1}\times\mathbf{R}^d\times\mathbf{R}^d} \phi_1(v')\phi_2(v)M_\beta(v)M_\beta(v_1)((v-v_1)\cdot\nu)_+ dvdv_1d\nu \\
&= \iiint_{\mathbf{S}^{d-1}\times\mathbf{R}^d\times\mathbf{R}^d} \phi_1(v')\phi_2(v)M_\beta(v')M_\beta(v_1')((v'-v_1')\cdot\nu)_+ dv'dv_1'd\nu \\
&= \langle \phi_1, \mathcal{K}\phi_2\rangle_{M_\beta}
\end{aligned}$$

Step 4. *Conclusion.*

Gathering the previous results leads to :

$$\operatorname{Im}\mathcal{L} = \operatorname{Ker}(\mathcal{L}^*)^\perp = (\operatorname{Ker}\mathcal{L})^\perp = \left\{ g \in L^2(\mathbf{R}^d, M_\beta dv),\ \int_{\mathbf{R}^d} g(v)M_\beta(v)dv = 0 \right\}$$

since $\operatorname{Ker}\mathcal{L}$ is reduced to constant functions. □

We can return to the proof of theorem 3.14. The first equation in (3.10) together with proposition 3.15 reflects the fact that $\tilde\rho_0$ does not depend on v. Thanks to proposition 3.15 we also know that the identity function on $L^2(\mathbf{R}^d; M_\beta dv)$ has a unique pre-image by $\mathcal{L}$ in $(\operatorname{Ker}\mathcal{L})^\perp$: let us define $b(v) \in \mathbf{R}^d$ as the solution to the Poisson equation :

$$\mathcal{L}b(v) = v \quad \text{and} \quad \int_{\mathbf{R}^d} b(v)M_\beta(v)dv = 0. \tag{3.12}$$

We will need the following lemma to conclude the proof :

Lemma 3.16. *There exists a measurable function $\gamma : \mathbf{R} \to \mathbf{R}$ such that :*

$$b(v) = \gamma(|v|)v.$$

PROOF. Let $Q \in \mathcal{O}_d(\mathbf{R})$. It follows from the definition of $\mathcal{L}$ that :

$$\mathcal{L}(Qb)(v) = Q\mathcal{L}b(v) = Qv \quad \text{and} \quad \int_{\mathbf{R}^d} Qb(v)M_\beta(v) = Q\int_{\mathbf{R}^d} b(v)M_\beta(v) = 0 \tag{3.13}$$

Let $\mathcal{O}_d(\mathbf{R})_v := \{Q \in \mathcal{O}_d(\mathbf{R}),\, Qv = v\}$ be the stabilizer of v. We know that $\mathcal{O}_d(\mathbf{R})_v$ is isomorphic to the group of orthogonal matrix of the hyperplan $(\mathbf{R}v)^\perp$ and therefore $\mathcal{O}_d(\mathbf{R})_v$ acts transitively on each sphere of $(\mathbf{R}v)^\perp$ centered at the origin. Moreover, if P denotes the orthogonal projection on v, one has for all $Q \in \mathcal{O}_d(\mathbf{R})_v$:

$$QP = PQP = PQ.$$

By definition of $b(v)$ as the unique solution to the Poisson equation in $(\operatorname{Ker}\mathcal{L})^\perp$ we have, thanks to (3.13) :

$$\forall Q \in \mathcal{O}_d(\mathbf{R})_v,\ b(v) = Qb(v)$$

and since P and Q commute : for all $Q \in \mathcal{O}_d(\mathbf{R})_v$, $(I-P)b(v) = Q(I-P)b(v)$. It implies that $b(v)$ and v are collinear : indeed, if not, we can find $Q \in \mathcal{O}_d(\mathbf{R})_v$ and $0 \neq w \in (\mathbf{R}v)^\perp$ such that

$$Q(I-P)b(v) = w \neq (I-P)b(v).$$

We deduce that $b(v) = \beta(v)v$ for some function β. In order to prove that β depends only on $|v|$, let us write for all $Q \in \mathcal{O}_d(\mathbf{R})$:

$$\begin{aligned}
\mathcal{L}(b\circ Q)(v) &= a_\beta(v)b(Qv) - \int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} b(Qv')M_\beta(v_1)((v-v_1)\cdot\nu)_+ dv_1 d\nu \\
&= a_\beta(v)b(Qv) - \int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} b(Qv')M_\beta(Qv_1)((Qv-Qv_1)\cdot Q\nu)_+ dv_1 d\nu \\
&= a_\beta(v)b(Qv) - \int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} b(Qv - \nu\cdot(Qv-v_1)\nu)M_\beta(v_1)((Qv-v_1)\cdot\nu)_+ dv_1 d\nu \\
&= \mathcal{L}b(Qv) = Qv
\end{aligned}$$

where the third line follows from the change of variables $(v_1,\nu) \mapsto (Qv_1, Q\nu)$. The same argument shows that

$$\int_{\mathbf{R}^d} b(Qv)M_\beta(v)dv = \int_{\mathbf{R}^d} b(Qv)M_\beta(Qv)dv = \int_{\mathbf{R}^d} b(v)M_\beta(v)dv = 0.$$

