Ride sharing platform Vs Taxi platform: the impact on the revenue

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Abstract. It is generally accepted that ride sharing, which consists in sharing a car with other people towards a specific destination, may significantly impact the traffic in major cities. A model has been introduced in [2] to study what would happen in a society in case of the introduction of a ride sharing platform. The authors used game theory, and more specifically the tools for equilibrium analysis (on an anonymous non-atomic game) to predict a long term behavior of the society. In that model, the population is represented as a group of agents identified by two parameters: their utility for using private transportation and their rate of income when they are working. Each agent may share rides for a given price (chosen by the platform), that the rider(s) pay to the driver. The behavior of each agent is defined by the strategy he chooses, like being a driver or a user, with respect to the ride sharing platform. Then, one can predict the behavior of the population in the case, for example, of a profit maximizing platform (that chooses the rental price so that its revenue is maximum). The aim of this paper is to study an extension of that model in which the users may choose between the ride sharing platform and a taxi platform according to the price they would pay. The main issue was to study the evolution of the revenue of the ride sharing platform with the introduction of the taxi platform.

Keywords: Ride sharing · Game theory · Equilibrium analysis

1 Introduction

Ride sharing refers to the habit for several people to share a car for a specific journey. The development of new technologies, specifically location services (such as GPS) and reputation system, has enabled this practice to become feasible even among strangers. For instance, people could enter their trip specifications (origin, destination, whether they need a ride or offer one) on an online platform (that will be called in the following the ride sharing platform) and get their expectations met. In parallel, these developments in technologies have also boosted the use of taxis, with the emergence of companies like Uber that allow to get a ride through an online platform (that will be called the taxi platform) with professional drivers.

With two platforms at their disposal, customers looking for a ride have more than one alternative to the use of public transportation or of one's own car. Therefore, it seems that enabling people to use taxis could reduce the use of ride sharing. However, this reduction highly depends on the rental prices set by the ride sharing and the taxi platforms. Indeed, if the price payed for the use of taxis is higher than the one payed for ride sharing, then the customers will, first, try to use the ride sharing platform. Therefore, it is possible that the opportunity of getting a ride through the taxi platform may act as an incentive not to own a car and to try to get a ride through these platforms. On the other hand, if the rental price of the taxi platform is lower than the rental price of the ride sharing platform, the users will probably be driven away from the ride sharing platform to the taxi platform. This paper aims to study these different possibilities.

In this paper, we study the competition between a ride sharing platform and a taxi platform. We use the model presented in [2] and extend it by adding the possibility for users to choose between a ride sharing platform and a taxi platform, depending on the price payed by users for the use of these two platforms. We consider a (heterogeneous) population of agents identified by the income the can get by working and their utility for private transportation (that is, not the public transport). Each agent can offer seats through the ride sharing platform (on a casual or a permanent basis) or try to get a seat through one of the two platforms. The choice made by each agent is assumed to be rational and utility maximizing. The whole set of choices of agents is modeled as an anonymous non-atomic game (that is a game in which we do not consider any specific individual but only fractions of the population). In the original game, the possible (Nash) equilibria that could be observed depended on the rental price (of the ride sharing platform), the cost of ownership and of usage (of a car), the number of available seats per car (supposed to be the same for every car) and the duration of a trip (supposed to be the same for every trip). For each value of these parameters, exactly one (modulo equivalence) equilibria could be observed. Overall, seven different type equilibria could occur.

In the new model, the rental price of the taxi platform and the number of taxis are two new parameters that also determine the outcome of the game. It has to be noted that the two situations where the rental price of the ride sharing platform is above the rental price of the taxi platform and when it is below yield two different games because the preference choice of each user changes.

We studied that new situation with a different point of view than the one used in the initial game. We did not establish clear results about the precise kind of equilibria that could happen. On the other hand, we studied more specifically the revenue of the ride sharing platform. We were able to show some properties verified by that revenue. For instance, we found some conditions on which the revenue could not increase by adding a taxi platform. We also used a numerical approach to apprehend the behavior of the population we could expect. For example, we were able to find some set of values for our parameters for which the revenue of the ride sharing platform could increase with the taxis. We could also look at the evolution of the prices if we allowed the two platforms to change their rental prices in order to maximize their revenue. However, our study is not entirely finished and there are still some unknown behavior of the population. These issues are discussed in the concluding remarks.

In section 2, we examine some related papers. In section 3, we present our model and the type of equilibria we consider. In section 4, we establish several theoretical results about the effect of introducing a taxi platform on the revenue of the ride sharing platform. In section 5, we study numerically some scenarios that could not be theoretically predicted. Finally, in section 6, we conclude with some remarks on our results. Additional proofs may be found in the appendix.

2 Related Literature

Shared mobility is a field that is increasingly dealt with in the literature. Some of that literature focus on the loan of a fleet of vehicle for a short-term period. For instance, [5, 3] concentrate on the optimization of the size of the fleet, or of the points of departure and arrival.

On the other hand, some papers consider the possibility for private owners to rent their car when they are not using them (that is a peer-to-peer system). For example, in [4], the authors examine how a car-producer should design its vehicle, in particular in terms of price and quality, if they were to be shared.

The notion of ride sharing, like considered in this paper, is not a very developed subject in literature. However, some papers do exist. For instance, [10, 1] look at ride sharing from a different point of view than ours since the authors discuss about an optimization algorithm to efficiently match demand and supply.

There are several papers that deal with taxis, and more generally with the impact of their introduction on society. For instance, in [9], the authors study the effect taxis have on the use of public transport, or how it changes the global distance traveled in vehicles in a city. However, to our knowledge, the question of the impact of taxis on a ride sharing platform has not been studied yet. More specifically, the topic of the evolution of the revenue of a ride sharing platform when competition appears (and the possibility that the revenue might increase) has not been studied yet (to our knowledge).

What we use to model a collective decision making is based on [11, 8]. It deals with anonymous games (with a continuum of players) in which we have guarantees about the existence of equilibria. Moreover, considering a continuum of players allows to avoid some inconvenience in the rules of the game, like the order to play [6]. In addition, any equilibrium we can get in this game is analogous to a ϵ-Nash equilibrium (an approximation of a Nash equilibrium) for a game with a finite number of players (games with a large number of players are discussed in [7]). However, we have no result about the uniqueness of equilibria.

Finally, this paper is mainly based on [2], since we study an extension of the model presented in that paper.

3 The Model

We present here the model we will adopt throughout this paper. This presentation is very similar to the one that can be found in [2]. However, there are some differences in the model considered, since we now also take into account a taxi platform.

3.1 Agents' possibilities

We model a population of individuals who alternate between a "transport" state and a "non-transport" state. Being in the transport state for an individual means that she is seeking a means of transportation. In terms of topology, our model is very simple since we assume we have, for every individual, only one pair origin-destination. An individual in the non-transport state can do whatever activity he wants. However, in our model we suppose that this activity generates an income. The mean duration of the transport state is $1/\lambda_t$ (in some units of time), and the mean duration of the "non-transport" state is $1/\lambda_n$. We assume that $1/\lambda_t + 1/\lambda_n = 1$, that is an individual go through both state in one unit of time (on average). This implies that $\lambda_t, \lambda_n > 1$, and that fixing one parameter, also fixes the other. λ_t is considered as the free parameter.

