Conway's surreal numbers
A particular case of Combinatorial Game Theory

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1. General background : combinatorial games
   - Definition
   - Conway Induction
   - Classifying games
   - Adding games
   - The GROUP of games

2. A particular kind of games : surreal numbers
   - Surreal numbers : definition
   - Multiplying numbers
   - The FIELD of numbers

Definition

Definition (Combinatorial game)

1. Let L and R be two sets of games. Then the ordered pair $\{L|R\}$ is a combinatorial game.
2. **(Descending Game Condition)** There is no infinite sequence of combinatorial games $(G_i)_{i\in\mathbb{N}} :\equiv (\{L_i|R_i\})_{i\in\mathbb{N}}$ such that $\forall i \in \mathbb{N}$, $G_{i+1} \in L_i \cup R_i$.

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Theorem (Conway induction)

Let $\mathcal{P}$ be a property which games might have, such that any game G has property $\mathcal{P}$ whenever all left and right options of G have this property. Then every game has property $\mathcal{P}$.

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$\mathcal{P}(G)$ holds whenever $\mathcal{P}(G^L)$ and $\mathcal{P}(G^R)$ hold. By Conway induction, then $\mathcal{P}(G)$ hols for any game G.

□

Conway Induction

Theorem (Generalised Conway induction)

For any $n \in \mathbb{N}^*$, let $\mathcal{P}$ be a property which any n-tuple of games might have. Suppose that $\mathcal{P}(G_1, \cdots, G_i, \cdots, G_n)$ holds whenever, for all $i \in 1, n$ and for all $G_i' \in L_i \cup R_i$ (where $G_i \equiv \{L_i | R_i\}$), $\mathcal{P}(G_1, \cdots, G_i', \cdots, G_n)$ holds. Then $\mathcal{P}(G_1, \cdots, G_n)$ holds for every n-tuple of games.

Conway Induction

Theorem (Targeted Conway induction)

Let $\mathcal{P}_{\mathcal{C}}$ be a hereditary property that games might have, and lLet then :

$$\mathcal{C} := \{G \text{ games} \mid \mathcal{P}_{\mathcal{C}}(G) \text{ is true}\}$$

Finally, let $\mathcal{P}$ be a property wich games in $\mathcal{C}$ might have, such that any game $G \in \mathcal{C}$ has property $\mathcal{P}$ whenever all left and right options of G have property $\mathcal{P}$. Then $\mathcal{P}$ holds for every game $G \in \mathcal{C}$.

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Definition (Order of games)

Let G be a game. Then :

1. $G \geq 0$ unless there is a right option $G^R \leq 0$ of G.
2. $G \leq 0$ unless there is a left option $G^L \geq 0$ of G.

Classifying games

Definition (Order of games, outcome classes)

	$\exists G^L \geq 0$	$\nexists G^L \geq 0$
$\exists G^R \leq 0$	$G \parallel 0$	$G < 0$
$\nexists G^R \leq 0$	$G > 0$	$G = 0$

Which means, with words :

		If *Left* starts then...	
		Left wins.	*Right* wins
If *Right* starts, then...	*Right* wins.	$G \parallel 0$	$G < 0$
	Left wins.	$G > 0$	$G = 0$

Adding games

Definition (Sum of games)

Let G and H be two games. Then the set of left options of $G+H$ is :

$$\left(\bigcup_{i\in I}(G^{L_i}+H)\right)\cup\left(\bigcup_{i'\in I'}(G+H^{L_{i'}})\right)$$

and the set of right options of $G+H$ is :

$$\left(\bigcup_{j\in J}(G^{R_j}+H)\right)\cup\left(\bigcup_{j'\in J'}(G+H^{R_{j'}})\right)$$

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So :

$$1 + 1 \equiv \{1; 1|\} \equiv \{1|\} :\equiv 2$$

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Let G be a game. Then :

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Definition (Substraction)

Let G and H be two games. Then we define :

$$G - H \equiv G + (-H)$$

Adding games

Definition (Equality of games)

Let G and H be two games. Then :

$$G = H \text{ if and only if } G - H = 0.$$

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Definition (Equality of games)

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- Equivalence classes of equal games

Theorem

Equal games are in the same outcome class.

The GROUP of games

Property

Addition :

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1. *Is compatible with the equivalence relation of equality : if* $G = G'$ *and* $H = H'$, *then* $G + H = G' + H'$ *and* $G = -G'$.

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4. *Has* $0 \equiv \{|\}$ *as zero element* $G + 0 \equiv G$.

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3. *Is commutative :* $G + H \equiv H + G$.
4. *Has* $0 \equiv \{|\}$ *as zero element* $G + 0 \equiv G$.
5. *Is such that the inverse equivalence class of* G *is* $-G$*, for all game* G.

The GROUP of games

Theorem

The equivalence classes formed by equal games form an additive abelian GROUP in which the zero element is represented by any game $G = 0$.

Surreal numbers : definition

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For any game G and for any left option G^L and any right option G^R of G :

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Definition (Surreal number)

Let x be a game. Then x is a surreal number if all left and right options of x are surreal numbers, and if, for all left option x^L and all right option x^R of x, then $x^L < x^R$.

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6. $* \equiv \{0|0\}$ is a game but NOT a number !

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- Behave as expected with comparison.

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As we want xy to remain a number, then we have to impose :

$$(xy)^L < xy < (xy)^R \tag{2}$$

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- $\begin{cases}(x - x^L) > 0 \\ (y - y^L) > 0\end{cases} \Rightarrow (x - x^L)(y - y^L) > 0 \Rightarrow \boxed{xy > xy^L + x^L y - x^L y^L}$

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- $\begin{cases}(x - x^R) < 0 \\ (y - y^R) < 0\end{cases} \Rightarrow (x - x^R)(y - y^R) > 0 \Rightarrow$ $\boxed{xy > xy^R + x^R y - x^R y^R}$

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The FIELD of numbers

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5. *Multiplication is commutative.*
6. *Multiplication associative and distributive uppon addition* ***when considered as an operation on equivalence classes of numbers***
7. *For all number* x*, there exists a number* y *such that* $xy = yx = 1$.

The FIELD of numbers

Theorem

The equivalence classes formed by equal numbers form a totaly ordered FIELD, in which the zero element for addition is represented by any number $x = 0$ and the neutral element for multiplication is represented by any number $y = 1$.