The unicity of the solution to the Poisson equation in $(\operatorname{Ker}\mathcal{L})^\perp$ therefore shows that for all $Q \in \mathcal{O}_d(\mathbf{R})$, $b(Qv) = Qb(v)$ which implies :

$$\forall Q \in \mathcal{O}_d(\mathbf{R}),\ \beta(Qv) = \beta(v)$$

and the conclusion follows. □

We can now go back to the proof. The second equation in (3.10) allows us to write $\tilde\rho_1 = \rho_1 + \bar\rho_1$ with

$$\rho_1(\tau,x,v) := -b(v)\cdot\nabla_x\tilde\rho_0(\tau,x) \ \text{ and } \ \bar\rho_1 \in \operatorname{Ker}\mathcal{L}.$$

The last equation in (3.10) tells us that $\partial_\tau \tilde\rho_0 + v \cdot \nabla_x \tilde\rho_1$ has to belong to the range of $\mathcal{L}$, which gives :

$$\partial_\tau \tilde\rho_0 + \int_{\mathbf{R}^d} v \cdot \nabla_x \tilde\rho_1(\tau, x, v) M_\beta(v) dv = 0.$$

Replacing $\tilde\rho_1$ by $\rho_1 + \bar\rho_1$ leads to :

$$\partial_\tau \tilde\rho_0 - \sum_{i,j} \partial_{x_i} \partial_{x_j} \tilde\rho_0 \int_{\mathbf{R}^d} v_i v_j \gamma(|v|) M_\beta(v) dv + \nabla_x \bar\rho_1 \cdot \int_{\mathbf{R}^d} v M_\beta(v) dv = 0$$

and we retrieve the heat equation after simplifications : $\tilde\rho_0(\tau, x) = \rho(\tau, x)$ where

$$\partial_\tau \rho - \kappa_d \Delta \tilde\rho = 0$$

with

$$\kappa_d := \frac{1}{d} \int_{\mathbf{R}^d} v \mathcal{L}^{-1}(v) M_\beta(v) dv = \frac{1}{d} \int_{\mathbf{R}^d} |v|^2 \gamma(|v|) M_\beta(v) dv. \tag{3.14}$$

Finally, an easy computation shows that

$$\tilde\rho_2(\tau, x, v) = \rho_2(\tau, x, v) + \bar\rho_2(\tau, x) - b(v) \cdot \nabla_x \bar\rho_1(\tau, x)$$

where

$$\rho_2(\tau, x, v) := D(v) : \operatorname{Hess} \rho(\tau, x) \quad \text{and} \quad \bar\rho_2 \in \operatorname{Ker} \mathcal{L}$$

where the matrix $D(v)$ is defined by

$$\mathcal{L} D(v) := v \otimes b(v) - \int_{\mathbf{R}^d} v \otimes b(v) M_\beta(v) dv \quad \text{and} \quad \int_{\mathbf{R}^d} D(v)_{k,l} M_\beta(v) dv = 0$$

PROOF (THEOREM 3.14). To conclude the proof, we need to check that $\tilde\varphi_\alpha$ can be approximated by the first three terms of the Hilbert's series expansion, which is a consequence of the maximum principles that hold for the heat and Liouville equations. More precisely, let us define

$$\Psi_\alpha(\tau, x, v) := \rho(\tau, x, v) + \frac{1}{\alpha} \rho_1(\tau, x, v) + \frac{1}{\alpha^2} \rho_2(\tau, x, v).$$

Then Ψ_α satisfies :

$$\partial_\tau \Psi_\alpha + \alpha v \cdot \nabla_x \Psi_\alpha + \alpha^2 \mathcal{L} \Psi_\alpha = S_\alpha$$

where

$$S_\alpha(\tau, x, v) = \frac{1}{\alpha} (\partial_\tau \rho_1(\tau, x, v) + v \cdot \nabla_x \rho_2(\tau, x, v) + \frac{1}{\alpha} \partial_\tau \rho_2(\tau, x, v)).$$

Defining $R_\alpha(\tau, x, v) := \Psi_\alpha(\tau, x, v) - \tilde\varphi_\alpha(\tau, x, v)$, we have

$$\partial_\tau R_\alpha + \alpha v \cdot \nabla_x R_\alpha + \alpha^2 \mathcal{L} R_\alpha = S_\alpha$$

and the maximum principle implies :

$$\|M_\beta R_\alpha\|_{L^\infty([0,T] \times \mathbf{T}^d \times \mathbf{R}^d)} \leq C(T) (\|M_\beta R_\alpha(0)\|_{L^\infty(\mathbf{T}^d \times \mathbf{R}^d)} + \|M_\beta S_\alpha\|_{L^\infty([0,T] \times \mathbf{T}^d \times \mathbf{R}^d)}).$$