The transport state There are several means of transportation for each individual:

Public transport: An individual can always opt for public transport since it is supposed to be always available.

Drive: An individual may choose to drive her car (provided that she owns one), she can offer the remaining seats through the ride sharing platform.

Ride share: An individual may want to get a ride through the ride sharing platform. The probability p of getting a ride depends on the supply and demand of the platform. With a probability $1 - p$, she does not get a ride and has to use one of three other means of transportation.

Taxi: An individual may try to get a taxi. The probability of getting a ride through the taxi platform p_t only depends on the demand in this platform. Like for the ride sharing platform, with a probability $1 - p_t$, she does not get a ride and has to use one of three other means of transportation.

We assume that everyone has to satisfy his needs of transportation. An individual has utility $\rho > 0$ (a characteristic that is heterogeneous between individuals) for using private transportation (that is, not public transportation) for any personal trip (that is, performed during the transport state). We assume that the utility for using public transport is null. By assumption, every individual performs one personal trip per unit time.

The non-transport state An individual has two options in the non-transport state:

Work: An individual may choose to work for a wage $\nu > 0$ (heterogeneous in the population). This gives to the individual utility ν/λ_n per unit time.

Provide rides: An individual can offer seats in the ride sharing platform and get payed by riders. By doing so they do not earn any utility with respect ot μ.

The type of an individual is denoted by a pair $\chi = (\rho, \nu)$. The types of the individuals are distributed in $X = (0, \infty)^2$ according to a distribution M. The density of this distribution is supposed to be positive on X.

3.2 Agents' strategies

We describe here every possible strategy an individual can choose, given the choice he can make in the transport and the non-transport state. We do not consider a strategy of the original paper [2] (Optimist) since individuals choosing that strategy only appeared in unrealistic situations (in terms of the rental prices chosen by the ride sharing platform). However, we consider a new strategy: the high users. The users of the original paper are replaced by a new strategy: the low users. Overall, five strategies are available:

(A) **The abstinent:** An abstinent works for a wage ν in the non-transport state and does not own a car. In the transport state, he uses only public transport.

(D) **The driver:** A driver works for a wage ν and owns a car. In the transport state, he uses it to fulfill his own needs and offer seats in the ride sharing platform.

(S) **The serive provider:** A service provider does not work in the non-transport state but instead offers seats in the ride sharing platform regardless of his own need for a ride. He obviously owns a car. During the transport state, he behaves like a driver.

(U_l) **The low user:** A low user works for a wage ν in the transport state and does not own a car. In the transport state, he tries to get a ride through the cheapest of the two platforms (taxi or ride sharing). If he does not get that ride, he chooses the public transport.

(U_h) **The high user:** A high user acts as a low user in the non-transport state, he also does not own a car. In the transport state, he tries to get a ride through the cheapest of the two platforms. If he does not get a ride, he tries the other platform. If his request is still unsuccessful, he chooses the public transport.

It has to be noted that being a taxi driver is not an available strategy here. We denote by $\Sigma = \{A, D, S, U_l, U_h\}$ the set of all strategies. For any $\sigma \in \Sigma$, μ_σ denotes the fraction of the population opting for strategy σ. The vector $\mu = (\mu_A, \mu_D, \mu_S, \mu_{U_l}, \mu_{U_h})$ gathers the distribution of all strategies.

3.3 The platforms

In this model, there are two platforms: the ride sharing platform and the taxi platform. There are supposed to be run by a third party (different for each platform) who matches supply and demand. The supply of the taxi platform is fixed: there is a given fleet of available taxis $n_t > 0$, whereas the supply in the ride sharing platform depends on the number of drivers and service providers. On the other hand, the demand in both platforms depends on the number of users (low or high). Every car in the population is supposed to have the capability of carrying at most $k > 0$ riders per trip (in addition to the rider). Whenever a platform matches a request with an empty seat, a given rental price (depending on the platform used) is payed by the rider(s) to the driver. Each platform chooses its rental price. These rental prices are noted r_1 and r_2 for the ride sharing platform and the taxi platform respectively.

Let us denote by p_l the probability of getting a ride through the cheapest platform, by p_h the probability of getting a ride through the other platform and by $\bar{p}$ the probability that a seat offered in the ride sharing platform (either by drivers or service providers) is sold. p_l, p_h and $\bar{p}$ can be expressed as functions of the vector distribution μ. Their expression depends on the inequality we have between r_1 and r_2 since the behavior of users depends on which is the cheapest platform. We assume (arbitrarily) that if $r_1 = r_2$ then users will choose first the ride sharing platform. We can now compute p_l, p_h and $\bar{p}$ without ambiguity:

If $r_1 \leq r_2$:

- $p_l = \frac{k(\mu_D + \lambda_t \mu_S)}{\mu_{U_l} + \mu_{U_h}} \wedge 1$
- $p_h = \frac{n_t}{(1-p_l)\mu_{U_h}} \wedge 1$
- $\bar{p} = \frac{\mu_{U_l} + \mu_{U_h}}{k(\mu_D + \lambda_t \mu_S)} \wedge 1$

If $r_1 > r_2$:

- $p_l = \frac{n_t}{\mu_{U_l} + \mu_{U_h}} \wedge 1$
- $p_h = \frac{k(\mu_D + \lambda_t \mu_S)}{(1-p_l)\mu_{U_h}} \wedge 1$
- $\bar{p} = \frac{(1-p_l)\mu_{U_h}}{k(\mu_D + \lambda_t \mu_S)} \wedge 1$

In case of the denominator being equal to 0 in one of these fractions, the corresponding expression would be equal to 1. Service providers "counts λ_t times more" than the drivers in terms of supply in the ride sharing platform because they ride during the transport and the non-transport state. Since $1/\lambda_t$ corresponds to one trip per unit time, then $1/\lambda_n + 1/\lambda_t = 1$ corresponds to $1/(1/\lambda_t) = \lambda_t$ trips per unit time. In fact, service providers rides their car at rate $\lambda_t > 0$ per unit time

3.4 Agents' payoffs

Per unit time, a utility ρ is received by each individual who performs there personal trip (during the transport state) by using private transportation. Moreover, anyone working during the non-transport state (that is, not the service providers) gets a utility ν/λ_n per unit time. Anyone who uses a platform to get a ride has to pay the corresponding rental price. The individuals using their car pay a cost $c > 0$ per trip to cover, for instance, the fuel cost (c is called the cost of usage). On top of that, any individuals with a car (drivers and service providers) have to pay the cost of ownership $\omega > 0$ per unit time. When drivers or service providers use their car and offer seats in the ride sharing platform, they get payed by the riders using their car. If their car is full (that is, if $\bar{p} = 1$), they will get kr_1 for each trip. However, if $\bar{p} < 1$, we assume that the amount of money payed by the riders is equally distributed between everyone offering seats. Therefore, at each trip, drivers and service providers get utility $k\bar{p}r_1$.

