ENS Rennes - Université Paris Diderot

Conway's surreal numbers
A particular case of Combinatorial Game Theory

Intership report

Clémentine Laurens

supervised by
Anatole Khelif

May - June 2018

Abstract

This intership report gives an introduction to Conway's sureal numbers as a particular case of combinatorial game theory. It focuses on showing how intuitive such abstract mathematical objects can become when looked at in the right way. To do so, it gives a step-by-step contruction of surreal numbers, based on intuition about the concept of *game* and analogy with standard mathematical objects. Simultaneously, it takes interest in the structures properties games and numbers have.

TABLE OF CONTENTS

CHAPTER 1

PRELIMINARIES : SET THEORY, CLASSICAL SETS OF NUMBERS

This chapter is not precisely part of the study led during the intership, but it is helpful to understand better Conway's construction of surreal numbers. It does not aim to precise all the foundations of the Mathematics we are going to deal with, but only to remind the reader about the way classical sets of numbers are contructed out from set theory, and about some classical proof methods. No proof will be given in this chapter : for mor details, one can refer to [Kri98, pp. 1-39] and [Rudon, pp. 1-20].

1 Set theory axioms : ZFC

In this paper, we use Zermelo–Fraenkel set theory axioms, including the axiom of choice (ZFC). Zermelo–Fraenkel set theory axioms include the axiom of extentionality, the axiom of pairing, the axiom of union, the axiom of power set, the axiom of infinity, the axiom schema of comprehension, the axiom of foundation and the axiom schema of replacement. To obtain ZFC set theory, one must add the axiom of choice.

2 The set $\mathbb{N}$ of natural numbers

2.1 Construction

The set $\mathbb{N}$ of natural numbers is recursively contructed out from these axioms of set theory as the smallest set (in the meaning of inclusion) verifying the two following properties.

1. $\emptyset \in \mathbb{N}$.
2. $\forall n \in \mathbb{N}$, $n^+ := (n \cup \{n\}) \in \mathbb{N}$.

EXAMPLES 1.2.1. The "first" natural numbers that can be constructed are :

- $0 := \emptyset$
- $1 := 0^+ = 0 \cup \{0\} = \{0\}$
- $2 := 1^+ = 1 \cup \{1\} = \{0, 1\}$
- $3 := 2^+ = 2 \cup \{2\} = \{0, 1, 2\}$
- $\cdots$

2.2 Mathematical induction

THEOREM 1.2.2 (Mathematical induction). *Let $\mathcal{P}$ be a property whose truth value depends on $n \in \mathbb{N}$. Suppose that $\mathcal{P}$ is such that :*

1. ***Initialisation** : $\mathcal{P}(0)$ is true.*
2. ***Induction** : $\forall n \in \mathbb{N}, (\mathcal{P}(n) \Rightarrow \mathcal{P}(n^+))$.*

Then $\forall n \in \mathbb{N}$, $\mathcal{P}(n)$ is true.

3 Ordinal numbers

3.1 Construction

Let α be a set. Then α is said to be an ordinal number if :

1. $(\alpha, \in)$ is a well ordered set : $\forall (x, y) \in \alpha^2$, either $x \in y$, $y \in x$ or $x = y$, and every non-empty subset of α has a least element (for the relation $\in$).
2. $\forall x \in \alpha$, $x \subset \alpha$.

EXAMPLES 1.3.1.

- $\forall n \in \mathbb{N}$, n is an ordinal.
- $\omega = \mathbb{N} = \{1, 2, 3, 4, \cdots\}$ is an ordinal.
- For all ordinal number α, $\alpha^+ := \alpha \cup \{\alpha\}$ is an ordinal. Ordinals of this forms are called *successor ordinals*.
- Some ordinal numbers are not successor ordinals. Such numbers are called *limit ordinals*. The smallest limit ordinal is $\omega = \mathbb{N}$ ("smallest" in the meaning of the relation $\in$).

The following property is very important, and is to be kept in mind to understand some structure properties that will be tackled in the next chapters.

Property 1.3.2. *Ordinal numbers form a proper CLASS.*

3.2 Transfinite induction

THEOREM 1.3.3 (Transfinite induction). *Let $\mathcal{P}$ be a property whose truth value depends on an ordinal number α. Suppose that $\mathcal{P}$ is such that :*

$$(\forall \beta \in \alpha, \mathcal{P}(\beta)) \Rightarrow \mathcal{P}(\alpha).$$

Then $\mathcal{P}(\alpha)$ is true for every ordinal number α.

Remark 1.3.4. The major difference between mathematical induction and transfinite induction is that the latter does not need explicit initialisation.

4 The set $\mathbb{R}$ of real numbers

4.1 The sets $\mathbb{Z}$ and $\mathbb{Q}$

In this section, we assume that addition and multiplication on natural numbers are known.

The set $\mathbb{Z}$ of relative numbers is built out from $\mathbb{N}$ as described below.

1. Let $\mathcal{R}$ be an equivalence relationship on $\mathbb{N} \times \mathbb{N}$ defined by : $\forall ((a, b), (c, d)) \in (\mathbb{N} \times \mathbb{N})^2$, $(a, b)\mathcal{R}(c, d) \iff a + d = c + b$.
2. $\mathbb{Z}$ is the quotient of $\mathbb{N} \times \mathbb{N}$ by $\mathcal{R}$, and the equivalence class of (a, b) represents the integer $(a - b)$.

The set $\mathbb{Q}$ of rational numbers is built out from $\mathbb{N}$ and $\mathbb{Z}$ as described below.