And the maximum principle for the heat equation implies that $M_\beta S_\alpha$ is bounded in L^∞ norm by α^{-1} provided that ρ is sufficiently smooth (up to regularizing ρ^0). It follows that

$$\|M_\beta (\Psi_\alpha - \tilde\varphi_\alpha)\|_{L^\infty([0,T] \times \mathbf{T}^d \times \mathbf{R}^d)} \leq \frac{C(T)}{\alpha}$$

□

3.5 Convergence to the Brownian motion

In the macroscopic limit, it is even possible to adapt theorem 3.1 to prove that the process describing the motion of the tagged particle defined by

$$\Xi(\tau) = x_1(\alpha\tau)$$

converges to a Brownian motion. To do so, one needs to check :

- the convergence of the finite-dimensional marginals sampled at different times :

$$\lim_{N\to+\infty} \mathbb{E}_N\Big(h_1\big(\Xi(\tau_1)\big)\dots h_\ell\big(\Xi(\tau_\ell)\big)\Big) = \mathbb{E}\Big(h_1\big(B(\tau_1)\big)\dots h_\ell\big(B(\tau_\ell)\big)\Big), \tag{3.15}$$

  where $\{h_1,\dots,h_\ell\}$ is a family of bounded continuous functions in $\mathbf{T}^d$ and B is a brownian motion.

- the tightness of the sequence, that is for any $\tau \in [0,T]$:

$$\forall \xi > 0,\ \ \lim_{\eta\to 0}\lim_{N\to+\infty} \mathbb{P}_N\left(\sup_{\tau<\sigma<\tau+\eta} |\Xi(\sigma)-\Xi(\tau)| \geq \xi\right) = 0$$

where $\mathbb{E}_N$ and $\mathbb{P}_N$ refer to the expectation and probability with respect to the initial data, that is to say that

$$\mathbb{E}_N\Big(h\big(\Xi(t)\big)\Big) := \int_{\mathbf{T}^{Nd}\times\mathbf{R}^{Nd}} dZ_N f_N^0(Z_N) h\big(\Xi(t)\big).$$

In order to prove the convergence (3.15), we are going to re-write the expectations in term of collision trees and then adapt the proof of theorem 3.1. First, let us consider an increasing collection of times $t_1 < \dots < t_\ell$ and $H_\ell = \{h_1,\dots,h_\ell\}$ a collection of bounded continuous functions on $\mathbf{T}^d$. Thanks to Radon-Nikodym theorem, we can define a biased distribution at time $t > t_\ell$ as follows :

$$\begin{aligned}\int_{\mathbf{T}^{Nd}\times\mathbf{R}^{Nd}} dZ_N f_{N,H_\ell}(t,Z_N)\Phi(Z_N) &:= \mathbb{E}_N\Big(h_1\big(x_1(t_1)\big)\dots h_\ell\big(x_1(t_\ell)\big)\Phi\big(Z_N(t)\big)\Big)\\ &= \int_{\mathbf{T}^{Nd}\times\mathbf{R}^{Nd}} dZ_N f_N^0(Z_N) h_1\big(x_1(t_1)\big)\dots h_\ell\big(x_1(t_\ell)\big)\Phi\big(Z_N(t)\big)\end{aligned}$$

for any test function Φ. By construction, f_{N,H_ℓ} satisfies the Liouville equation for $t > t_\ell$ and its marginals obey the BBGKY hierarchy :

$$f^{(s)}_{N,H_\ell}(t,Z_s) = \sum_{m=0}^{N-s} Q_{s,s+m}(t-t_\ell) f^{(s+m)}_{N,H_\ell}(t_\ell).$$

But, since $f_{N,H_\ell}(t_\ell,Z_N) = f_{N,H_{\ell-1}}(t_\ell,Z_N)h_\ell(z_1)$, we can write the first marginal $f^{(1)}_{N,H_\ell}$ in terms of a weighted collision tree :

$$f^{(1)}_{N,H_\ell}(t) = \sum_{m_1+\dots+m_\ell=0}^{N-1} Q_{1,1+m_1}(t-t_\ell)\Big(h_\ell Q_{1+m_1,1+m_2}(t_\ell-t_{\ell-1})\Big(h_{\ell-1}\dots$$
$$Q_{1+m_1+\dots+m_{\ell-1},1+m_1+\dots+m_\ell}(t_1)\Big)f_N^{(1+m_1+\dots+m_\ell)}(0). \tag{3.16}$$

In the Boltzmann-Grad limit and with the appropriate time scaling, the behaviour of the tagged particle can be approximated by a Markov process which converges to a Brownian motion. Let us define the limit process as follows :