Then, the payoff of a given strategy depends on the type of the individual $\chi = (\rho, \nu)$ and vector distribution μ. Here are the payoffs associated to each strategy (per unit time):

(A) The only source of revenue for an abstinent comes from his work. Therefore, the payoff of an abstinent of type (χ, ν) is:

$$\pi_A(\rho, \nu) = \nu/\lambda_n$$

(D) A driver always gets a ride in the transport state since he uses his own car, therefore he gets ρ per unit time. Owning a car costs him ω per unit time. Since he drives once per unit time, he has to pay the cost of usage c, but also gets payed $k\bar{p}r_1$ by the users. On top of that, he works in the non-transport state. Hence, the payoff of a driver is:

$$\pi_D(\rho, \nu) = \nu/\lambda_n + \rho - \omega + k\bar{p}(\mu)r_1 - c$$

(S) A service provider gets ρ per unit for his trip in the transport state. He also has to pay the cost of ownership ω. At each ride, he gets $k\bar{p}r_1 - c$. He rides his car during the transport and the non-transport state. Therefore, he rides his car at rate $\lambda_t > 0$ per unit time. Hence, the payoff of a service provider is:

$$\pi_S(\rho, \nu) = \rho - \omega + \lambda_t(k\bar{p}(\mu)r_1 - c)$$

(U_l) A low user works in the non-transport state, therefore gets ν/λ_n per unit time. In the transport state, he gets a ride with probability p_l (since he chooses the cheapest platform), and in that case gets utility $\rho - min(r_1, r_2)$. Therefore, the payoff of a low user is given by:

$$\pi_{U_l}(\rho, \nu) = \nu/\lambda_n + p_l(\mu)(\rho - min(r_1, r_2))$$

That payoff is the same as the payoff of the users in the original paper.

(U_h) A high user is identical to a low user in the non-transport state. In the transport state, he gets a ride in the cheapest platform with probability p_l and, in that case, gets $\rho - min(r_1, r_2)$. If he does not get that ride (so with probability $1 - p_l$), then he tries the other platform, gets a ride there with probability p_h and, if he does, gets utility $\rho - max(r_1, r_2)$. The payoff of a high user is:

$$\pi_{U_h}(\rho, \nu) = \nu/\lambda_n + p_l(\mu)(\rho - min(r_1, r_2)) + (1 - p_l(\mu))p_h(\mu)(\rho - max(r_1, r_2))$$

3.5 Equilibrium

The type of equilibrium we consider in this paper is the Nash equilibria. This kind of equilibria allows to predict the behavior of the population modeled by that game. A Nash equilibria characterizes a situation of the game where every players has chosen a strategy and where it is not in the interest of any of these players (in terms of maximizing his payoff) to unilaterally change his strategy. More formally, let us assume that the strategy of any player is given by a function $\chi : X \to \Sigma$. Then, the situation is a Nash equilibrium if:

$$\forall \chi = (\rho, \nu) \in X, \forall \sigma \in \Sigma, \pi_{\sigma(\chi)}(\rho, \nu) \geq \pi_\sigma(\rho, \nu)$$

A function assigning each player to a strategy is, in fact, a partition of the type space into sets of players choosing the same strategy. For each strategy $\sigma \in \Sigma$, we can define $P_\sigma \subset X$ as the set of players adopting strategy σ. Then, the vector distribution μ can be computed:

$$\forall \sigma \in \Sigma, \mu_\sigma = M(P_\sigma) = \int_{P_\sigma} m(\chi) d\chi$$

In our situation, the type space is $X = (0, \infty)^2$. Therefore, a partition of the players can be seen geometrically. In fact, because the payoffs are affine functions of the type of the players $\chi = (\rho, \nu)$, we know (from the original paper [2]) that, at any equilibrium, it is possible to partition X into convex sets (if we do not consider the boundaries of measure 0) where every player prefers the same strategy, or is indifferent to the same set of strategies (meaning that they have the same payoffs for several strategies).

In this paper, we do not focus on finding the exact equilibrium we get for any value of our parameters. However, let us show an example of values of our parameters for which the equilibrium is quite easy to obtain, to illustrate what an equilibrium looks like.

Let us consider our parameters $\omega, c, k, \lambda_t, n_t, r_1$ and r_2 and assume that $\omega + c < r_1 \leq r_2$.

We have that, for $\rho < r_1$, $\pi_A > \pi_{U_l}, \pi_{U_h}$. Moreover, because $\omega + c < r_1 \leq r_2$, we can show that, whenever $\rho \geq r_1$, $\pi_D > \pi_{U_l}, \pi_{U_h}$. That means that no player will choose to be a user. Therefore, $\mu_{U_l} = \mu_{U_h} = 0$, which leads to $\bar{p}(\mu) = \frac{\mu_{U_l} + \mu_{U_h}}{k(\mu_D + \lambda_t \mu_S)} \wedge 1 = 0$. This implies that, for any player, $\pi_D > \pi_S$. We can deduce that, at equilibrium, there are only drivers and abstinent.

To conclude, at equilibrium, $P_{U_l} = P_{U_h} = P_S = \emptyset$ and:

$$E_{A_1} = \{\chi \in X, \rho < \omega + c\} \subset P_A \subset \{\chi \in X, \rho \leq \omega + c\} = E_{A_2}$$

$$E_{D_1} = \{\chi \in X, \omega + c < \rho\} \subset P_D \subset \{\chi \in X, \omega + c \leq \rho\} = E_{D_2}$$

Because $M(\{\chi \in X, \omega + c = \rho\}) = 0$, we have $M(E_{A_1}) = M(E_{A_2})$ and $M(E_{D_1}) = M(E_{D_2})$. Therefore the distribution of the strategies is defined without ambiguity. A schema of the resulting equilibria can be seen in figure 1 where $\mu_A = M(P_A) = 0.5 = M(P_D) = \mu_D$.

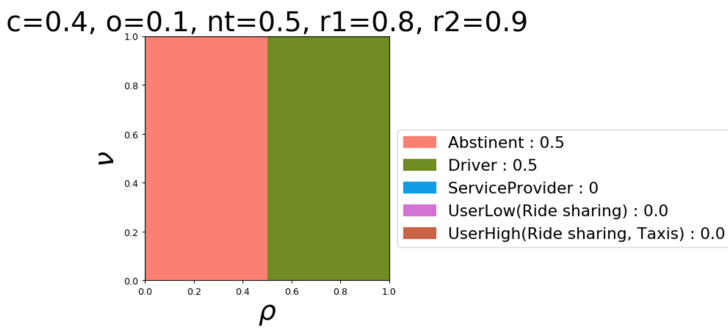

Fig. 1. The shape of the equilibrium with parameters $\lambda_t = 6, k = 2$ (o stands for ω).

4 Theoretical analysis

The object of study of this paper is the revenue of the ride sharing platform. In our model, this platform earns something proportional to its rental price each time a seat is sold. It has to be noted that the expression of this revenue changes whether $r_1 \leq r_2$ or not. For a distribution vector μ, the revenue of the platform is given by:

- $\mathcal{R} = r_1 \times p_l \times (\mu_{U_l} + \mu_{U_h})$ if $r_1 \leq r_2$;
- $\mathcal{R} = r_1 \times p_h \times (1 - p_l)\mu_{U_h}$ otherwise.