- $\{\widetilde{v}_1(t),\ t \geq 0\}$ is a Markov process with generator[2] $-\mathcal{L}$, initially distributed according to $M_\beta(v)$.
- The rescaled process $\overline{v}_1(\tau) = \widetilde{v}_1(\alpha\tau)$ is a Markov process with generator $-\alpha\mathcal{L}$ initially distributed according to $M_\beta(v)$.
- In the macroscopic limit, the behaviour of the tagged particle $\Xi(\tau)$ is equivalent to the one of $\overline{x}_1(\alpha\tau)$ where $\overline{x}_1(t)$ is defined as an additive functional of the Markov process $\overline{v}_1(t)$

$$\overline{x}_1(t) := \overline{x}_1(0) + \int_0^t \overline{v}_1(s)ds$$

where $(\overline{x}_1(0), \overline{v}_1(0))$ is initially distributed according to $\rho^0(x)M_\beta(v)$. The expectation associated to the Markov process $\overline{z}_1(t) = (\overline{x}_1(t), \overline{v}_1(t))$ is denoted by $\mathbb{E}_{M_\beta}$.

A biased distribution g_{α,H_ℓ} of this Markov process can be defined as before as :

$$\int_{\mathbf{T}^d\times\mathbf{R}^d} g_{\alpha,H_\ell}(t,z)\Phi(z)dz := \mathbb{E}_{M_\beta}\Big(h_1\big(\overline{x}_1(t_1)\big)\dots h_\ell\big(\overline{x}_1(t_\ell)\big)\Phi\big(\overline{z}_1(t)\big)\Big)$$

for any test function Φ. The marginals of this measure are

$$g^{(s)}_{\alpha,H_\ell}(t, Z_s) := g_{\alpha,H_\ell}(t, z_1)\prod_{i=2}^{s} M_\beta(v_i).$$

Note that by construction, $g_{\alpha,H_\ell}(t,z)M_\beta(v)^{-1}$ satisfies the linear Boltzmann equation and then, as in (3.16) :

$$g_{\alpha,H_\ell}(t) = \sum_{m_1+\ldots+m_\ell=0}^{\infty} Q^0_{1,1+m_1}(t-t_\ell)\Big(h_\ell Q^0_{1+m_1,1+m_2}(t_\ell - t_{\ell-1})\Big(h_{\ell-1}\dots Q^0_{1+m_1+\ldots+m_{\ell-1},1+m_1+\ldots+m_\ell}(t_1)\Big)g^{(1+m_1+\ldots+m_\ell)}_{\alpha,H_\ell}(0). \quad (3.17)$$

Suppose now that the collection H_ℓ satisfies the uniform bounds on $\mathbf{T}^d$:

$$\forall i \leq \ell,\ \ 0 \leq h_i(x_1) \leq m$$

then the marginals $f^{(s)}_{N,H_\ell}$ satisfy the maximum principle 3.3 with an extra factor m^ℓ and it is therefore possible to compare $f^{(1)}_{N,H_\ell}$ and g_{α,H_ℓ} in the same way as before. The conclusion of theorem 3.1 leads to the following bounds :

$$\|f^{(1)}_{N,H_\ell}(\alpha\tau,x,v) - g_{\alpha,H_\ell}(\alpha\tau,x,v)\|_{L^\infty(\mathbf{T}^d\times\mathbf{R}^d)} \leq Cm^\ell\left[\frac{\tau\alpha^2}{(\log\log N)^{\frac{A-1}{A}}}\right]^{\frac{A^2}{A-1}}$$

which implies the convergence :

$$\lim_{\alpha\to+\infty}\mathbb{E}_{M_\beta}\Big(h_1\big(\overline{x}_1(\alpha\tau_1)\big)\dots h_\ell\big(\overline{x}_1(\alpha\tau_\ell)\big)\Big) - \mathbb{E}_N\Big(h_1\big(x_1(\alpha\tau_1)\big)\dots h_\ell\big(x_1(\alpha\tau_\ell)\big)\Big) = 0. \quad (3.18)$$

[2]Note that the proof of proposition 3.15 also shows that $-\mathcal{L}$ satisfies all the hypotheses of the Hille-Yosida and Lumer-Phillips theorems C.9 and C.10

Finally, (3.15) follows from (3.18) and a central limit theorem[3] for additive functionals that states that :

$$\frac{1}{\alpha}\int_0^{\alpha^2\tau} \widetilde{v}_1(s)ds = \int_0^{\alpha\tau} \overline{v}_1(s)ds$$

converges in distribution to a Brownian motion of zero mean and variance given by (3.14), which implies that :

$$\lim_{\alpha\to+\infty} \mathbb{E}_{M_\beta}\Big(h_1\big(\overline{x}_1(\alpha\tau_1)\big)\dots h_\ell\big(\overline{x}_1(\alpha\tau_\ell)\big)\Big) = \mathbb{E}_N\Big(h_1\big(B(\tau_1)\big)\dots h_\ell\big(B(\tau_\ell)\big)\Big).$$

[3] See [6] theorem 2.32 for a precise statement. The hypotheses are verified by the proof of proposition 3.15.

Appendix A

Proof of lemma 1.4

We want to show that the change of variables

$$\Phi^{-,i}: \left| \begin{array}{rcl} \mathcal{D}_s \times \mathbf{R}^d \times [0,\delta] \times \mathbf{S}_1^{d-1} & \longrightarrow & \mathbf{R}^{2d(s+1)} \\ (Z_s, v_{s+1}, t, \omega) & \longmapsto & Z_{s+1} := (X_s - tV_s, V_s, x_i + \varepsilon\omega - tv_{s+1}, v_{s+1}) \end{array} \right.$$

maps the measure

$$d\mu_i^- := [\omega \cdot (v_{s+1} - v_i)]_- dZ_s dt d\omega dv_{s+1}$$

on the Lebesgue measure dZ_{s+1}. The idea is to use the d-dimensional spherical coordinates to compute explicitly the jacobian. If $\omega \in \mathbf{S}^{d-1}$, one can write :

$$\omega = \phi(\theta_1, \ldots, \theta_{d-1}) = \begin{pmatrix} \cos(\theta_1)\cos(\theta_2)\ldots\cos(\theta_{d-1}) \\ \sin(\theta_1)\cos(\theta_2)\ldots\cos(\theta_{d-1}) \\ \sin(\theta_2)\cos(\theta_3)\ldots\cos(\theta_{d-1}) \\ \ldots \\ \sin(\theta_{d-2})\cos(\theta_{d-1}) \\ \sin(\theta_{d-1}) \end{pmatrix}$$

$$(\theta_1, \theta_2, \ldots, \theta_{d-1}) \in [0, 2\pi) \times (-\pi/2, \pi/2)^{d-2}.$$

So that :

$$\int_{\mathbf{S}^{d-1}} u(\omega) d\omega = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} u(\phi(\theta_1, \ldots, \theta_{d-1})) |J| d\theta_1 \ldots d\theta_{d-1}$$

where

$$|J| = \cos(\theta_2)\cos(\theta_3)^2 \ldots \cos(\theta_{d-1})^{d-2}.$$

With a slight abuse of notations, we will write

$$Z_s = (x_1, x_2, \ldots, x_s, v_1, \ldots, v_s).$$

And the jacobian determinant of $\Phi^{-,i}$ is therefore:

$$
d_\Phi = |\det| \left(\begin{array}{ccc|c|cc|c|c|c|c}
I_d & & & & & & & & -v_1 & 0 \\
 & I_d & & & & & & & -v_2 & \vdots \\
 & & \ddots & & & & -tI_{ds} & & \vdots & \vdots \\
 & & & I_d & & & & & \vdots & \vdots \\
 & & & & \ddots & & & & \vdots & \vdots \\
 & & & & & I_d & & & -v_s & 0 \\
\hline
 & & & & & & & & 0 & 0 \\
 & & & & & & I_{ds} & & \vdots & \vdots \\
 & & & & & & & & 0 & 0 \\
\hline
 & & & I_d & & & & -tI_d & -v_{s+1} & M \\
\hline
 & & & & & & & I_d & 0 & 0
\end{array}\right)
$$

where M is the jacobian matrix of ϕ :

$$
M = \begin{pmatrix}
-s_1c_2\dots c_{d-1} & -c_1s_2c_3\dots c_{d-1} & & -c_1c_2\dots s_{d-1} \\
c_1c_2\dots c_{d-1} & s_1s_2c_3\dots c_{d-1} & & -s_1c_s\dots s_{d-1} \\
0 & c_2c_3\dots c_{d-1} & \dots & -s_2c_3\dots s_{d-1} \\
\vdots & 0 & & \vdots \\
\vdots & \vdots & & \vdots \\
\vdots & \vdots & & -s_{d-2}s_{d-1} \\
0 & 0 & & c_{d-1}
\end{pmatrix}
$$

with $c_k := \cos(\theta_k)$ and $s_k := \sin(\theta_k)$. The size of M is $d \times (d-1)$.

To compute d_Φ, we write

$$
d_\phi = |\det| \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)
$$

where A, B, C, D are square matrices of size $d(s+1)$. Since A is invertible and commutes with C, one an write

$$
d_\Phi = |\det(AD - BC)|
$$

and a complicated calculation shows that

$$
d_\Phi = |\det| \begin{pmatrix}
v_i^1 - v_{s+1}^1 & -s_1c_2\dots c_{d-1} & -c_1s_2c_3\dots c_{d-1} & & -c_1c_2\dots s_{d-1} \\
v_i^2 - v_{s+1}^2 & -c_1c_2\dots c_{d-1} & s_1s_2c_3\dots c_{d-1} & & -s_1c_s\dots s_{d-1} \\
\vdots & 0 & c_2c_3\dots c_{d-1} & \dots & -s_2c_3\dots s_{d-1} \\
\vdots & \vdots & 0 & & \vdots \\
\vdots & \vdots & & \vdots & \\
\vdots & \vdots & \vdots & & -s_{d-2}s_{d-1} \\
v_i^d - v_{s+1}^d & 0 & 0 & & c_{d-1}
\end{pmatrix}
$$

where v_i^k is the k-th coordinate of $v_i \in \mathbf{R}^d$. Finally we can prove by induction that

$$d_\Phi = |(v_i - v_{s+1}) \cdot \omega||J|$$

and the conclusion follows.