In the original paper, the revenue of the ride sharing platform has also been studied. However, even there, with the exact equilibrium known for every value of the parameters, the authors could not find an analytical expression of that revenue. Moreover, the numerical experiments showed that the revenue was not monotonous if it is expressed as a function of the rental price, every other parameters being fixed. Therefore, we do not aim at finding an expression of that revenue.

The model we consider in this paper changes a lot between the two cases where $r_1 \leq r_2$ and where $r_1 > r_2$. So, let us study these two cases separately.

4.1 Case where $r_1 \leq r_2$

That case is the easier one to study since the payoffs have not changed a lot. In fact, π_A, π_D, π_S and π_{U_l} have not changed at all. So, if at equilibrium there are no high users or if high users have the same payoffs as low users, then we have the same equilibrium as without taxis.

The first result we have (whose proof can be found in appendix A) concerns a condition on which adding taxis does not change the equilibrium.

Theorem 1. *If $r_1 \leq r_2$ and $r_1 \geq \frac{\omega+c}{k+1}$, then the equilibrium is the same as in the original game (without taxis).*

This result can be explained by the fact that the condition $r_1 \geq \frac{\omega+c}{k+1}$ was sufficient, in the original game, to establish that $p_l = 1$. In the new game, if $p_l = 1$ at equilibrium, then we have the same equilibrium as without taxis. Indeed, $p_l = 1$ means that the supply exceeds the demand in the ride sharing platform, which implies that no user gets to the taxi platform. Therefore the game with taxis is analogous to the game without taxis (in terms of payoffs, $p_l = 1$ implies that $\pi_{U_l} = \pi_{U_h}$). It follows that whenever $p_l = 1$, the revenue does not change. Let us now consider an equilibrium with taxis where $p_l < 1$ (and therefore, where $r_1 > \frac{\omega+c}{k+1}$).

Our idea of why it could be possible to increase the revenue by adding competition via a taxi platform was that the perspective of getting a ride through another platform could be an incentive to have more users, and therefore, more rides. In fact, this phenomenon occurs and we effectively see an increase of the number of users. However, when $p_l < 1$, we have that the revenue $\mathcal{R}$ of the ride sharing platform is equal to $\mathcal{R} = r_1 \times p_l \times (\mu_{U_l} + \mu_{U_h}) = r_1 k(\mu_D + \lambda_t \mu_S)$. So, to increase the revenue, we need to increase the supply (that is, the number of drivers and service providers), not the demand. What we thought could be the reason why the revenue could increase, is in fact the reason why it decreases. Indeed, it is possible to show that adding taxis increases the demand, reduces the supply and leads to a decrease of the revenue:

Theorem 2. *If $\omega \leq c/k$ and $r_1 \leq r_2$ then adding the taxi platform can not increase the revenue of the ride sharing platform.*

The proof of that theorem can be found in appendix B. The condition $\omega \leq c/k$ avoid the existence of service providers in the original and the new game. The proof of this theorem relies on the fact that the number of abstinent does not change by adding taxis, but the number of users increases, which leads to a decrease of the number of drivers.

It has to be noted that this result (probably) does not hold if it is possible to have service providers at equilibrium (i.e. if $c/k < \omega$). We found a example of values for our parameters for which the revenue of the ride sharing platform increased. However, we found it numerically, so we will talk about it in the next section.

4.2 Case where $r_1 > r_2$

That case is a lot harder to study because it is not possible to draw aspiration from the original paper since the payoffs of users have entirely changed. However, we were able to find a necessary and sufficient condition on which $p_l = 1$ (when $r_1 > r_2$, $p_l = 1$ does not mean anymore that we have the same equilibrium as in the original paper).

Theorem 3. *Let us assume that $r_1 > r_2$. Then, $p_l = 1$ if and only if $r_2 \geq \omega + c$ or $n_t \geq \int_{\{\rho > r_2\}} m(\chi) d\chi$.*

The proof of that theorem can be found in appendix C. It has to be noted that if $p_l = 1$ then no user gets to the ride sharing platform, which means that its revenue is equal to 0. Now, let us assume that $p_l < 1$. We can consider the revenue of the taxi platform. If $r_1 > r_2$, the revenue of this platform is $\mathcal{R}_t = r_2 \times p_l \times (\mu_{U_l} + \mu_{U_h}) = r_2 \times n_t$ (because, by assumption, $p_l < 1$). Then, we can see that the taxi platform would prefer to have a rental price as high as possible, while still ensuring $r_2 < r_1$ and $r_2 < \omega + c$ (if $r_2 \geq \omega + c$, then $p_l = 1$ and we get the equilibrium from figure 1, where the revenue is null for both of the platforms).

It has to be noted that, when $r_1 > r_2$, every user tries first the taxi platform. Then, some of them may try the ride sharing platform. Therefore, it seems that the revenue of the ride sharing can not increase by adding taxis. However, we were able to find an example where the revenue of the platform does increase. In fact, we used the fact that for r_1 high enough, there are no users at all (without taxis) because the price of transportation is too high. Having no user at equilibrium implies that the revenue of the ride sharing platform is null. However, if we add a taxi platform with a low enough rental price, then the average transportation price could be low enough for (high) users to appear. If they do, the revenue of the ride sharing platform becomes strictly positive.

Theorem 4. *Let us consider the parameters* $\omega = 0.4, c = 0.1, k = 1, \lambda_t = 6., r_1 = 0.6, r_2 = 0.1, n_t = 0.2$. *Then, the revenue of the ride sharing platform strictly increases by adding taxis, for the uniform distribution over* $[0, 1]^2$.

The proof can be found in appendix D. That situation is not very realistic since the ride sharing platform has no interest in pricing that high (and similarly we have seen that the taxi platform should price a lot higher–closer to $\omega + c$–to increase its revenue). However, it shows that, in some cases, competition may be positive for the ride sharing platform.

Some other attempts are discussed in section 6.

5 Numerical analysis

Because the theoretical analysis is difficult, we tried to do some numerical experiment to see what kind of effect we could expect from adding the taxi platform. The main algorithm we used is the best response dynamics algorithm. We give the pseudo code of this algorithm in the following (algorithm 1), with N players in $PlayerSet$, the strategies in Σ, and the payoff noted π.

Basically, the idea is to update the strategy of every player by choosing the best strategy (in order to maximize his payoff) while the choice of every other player is fixed (every player chooses the 'best response' to the choice of the other players). It has to be noted that this algorithm does not necessarily terminate. However, its huge asset is that, whenever it does, it converges to a Nash equilibrium[3] (obviously, we only consider the result of this algorithm when

[3] In a game with a finite number of player (i.e. where it is possible to use the best response dynamics algorithm) the distribution vector has to be updated when a player

Algorithm 1 Best response dynamics

Initialize distribution vector $\mu = (0, .., 0)$
for pl *in* $SetPlayer$ **do**
    Randomly choose a strategy $\sigma \in \Sigma$, $\sigma(p_l) = \sigma$
    $\mu_\sigma \ += 1/N$
$Converge = False$
while *not* $Converge$ **do**
    $Converge = True$
    for pl *in* $SetPlayer$ **do**
        $\sigma = \sigma(pl)$
        $\sigma(pl) \leftarrow argmax_{\sigma' \in \Sigma}(\pi_{\sigma'}(pl, \mu[\mu'_\sigma += 1/N, \mu_\sigma -= 1/N]))$
        $\mu_{\sigma(pl)} \ += 1/N, \mu_\sigma \ -= 1/N$
        if $\sigma(pl) \neq \sigma$ **then**
            $Converge = False$

it terminates). The equilibria we obtain with this algorithm are equilibria in a game with a finite number of players (finite game), that is different from our non-atomic game. However, the result we mentioned in section 2 about ϵ-Nash equilibria ensures that the equilibria we get in the finite game are as close as we want from the equilibria in the non-atomic game for a large enough number of players. We implemented a tool that allowed us to compute the outcome of the game for any value of our parameters using this algorithm. Once the equilibrium is computed, obtaining the revenue of the ride sharing platform is straightforward.

With that tool, we were able to find an example of values of our parameters (with $r_1 \leq r_2$) for which the revenue of the ride sharing platform increases with the taxis. That example is shown in figure 2 where R denotes the revenue of the ride sharing platform. The location of the players is generated uniformly in $[0, 1]^2$, which corresponds to a distribution M uniform in $[0, 1]^2$. Because the generation of players is random, only one experiment is not enough to be sure that the difference we find in the revenue is significant. After 100 experiments, we found that (on average) $R_{without} = 0.04342, R_{with} = 0.04369$ (with a standard deviation of respectively 5.6×10^{-5} and 6.3×10^{-5}). Even if it seems conclusive, we still need to theoretically confirm this result.

In a totally different aspect, we studied the dynamics of the prices. More specifically, if we assume that the two platforms are profit-maximizing (that is, they choose their price in order to maximize their profit), then it is possible to see how the prices of the platforms will evolve. An example of that can be seen in figure 2. The red curve can be read as it is displayed because it shows the function $r_1(Yaxis) = f(r_2)(Xaxis)$. However, the blue curve shows

looks at another strategy because if he changes his strategy, the distribution vector will also change. On the other hand, the definition we gave of a Nash equilibrium (in section 3) holds in a game with a continuum of players. In such a game, any specific player has no influence on the distribution which does not need to be updated when one player changes its strategy.

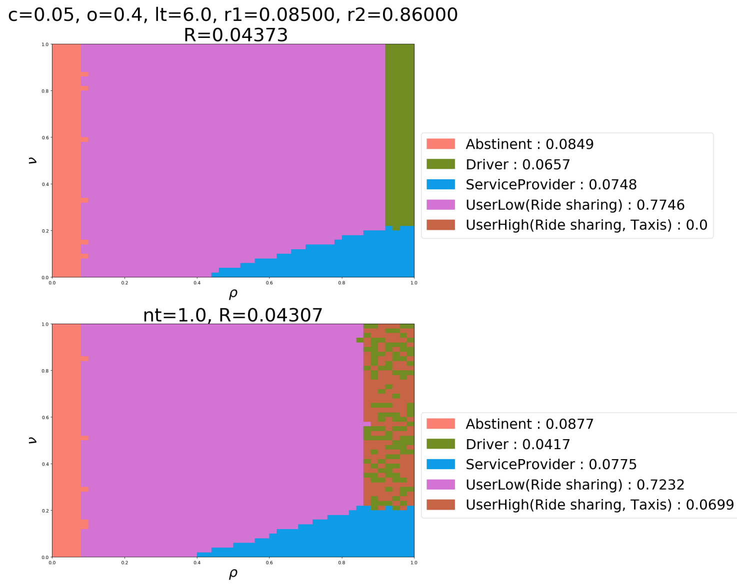

Fig. 2. The equilibria without taxis (i.e. $n_t = 0$, first picture) and with taxis (second picture), computed with 10 000 players and $k = 1$. In the second picture, high users and drivers are mixed because they have exactly the same payoff. What is drawn in these pictures is a tilling of the $[0, 1]^2$ square by 2 500 smaller squares whose color are chosen according to strategy chosen by the closest player (any of the 10 000 players corresponding to a specific point in the $[0, 1]^2$ square).

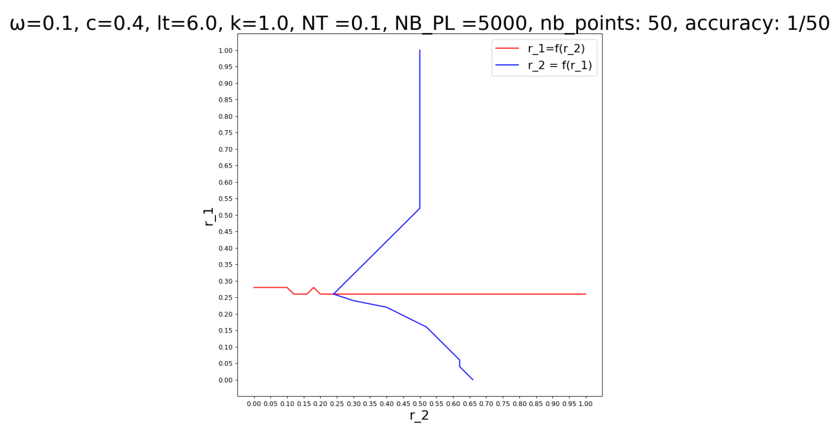

Fig. 3. These curves are drawn with 5 000 players. The red curves draws the best response for the ride sharing platform when the taxi platform has chosen its price r_2. Similarly, the blue curve draws the best response of the taxi platform when the ride sharing platform has chosen its price r_1. Each curves are drawn with $nb_{points} = 50$ points. For every of r_2–for the red curve–(or r_1, for the blue curve), the best response is chosen among 50 (the accuracy being 1/50) possible values between 0 and 1.

$r_2(Xaxis) = g(r_1)(Yaxis)$, therefore we have to be careful when we read it. This way of displaying these curves is useful because it allows to see if (and where) there is an equilibrium in the game between the two platforms (continuous in terms of strategies: picking a rental price between 0 and 1). Such an equilibrium corresponds to an intersection of the curves. Indeed, if both platforms chooses the rental prices corresponding to an intersection of the curves, then no platform has any interest in changing its price.

This drawing seems to show that there exists a unique equilibrium in that game between the two platforms. However, precisely understanding the shape of the curves is harder. We are still trying to (theoretically) understand why the blue curves suddenly changes its evolution and starts increasing. Moreover, this situation is very nice in terms of monotonicity and shape of the curves. However, we did several other experiments and the results were a lot less clean in terms of monotonicity of the functions drawn (especially when there are service providers, that are not present in any equilibrium that occurs in figure 2).