Appendix B

About spaces of probability measures and the Hewitt-Savage theorem

This appendix takes the results of [7].

Let E be a polish space (*i.e.* a separable completely metrizable topological space) and $P(E)$ the space of probability measures on E. A sequence $(\rho_n)_n$ in $P(E)$ is said to *converge weakly* to $\rho \in P(E)$ as $n \to +\infty$ when for all $\varphi \in C_b(E)$, one has $\langle \rho_n, \varphi \rangle \to \langle \rho, \varphi \rangle$ where

$$\langle \rho, \varphi \rangle = \int_E \varphi \, d\rho.$$

Definition B.1 (Lévy-Prokhorov metric)**.** *The Lévy-Prokhorov distance on $P(E)$ is defined by*

$$D_{LP}(\rho_1, \rho_2) = \inf \{\varepsilon > 0 : \rho_1(A) \leq \rho_2(A^\varepsilon) + \varepsilon, \text{ and } \rho_2(A) \leq \rho_1(A^\varepsilon) + \varepsilon \text{ for all } A \in \mathscr{B}_E\}$$

where $\mathscr{B}_E$ is the σ-algebra of Borel sets of (E, d) and

$$A^\varepsilon = \{x, \ d(x, A) < \varepsilon\}.$$

Theorem B.2. *$(P(E), D_{LP})$ is a polish space and D_{LP} is a metrization of the topology of weak convergence on $P(E)$, i.e :*

$$D_{LP}(\rho_n, \rho) \to 0 \quad \text{if and only if} \quad \forall \varphi \in C_b(E), \ \int_E \varphi \, d\rho_n \to \int_E \varphi \, d\rho.$$

Remark B.3. *There exist other metrizations of the weak convergence on $P(E)$, for instance the Monge-Kantorovich-Wasserstein and Zolotarev distances.*

Proposition B.4. *The two following σ-algebra on $P(E)$ are identical :*

(i) The σ-algebra of Borel sets $\mathscr{B}_{P(E)}$ associated to the Lévy-Prokhorov distance D_{LP}.

(ii) The σ-algebra formed from the sets

$$C_{A,\lambda} := \{\rho \in P(E), \ \rho(A) < \lambda\} \quad \text{or} \quad C'_{A,\lambda} := \{\rho \in P(E), \ \rho(A) \leq \lambda\}$$

with $A \in \mathscr{B}_E$ and $\lambda \in [0, 1]$.

Definition B.5. *Let E be a polish space. E^∞ denotes the space $E^{\mathbb{N}}$ of infinite sequences on E.*

- *A sequence* $(\pi_k)_{k\geq 1}$ *of measures of* $P(E^k)$ *is said to be compatible when :*

$$\forall k \leq N,\ \ \Pi_{N,k}\pi_N = \pi_k$$

  where $\Pi_{N,k}\pi_N = \Pi_k\pi_N$ *is the* k*-th marginal of* π_N *defined by*

$$\forall A \in \mathscr{B}_{E^k},\ \ (\Pi_k\pi_N)(A) = \pi_N(A \times E^{N-k}).$$

- *A measure* $m \in P(E^N)$*,* $N \in \mathbb{N}^*$ *is said to be symmetric when*

$$\forall A = A_1 \times \ldots \times A_N \in \mathscr{B}_E^{\otimes N}, \forall \sigma \in \mathfrak{S}_N,\ \ m(A_\sigma) = m(A)$$

  where $A_\sigma = A_{\sigma 1} \times \ldots \times A_{\sigma(N)}$*.* $P_{sym}(E^N)$ *will denote the space of symmetric measures on* E^N*.*

- *A measure* $m \in P(E^\infty)$ *is said to be symmetric when all its marginals are symmetric.* $P_{sym}(E^\infty)$ *will denote the space of symmetric measures on* E^∞*.*

- *A measure* $m \in P(E^\infty)$ *is said to be a product measure when there exists a measure* $\mu \in P(E)$ *such that for all* $N \in \mathbb{N}^*$*,*

$$\forall A = A_1 \times \ldots \times A_N \in \mathscr{B}_E^{\otimes N},\ \ m(C_A) = \prod_{j=1}^{N} \mu(A_j)$$

  where $C_A = A \times E \times E \times \ldots \in \mathscr{B}_{E^\infty}$ *is the cylinder of basis* A*.* $\tilde{P}(E^\infty)$ *will denote the space of product measures on* E^∞*. Note that* $\tilde{P}(E^\infty) \subset P_{sym}(E)$*.*