6 Concluding remarks

In this paper, we have studied an extension of a model that focuses on competition between a ride sharing platform and a taxi platform. Our main subject of interest was the evolution of the revenue of the ride sharing platform and more specifically whether or not competition (in this case, a taxi platform) could increase that revenue. Although we did not determine what type of equilibrium we have for every values or our parameters, we were able to prove some results about the conditions on which an increase of the revenue could occur. In addition, numerical simulations allowed us to surmise the behavior we could expect from the platforms assuming they are profit maximizing.

However, we did not present all of our findings. Because we could not directly use them to study the evolution of the revenue, we did not mention any result on the equilibria we could obtain. For example, an interesting result in the original paper was a necessary and sufficient condition for the existence of service providers (which was $c/k < r_1 < \omega$). In the case where $r_2 < r_1$, we were also able to find a necessary and sufficient condition for the existence of service providers (which is $c/k < r_1$ and $\frac{r_1-\omega}{r_1-r_2} < p_l < 1$). However, our condition involves p_l that does not entirely depend on our parameters. Indeed, $p_l = \frac{nt}{\mu_{U_l}+\mu_{U_h}} \wedge 1$, which means that its value depends on the number of users, and therefore on the distribution M. The only thing we could say is that there exists a value of nt such that p_l is high enough to get service providers. Therefore, this result can not be used to predict the equilibrium (we do no provide the proof of that result here because we do not use it in this paper). A lead for future work could be to extend these research to find nicer conditions for the existence of service providers, that could be used to make some assumptions about how the revenue of the ride sharing platform evolves.

Some other extensions of the initial model may be looked at. An example would be to take into account several origin-destination pairs which would make the model more realistic. However, such an extension would create a more elaborate and obtaining clear results would be that much more difficult.

Before looking at the taxi platform, we also looked at some other possible extensions. We opted for the taxi platform because it seemed to be the most promising extension to look at among those we have tried. However, some other possibilities could be considered. For instance, examining if adding some taxes could increase the social welfare (payoffs of the total population) or the revenue of the platform could be interesting. For example, taxing drivers and service providers who do not entirely fill up their car could be an incentive to exactly match supply and demand, which could benefit the population or the platform.

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A If $r_1 \leq r_2$ and $r_1 \geq \frac{\omega+c}{k+1}$, then the equilibrium is the same as in the original game

First, a quick reminder of our functions of interest when $r_1 \leq r_2$:

- $\pi_A = \frac{\nu}{\lambda_n}$
- $\pi_D = \frac{\nu}{\lambda_n} + \rho - \omega + k\bar{p}r_1 - c$
- $\pi_S = \rho - \omega + \lambda_t(k\bar{p}r_1 - c)$
- $\pi_{U_l} = \frac{\nu}{\lambda_n} + p_l(\rho - r_1)$
- $\pi_{U_h} = \frac{\nu}{\lambda_n} + p_l(\rho - r_1) + (1 - p_l)p_h(\rho - r_2)$
- $p_l = \frac{k(\mu_D + \lambda_t\mu_S)}{\mu_{U_l} + \mu_{U_h}} \wedge 1$
- $p_h = \frac{n_t}{(1-p_l)\mu_{U_h}} \wedge 1$
- $\bar{p} = \frac{\mu_{U_l} + \mu_{U_h}}{k(\mu_D + \lambda_t\mu_S)} \wedge 1$
- The revenue of the ride sharing platform: $R = r \times (\mu_{U_l} + \mu_{U_h}) \times p_l$

Lemma 1. *If $r_1 > \frac{\omega+c}{k+1}$ or $r_1 = \frac{\omega+c}{k+1}$ and $r_1 < r_2$, the equilibrium is the same as in the original paper.*

Let us assume that $p_l < 1$ (therefore, $\bar{p} = 1$). Then, we have:

$$\begin{aligned}
\pi_D > \pi_{U_h} &\Leftrightarrow \rho - \omega + kr_1 - c > p_l(\rho - r_1) + (1 - p_l)p_h(\rho - r_2) \\
&\Leftrightarrow \rho(1 - p_l)(1 - p_h) > \\
&\qquad \omega + c - kr_1 - p_l r_1 - (1 - p_l)p_h r_2 \\
&\qquad + (r_1 - (1 - pl)p_h r_1 - r_1 + (1 - pl)p_h r_1) \\
&\Leftrightarrow \rho(1 - p_l)(1 - p_h) > \\
&\qquad (1 - p_l)(1 - p_h)r_1 + \omega + c - (k + 1)r_1 + (1 - p_l)ph(r_1 - r_2)
\end{aligned}$$

If $p_h = 1$, this is equivalent to: $0 > \omega + c - (k+1)r_1 + (1-p_l)(r_1 - r_2)$ which is true by hypothesis. Therefore, high users are strictly dominated by drivers.

Now, if $p_h < 1$, we have:

$$\pi_D > \pi_{U_h} \Leftrightarrow \rho > \underbrace{r_1 + \frac{\omega + c - (k + 1)r_1 + (1 - p_l)ph(r_1 - r_2)}{(1 - p_l)(1 - p_h)}}_{\text{noted } \rho^*}$$

Because of the inequalities we have on r_1 and r_2, we have $\rho^* < r_1$. This means that for any $\rho \geq r_1$, $\pi_D > \pi_{U_h}$. Moreover, since $r_1 \leq r_2$, it is obvious that for any $\rho < r_1$, $\pi_A > \pi_{U_h}$. Therefore, in any case, high users are strictly dominated either by drivers or by abstinent.

Therefore, if $p_l < 1$, there are no high users at equilibrium, which means that we will get the same equilibrium as in the original game if we consider that users are low users (we can also consider high users because there will not be any).

Moreover, $p_l = 1$ implies that $\pi_{U_l} = \pi_{U_h}$. That is, there will be the same equilibrium as in the original paper if we consider that users are either low users or high users.

Lemma 2. *If $r_1 = r_2 = \frac{\omega+c}{k+1}$, the equilibrium is the same as in the original paper.*

If $p_l = 1$, we get the same equilibrium as in the original paper.

Let us now assume that $p_l < 1$ ($\bar{p} = 1$). In that case, $\mu_{U_l} = 0$ because low users are dominated by high users and abstinent (they may only exist for $\rho = r_1$). If $p_h < 1$, then $\mu_{U_h} = 0$ because high users are dominated by abstinent and drivers (again, they may only exist for $\rho = r_1$). However, this is not possible because $p_l < 1$ and we already have $\mu_{U_l} = 0$. Therefore, $p_h = 1$ and $\pi_D = \pi_{U_h}$ for any ρ. In that case, we get the same equilibrium as in the original paper with users being high users and $\pi_{U_h} = \nu/\lambda_n + p_l(\rho - r_1) + (1-p_l)p_h(\rho - r_1) = \nu/\lambda_n + \rho - r_1$ because $p_h = 1$. That is the payoffs of users in the original paper (because if there are some users and $r_1 \geq \frac{\omega+c}{k+1}$ then $p_l = 1$).

Lemma 3. *If $r_1 \geq \frac{\omega+c}{k+1}$, the equilibrium is the same as in the original paper.*

The two previous lemmas give the result by considering that low and high users in the new game are the users of the original game. It has to be noted that, in this situation the only possible equilibrium is the equilibrium of the original paper. That implies that the equilibrium is unique in that situation.