Theorem B.6 (Hewitt-Savage)**.** *Let* E *be a locally compact polish space. Let* $(\pi_k)_{k\geq 1}$ *be a sequence of measures of* $P(E^k)$*. The following assertions are equivalent :*

(i) $(\pi_k)_k$ *is symmetric and compatible.*

(ii) There exists $\pi \in P_{sym}(E^\infty)$ *such that for all* $k \geq 1$*,* $\pi_k = \Pi_k\pi$*.*

(iii) There exists $\hat{\pi} \in P(P(E))$ *such that for all* $k \geq 1$ *:*

$$\pi_k = \int_{P(E)} \rho^{\otimes k} d\hat{\pi}(\rho).$$

(iv) There exists $\tilde{\pi} \in P(\tilde{P}(E^\infty))$ *such that for all* $k \geq 1$ *:*

$$\pi_k = \int_{\tilde{P}(E^\infty)} (\Pi_k\alpha) d\tilde{\pi}(\alpha).$$

Proof (Idea). The equivalence between (i) and (ii) is given by Kolmogorov's extension theorem, the implication $(iii) \Rightarrow (i)$ is trivial and the equivalence $(iii) \Leftrightarrow (iv)$ relies on some more standard arguments of measure theory. There are two ways of finishing the proof :

1. One can prove the implication $(ii) \Rightarrow (iii)$ using Stone-Weierstrass theorem with the family of *polynoms* :

$$R_\varphi : P(E) \to \mathbf{R},\ m \mapsto R_\varphi(m) := \int_{E^k} \varphi(x_1, \ldots, x_k) m(dx_1) \ldots m(dx_k)$$

   with $\varphi \in C_b(E^k)$ and $k \in \mathbb{N}^*$.

2. One can use the Krein-Milman theorem to prove the implication $(ii) \Rightarrow (iv)$.

□

Remark B.7. *The integrals in* (iii) *and* (iv) *have to be understood in the following sense :*

$$\forall \varphi \in C_b(E^k), \ \ \langle \pi_k, \varphi \rangle = \int_{P(E)} \langle \rho^{\otimes k}, \varphi \rangle d\hat{\pi}(\rho)$$

where the function $\rho \mapsto \langle \rho^{\otimes k}, \varphi \rangle$ *is measurable (in fact it is even continuous for the topology previously defined) so that the Lebesgue integral is well-defined.*

Remark B.8. *When* $E = \mathbf{R}^d$ *and when* $(\pi_k)_k$ *is a sequence of probability density functions, one can prove using Radon-Nikodym theorem that* $\operatorname{Supp} \hat{\pi}$ *is a subset of the space of probability density functions. Indeed, let* $A \subset \mathbf{R}^d$ *be a subset of zero Lebesgue measure. We can write for the first marginal :*

$$0 = \langle \pi_1, \mathbf{1}_A \rangle = \int_{P(\mathbf{R}^d)} \langle \rho, \mathbf{1}_A \rangle d\hat{\pi}(\rho)$$

where $\rho \mapsto \langle \rho, \mathbf{1}_A \rangle = \rho(A)$ *is a non-negative continuous function for the Lévy-Prokhorov distance. We deduce that for all* ρ *in* $\operatorname{Supp} \hat{\pi}$, $\langle \rho, \mathbf{1}_A \rangle = 0$, *that is to say that* ρ *is absolutely continuous with respect to the Lebesgue measure and the conclusion follows.*

Appendix C

About Markov processes and their generators

This appendix sums up elementary results that can be found in [10] and [5].

Definition C.1 (Transition probability). *Let $(E, \mathscr{E})$ be a measurable space. A transition probability π is a map from $E \times \mathscr{E}$ into $\mathbf{R}_+ \cup \{+\infty\}$ such that :*

1. *for every $x \in E$, the map $A \mapsto \pi(x, A)$ is a positive measure on $\mathscr{E}$*
2. *for every $A \in \mathscr{E}$, the map $x \mapsto \pi(x, A)$ is $\mathscr{E}$-measurable*
3. *for every $x \in E$, $\pi(x, E) = 1$*

If f is a positive $\mathscr{E}$-measurable function, we define the function πf on E by

$$\pi f(x) = \int_E \pi(x, dy) f(y).$$

Definition C.2 (Homogeneous transition function). *A transition function on $(E, \mathscr{E})$ is a family $P_{s,t}$, $0 \leq s < t$ of transition probabilities on $(E, \mathscr{E})$ such that for every $s < t < v$, the Chapman-Kolmogorov equation holds :*

$$\forall x \in E,\ \forall A \in \mathscr{E},\quad \int P_{s,t}(x, dy) P_{t,v}(y, A) = P_{s,v}(x, A).$$

The transition function is said to be homogeneous if $P_{s,t}$ depends on s and t only through the difference $t - s$. In this case, we write P_t for $P_{0,t}$. The Chapman-Kolmogorov equation then says that the family $\{P_t,\ t \geq 0\}$ forms a semi-group of operators which acts on positive $\mathscr{E}$-measurable functions.