B If $\omega \leq c/k$ and $r_1 \leq r_2$ then adding the taxi platform can not increase the revenue of the ride sharing platform.

The expressions of interest are reminded at the beginning of the previous section (we are still in a case where $r_1 < r_2$).

Lemma 4. *If $r_1 \leq r_2$ and $\omega \leq c/k$, there are no service providers at equilibrium (with or without taxis).*

We know from the original paper[2] that, without the taxi platform, if $\omega \leq c/k$, it is not possible to have service providers at equilibrium.

We now consider the game with taxis. Let us assume that we have service providers at equilibrium. Then, we have:

$$\pi_S > \pi_D \Leftrightarrow \nu < \lambda_n(\lambda_t - 1)(k\bar{p}r_1 - c)$$

We can deduce that $k\bar{p}r_1 - c > 0$, otherwise service providers would always be dominated by drivers. In particular, this implies that $r_1 > c/k$, because $\bar{p} \leq 1$.

Let us now assume towards a contradiction that $\omega \leq r_1$. Then, for $\rho < r_1$, $\pi_A > \pi_{U_l}, \pi_{U_h}$ (because $p_l > 0$ since there are service providers). In addition, for $\rho \geq r_1$, we have:

$$\pi_{U_l}, \pi_{U_h} \leq \frac{\nu}{\lambda_n} + \rho - r_1 \leq \frac{\nu}{\lambda_n} + \rho - \omega < \frac{\nu}{\lambda_n} + \rho - \omega + k\bar{p}r_1 - c = \pi_D$$

Therefore, users are dominated by drivers or abstinent, which implies that $\mu_{U_l} = \mu_{U_h} = 0$, and therefore $\bar{p} = 0$. Then, $k\bar{p}r_1 - c = -c \leq 0$, hence the contradiction. In fact, $r_1 < \omega$.

To conclude, we have $c/k < r_1 < \omega$, which implies $c/k < \omega$. Therefore, if $\omega \leq c/k$, there are no service providers.

Lemma 5. *If $r_1 \leq r_2$ and if $\omega \leq c/k$, then the revenue of the ride sharing platform can not increase by adding taxis.*

From the previous lemma, we know that there are no service providers at equilibrium.

Let us consider a set of values for our parameters: $r_1, r_2, \omega, c, k, \lambda_t, n_t$ satisfying $r_1 \leq r_2$ and $r_1 < \frac{\omega+c}{k+1}$ (we know from theorem 1 that if $r_1 \geq \frac{\omega+c}{k+1}$, the equilibrium does not change, which means that the revenue of the ride sharing platform do not change either).

Let us consider the (unique) equilibrium without taxis (original game) and an equilibrium with taxis (new game). Clearly, if $p_l = 0$ with taxis, then $R = 0$ and the revenue has not increased. In addition, if $p_l = 1$, we obtain the same equilibrium as without taxis (see previous section). Therefore, we will assume that: $0 < p_l < 1$, $\bar{p} = 1$.

First let us examine the abstinent. We have:

- $\pi_A - \pi_D > 0 \Leftrightarrow \rho < \omega + c - kr_1$
- $\pi_A - \pi_{U_l} > 0 \Leftrightarrow \rho < r_1$ (because $p_l > 0$)
- $\pi_A - \pi_{U_h} > 0 \Leftrightarrow \rho < \frac{p_l r_1 + (1-p_l)p_h r_2}{p_l + (1-p_l)p_h}$

According to the inequalities on r_1, we have $min(r_1, \frac{p_l r_1 + (1-p_l)p_h r_2}{p_l + (1-p_l)p_h}, \omega + c - kr_1) = r_1$. Therefore: $\{\chi \in X : \rho < r_1\} \subseteq P_A \subseteq \{\chi \in X : \rho \leq r_1\}$. This result is the same as without taxis. Hence, the number of abstinent is the same with or without taxis. Because, at any point, we have: $\mu_{U_l} + \mu_{U_h} + \mu_D + \mu_A = 1$ (since $\mu_S = 0$), we can conclude that there exist a constant nb_a such that, with and without taxis: $\mu_{U_l} + \mu_{U_h} + \mu_D = nb_A$.

Let us now assume towards a contradiction that p_l has increased from the equilibrium without taxis to the equilibrium with taxis we consider (it implies that without taxis we also have $p_l < 1, \bar{p} = 1$).

It is clear that, for $\rho < r_1$, there are no users, with or without taxis (see the above inequalities). Moreover, whenever $\rho \geq r_1$, the payoff of users (that is, the maximum between the payoffs of high and low users) has increased because p_l has increased. In addition, the payoffs of drivers has not changed (since $\bar{p} = 1$ with and without taxis). Therefore, the total number of users can not have decreased by adding taxis, and the number of drivers can not have increased. Because $p_l = \frac{k\mu_D}{\mu_{U_l} + \mu_{U_h}}$ (there are no service providers), we can conclude that p_l can not have increased. Hence, the contradiction.

In fact, p_l has decreased (in a large sense, it may have not changed). To summarize, we know that:

- $p_l = \frac{k\mu_D}{\mu_{U_l} + \mu_{U_h}}$ has decreased

– $\mu_{U_l} + \mu_{U_h} + \mu_D$ has not changed

Therefore, it follows that $\mu_{U_l} + \mu_{U_h}$ has increased (large sense) and that μ_D has decreased (large sense). Therefore, the revenue of the ride sharing platform $R = r_1 \times k\mu_D$ can not have increased.

C Let us assume that $r_1 > r_2$. Then, $p_l = 1$ if and only if $r_2 \geq \omega + c$ or $n_t \geq \int_{\{\rho > r_2\}} m(\chi) d\chi$.

We are in the case where $r_1 > r_2$, our functions of interest are:

– $\pi_A = \frac{\nu}{\lambda_n}$

– $\pi_D = \frac{\nu}{\lambda_n} + \rho - \omega + k\bar{p}r_1 - c$

– $\pi_S = \rho - \omega + \lambda_t(k\bar{p}r_1 - c)$

– $\pi_{U_l} = \frac{\nu}{\lambda_n} + p_l(\rho - r_2)$

– $\pi_{U_h} = \frac{\nu}{\lambda_n} + p_l(\rho - r_2) + (1 - p_l)p_h(\rho - r_1)$

– $p_l = \frac{n_t}{\mu_{U_l} + \mu_{U_h}} \wedge 1 > 0$ (because $n_t > 0$)

– $p_h = \frac{k(\mu_D + \lambda_t \mu_S)}{(1-p_l)\mu_{U_h}} \wedge 1$

– $\bar{p} = \frac{(1-p_l)\mu_{U_h}}{k(\mu_D + \lambda_t \mu_S)} \wedge 1$

Lemma 6. *If $r_1 > r_2 \geq \omega + c$, then $p_l = 1$ at equilibrium.*

Let us assume towards a contradiction that $p_l < 1$. Then, for $\rho < r_2$, $\pi_A > \pi_{U_l}, \pi_{U_h}$. For for $\rho > r_2$, we have:

$$\pi_{U_l}, \pi_{U_h} < \frac{\nu}{\lambda_n} + \rho - r_2 \leq \frac{\nu}{\lambda_n} + \rho - (\omega + c) \leq \frac{\nu}{\lambda_n} + \rho - \omega + k\bar{p}r_1 - c = \pi_D$$

This implies $\mu_{U_l} = \mu_{U_h} = 0$ and therefore $p_l = 1$. Hence the contradiction. In fact, $p_l = 1$.

Lemma 7. *Let us assume that $r_1 > r_2$ and $r_2 < \omega + c$. Then, $p_l = 1 \Leftrightarrow n_t \geq \int_{\{\rho > r_2\}} m(\chi) d\chi$.*

Let us assume that $p_l = 1$ (that implies $\bar{p} = 0$). Therefore, at equilibrium there are no service providers (because $k\bar{p}r_1 - c = -c \leq 0$, see the proof of lemma 4). Moreover, $\pi_{U_l} = \pi_{U_h} = \frac{\nu}{\lambda_n} + \rho - r_2 > \frac{\nu}{\lambda_n} + \rho - (\omega + c)$. Therefore, there are no drivers either. Therefore, at equilibrium, for $\rho < r_2$, there are only abstinent, and for $\rho > r_2$, there are only users (low and high). Hence, $\mu_{U_l} + \mu_{U_h} = M(P_{U_l} \cup P_{U_h}) = M(\{\chi \in X, \rho > r_2\}) = \int_{\{\rho > r_2\}} m(\chi) d\chi$. Then $p_l = 1$ implies that $nt > \mu_{U_l} + \mu_{U_h} = \int_{\{\rho > r_2\}} m(\chi) d\chi$.

Let us now assume that $n_t \geq \int_{\{\rho > r_2\}} m(\chi) d\chi$. We know that $p_l > 0$. Therefore, for $\rho < r_2$, $\pi_A > \pi_{U_l}, \pi_{U_h}$. This implies that $\{\chi \in X, \rho < r_2\} \cap (P_{U_l} \cup P_{U_h}) = \emptyset$, or $P_{U_l} \cup P_{U_h} \subset (X \setminus \{\chi \in X, \rho < r_2\}) = \{\chi \in X, \rho > r_2\}$. Hence, $\mu_{U_l} + \mu_{U_h} = M(P_{U_l} \cup P_{U_h}) \leq M(\{\chi \in X, \rho > r_2\}) = \int_{\{\rho > r_2\}} m(\chi) d\chi \leq n_t$. Therefore, $\mu_{U_l} + \mu_{U_h} \leq n_t$ and $p_l = 1$.

That concludes the proof of the theorem.

D There exists some values of our parameters with $r_1 > r_2$ for which the revenue of the ride sharing platform strictly increases

Let us consider the parameters $\omega = 0.4, c = 0.1, k = 1, \lambda_t = 6, r_1 = 0.6, r_2 = 0.1, n_t = 0.2$ and a uniform distribution of players over $[0,1]^2$.

From the original paper[2], we know that without taxis, for $r_1 > \omega + c$, there are no users at equilibrium (like in figure 1). Therefore, without taxis, the revenue of the ride sharing platform is null.

Let us now consider the taxi platform. With such parameters, the payoffs are:

- $\pi_A = \frac{\nu}{1.2}$
- $\pi_D = \frac{\nu}{1.2} + \rho - 0.5 + \bar{p} \times 0.6$
- $\pi_S = \rho - 1 + \bar{p} \times 3.6$
- $\pi_{U_l} = \frac{\nu}{1.2} + p_l(\rho - 0.1)$
- $\pi_{U_h} = \frac{\nu}{1.2} + p_l(\rho - 0.1) + (1 - p_l) p_h (\rho - 0.6)$
- $p_l = \frac{n_t}{\mu_{U_l} + \mu_{U_h}} \wedge 1 > 0$ (because $n_t > 0$)
- $p_h = \frac{k(\mu_D + \lambda_t \mu_S)}{(1 - p_l)\mu_{U_h}} \wedge 1$
- $\bar{p} = \frac{(1 - p_l)\mu_{U_h}}{k(\mu_D + \lambda_t \mu_S)} \wedge 1$
- The revenue of the ride sharing platform: $R = r_1 \times (1 - p_l)\mu_{U_h} \times p_h$

From the previous theorem, we know that $p_l < 1$ because $n_t = 0.2 < 0.9 = \int_{\{\rho > 0.1\}} m(\chi) d\chi$ and $r_2 = 0.1 < 0.5 = \omega + c$.

In addition, if $\bar{p} = 1$, then $\pi_D = \frac{\nu}{1.2} + \rho + 0.1 > \frac{\nu}{1.2} + \rho - 0.1 \geq \pi_{U_l}, \pi_{U_h}$. Therefore, there are no users ($\mu_{U_l} = \mu_{U_h} = 0$). This leads us to $p_l = 1$. Hence the contradiction. In fact, $\bar{p} < 1$ and $p_h = 1$.

Let us now assume towards a contradiction that $\mu_{U_h} = 0$. Then, $\bar{p} = 0$ and $\mu_S = 0$ (because $k\bar{p}r_1 - c < 0$). Therefore, there are only abstinent, drivers and low users, who get the same utility from the wage ν. Therefore, the boundaries between players opting for these strategies only depend on ρ. The payoffs of interest now are:

- $\pi_A = \frac{\nu}{1.2}$
- $\pi_D = \frac{\nu}{1.2} + \rho - 0.5$
- $\pi_{U_l} = \frac{\nu}{1.2} + p_l(\rho - 0.1)$

From this, we can deduce that:

- $\pi_A - \pi_D > 0 \Leftrightarrow \rho < 0.5$
- $\pi_A - \pi_{U_l} > 0 \Leftrightarrow \rho < 0.1$
- $\pi_{U_l} - \pi_D > 0 \Leftrightarrow \rho < \frac{0.5-0.1p_l}{1-p_l}$

Because we have a uniform distribution, we can now compute p_l. Indeed: $\mu_{U_l} = M(\{\chi \in X, 0.1 < \rho < \frac{0.5-0.1p_l}{1-p_l}\}) = \frac{0.5-0.1p_l}{1-p_l} - 0.1 = \frac{0.4}{1-p_l}$. Then:

$$p_l = \frac{n_t}{\mu_{U_l}} = (1-p_l)\frac{0.2}{0.4} \Leftrightarrow p_l = \frac{1}{3}$$

Now, we have (because $p_h = 1$): $\pi_{U_h} = \frac{\nu}{1.2} + \rho - \frac{7}{30} > \frac{\nu}{1.2} + \rho - 0.5 = \pi_D$. Therefore, the high users dominate the drivers. Hence, the contradiction. In fact, $\mu_{U_h} > 0$.

Therefore, the revenue of the ride sharing platform $R = r_1 \times (1-p_l)\mu_{U_h} \times p_h > 0$. This proves that the revenue of the ride sharing platform can strictly increase even if $r_1 > r_2$.