Definition C.3 (Markov process). *Let $(\Omega, \mathscr{F}, (\mathscr{G}_t), Q)$ a filtered probability space. A time homogeneous Markov process with respect to $(\mathscr{G}_t)$ with transition function P_t is a process $(X_t)_{t \geq 0}$ that satisfies :*

$$\mathbb{E}[f(X_t) | \mathscr{G}_s] = P_{t-s} f(X_s)$$

for any positive $\mathscr{E}$-measurable function f and any pair $s < t$.

Assume that initially X_0 is distributed according to a measure ν. The expectation associated to a time homogeneous Markov process with initial distribution ν is defined by

$$\mathbb{E}_\nu[f(X_t)] := \int_E \nu(dx) P_t f(x).$$

Definition C.4 (Feller semi-group). *A Feller semi-group on $C(E)$ is a family T_t, $t \geq 0$ of linear operators on $C(E)$ such that* *(i)* $T_0 = Id$ *and* $\|T_t\| \leq 1$ *for every* t

(ii) $T_{t+s} = T_t \circ T_s$ *for any pair* $s, t \geq 0$

(iii) $\lim_{t\downarrow 0} \|T_t f - f\| = 0$ *for every* $f \in C(E)$

More generally, when $\{T_t,\ t \geq 0\}$ is a family of linear operators on a Banach space X satisfying (i), (ii), (iii), we say that $\{T_t,\ t \geq 0\}$ is a strongly continuous contraction semi-group.

Proposition C.5. *With each Feller semi-group on* E*, one can associate a unique homogeneous transition function* P_t*,* $t \geq 0$ *on* $(E, \mathscr{E})$ *such that*

$$T_t f(x) = P_t f(x)$$

for every $f \in C(E)$ *and every* x *in* E*.*

PROOF. Since for any $x \in E$, the map $f \mapsto T_t f(x)$ is a positive linear form on $C(E)$, the existence of P_t follows from Riesz's theorem. The fact that P_t is a transition function is a consequence of the semigroup property of T_t and of the monotone class theorem. □

Definition C.6 (Feller process)**.** *A transition function associated to a Feller semigroup is called a Feller transition function. A Markov process having a Feller transition function is called a Feller process.*

Definition C.7 (Infinitesimal generator)**.** *Let* X_t*,* $t \geq 0$ *be a Feller process. The infinitesimal generator of* X_t *is the operator* $A : \mathscr{D}(A) \subset C(E) \to C(E)$ *defined by*

$$\mathscr{D}(A) = \left\{ x \in E,\ \lim_{t\downarrow 0} \frac{1}{t}(P_t f - f) \ \textit{exists} \right\}$$

$$\forall f \in \mathscr{D}(A),\ \ Af = \lim_{t\downarrow 0} \frac{1}{t}(P_t f - f)$$

If $f \in \mathscr{D}(A)$. we may also write :

$$Af(X_t) = \lim_{h\downarrow 0} \frac{1}{h}\mathbb{E}\Big[f(X_{t+h}) - f(X_t)\,\big|\,\mathscr{G}_t\Big].$$

Proposition C.8. *If* $f \in \mathscr{D}(A)$*, then*

(i) $P_t f \in \mathscr{D}(A)$ *for every* t

(ii) the function $t \mapsto P_t f$ *is differentiable and*

$$\frac{d}{dt}P_t f = AP_t f = P_t A f.$$

Given a linear operator A on $C(E)$ it could be useful to know whether A is the infinitesimal generator of a Feller semigroup. The answer is generally given by the Hille-Yosida theorem (see [8]).

Theorem C.9 (Hille-Yosida)**.** *Let* A *be a closed linear operator defined on a linear subspace* $\mathscr{D}(A)$ *of a Banach space* X*. Then* A *is the infinitesimal generator of a unique strongly continuous contraction semigroup if and only if :*

1. *$\mathscr{D}(A)$ is dense in X*
2. *Every real $\lambda > 0$ belongs to the resolvant set of A and for such λ*

$$\|(\lambda Id - A)^{-1}\| \leq \frac{1}{\lambda}.$$

If X is a Hilbert space, the Lumer-Phillips theorem is a special case of the previous one, as explained in [8].

Theorem C.10 (Lumer-Phillips)**.** *Assume that X is a Hilbert space. Let A be a linear operator with dense domain $\mathscr{D}(A)$ in X. Then A is the inifnitesimal generator of a unique strongly continuous semigroup of contractions if and only of A is dissipative i.e. :*

$$\forall x \in X, \;\; \langle Ax, x\rangle \leq 0.$$

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