1. Let $\mathcal{R}'$ be an equivalence relationship on $\mathbb{Z} \times \mathbb{N}^*$ defined by : $\forall ((a, b), (c, d)) \in (\mathbb{Z} \times \mathbb{N}^*)^2$, $(a, b)\mathcal{R}'(c, d) \iff ad = bc$.
2. $\mathbb{Q}$ is the quotient of $\mathbb{Z} \times \mathbb{N}^*$ by $\mathcal{R}'$, and the equivalence class of (a, b) represents the rational number $\frac{a}{b}$.

4.2 Construction of $\mathbb{R}$ with Dedekind cuts

The set $\mathbb{R}$ of real numbers can be built out from the set $\mathbb{Q}$ in various ways. The one we are interested in is described below.

1. Let C be a subset of $\mathbb{Q}$ such that :
   (i) $C \neq \emptyset$.
   (ii) $C \neq \mathbb{Q}$.
   (iii) $\forall x \in C$, $\forall y \in \mathbb{Q} \backslash C$, then $y > x$.
   (iv) C does not have a biggest element.

   Then the couple $(C\ ,\ \mathbb{Q} \backslash C)$ is said to be a *Dedekind cut*.
2. $\mathbb{R}$ is defined by : $\mathbb{R} = \{C \subset \mathbb{Q} \mid (C, \mathbb{Q} \backslash C) \text{ is a Dedekind cut}\}$.

Remark 1.4.1. The fundamental idea lying behind this construction is that any real number x can be defined by the set of rational numbers q_- such that $q_- < x$ and the set of rational numbers q_+ such that $q_+ > x$.

Examples 1.4.2.

- For all $a \in \mathbb{Q}$, $\{x \in \mathbb{Q} \mid x < a\}$ is the real number a.
- $\{x \in \mathbb{Q}_+ \mid x^2 < 2\} \cup \mathbb{Q}_-$ is the real number called $\sqrt{2}$.

These definitions and construction methods should be kept in mind while reading the rest of this report. Indeed, some mathematical objects defined and studied below might look familiar and will be easier to understand if compared to the ones described in this first chapter.

CHAPTER 2

GENERAL BACKGROUND : COMBINATORIAL GAMES

Conway's surreal numbers seem to "appear" quite naturally while studying combinatorial games. In fact, they *are*, by definition, a particular kind of such games. Before defining and studying them more precisely, this chapter aims to introduce the general notions, tools and properties from combinatorial game theory that will be needed further on. It also tries to highlight the fact that, although this theory is really powerfull and can reach a high level of abstraction, it is incredibly intuitive, when looked at in the right way, .

1 Combinatorial games : definition

The games we are interested in are similar to Go : two-person strictly competitive[1] games with complete information[2]. In particular, this means that no chance can influence the game. We also assume that the players take their turn one after the other, and that the game cannot end in a draw.

The fundamental idea used to give the definition below is that a game can be seen as a succession of playing situations, called *positions* of the game, for each of which a set of legal *moves* is defined for each player. Executing one of those legal moves brings the game from one position to another. The positions that can be reached in one move from the starting position of the game on are called *options* of the game.

Moreover, our intuition tells us that a game *must* end at some point. We will consider that the player who cannot play anymore, because they have no legal move left in the situation reached, loses the game : the winner is the one who makes the last move. This convention is called the *Normal Play Convention*.

Remark 2.1.1. Another convention that could be used is the one in which the player who makes the last move loses the game. This is called the *Misère Play Convention*. WARNING : using this convention would change completely the theory described below (see **??**).

Notation 2.1.2. The two players are usualy called *Left* and *Right*. By convention, we will use this vocabulary in the rest of this document.

[1] Which means that the two players play one *against* the other : they have opposite goals.

[2] Which means that each player knows, at any moment of the game, what playing options they have, what playing options their opponent has, and the "history" of the game (its evolution from the beginning on).

The following definition formalises what has just been described here above.

Definition 2.1.3 (Combinatorial game).

1. *Let L and R be two sets of games. Then the ordered pair $\{L|R\}$ is a combinatorial game.*
2. ***(Descending Game Condition)*** *There is no infinite sequence of combinatorial games $(G_i)_{i\in\mathbb{N}} :\equiv (\{L_i|R_i\})_{i\in\mathbb{N}}$ such that $\forall i \in \mathbb{N}$, $G_{i+1} \in L_i \cup R_i$.*

Notation 2.1.4. In the rest of this document, "combinatorial game" will be abbreviated in "game".

Notation 2.1.5. We will use capital letters G, H, K,... to denote numbers.

Definition 2.1.6 (Options, positions). *Let $G :\equiv \{L|R\}$ be a game.*

1. *The elements of L and R are called left resp. right options of the game G.*
2. *The positions of G are G and all the positions of any option of G.*

Definition 2.1.7 (Identical games). *Let G and H be two games. Then G and H are said to be identical, written $G \equiv H$, if and only if G and H have the same left and right options.*

Remark 2.1.8. The notation "$G = H$" ("G *equals* H") has to be set aside for another property, which will be defined later on.

Remark 2.1.9. In a game $G :\equiv \{L|R\}$, L (resp. R) is the set of positions that can be reached, from the starting position of G on, through a legal move for the *Left* (resp. *Right*) player. Note that those sets might be empty or infinite, or even uncountable.

Remark 2.1.10. This definition shows that, from a mathematical point of view, all that matters is the starting position of the game and the sets of options for both players. The idea lying behind this is that, to play a game, the player whose turn it is has to chose one of their available options. By doing so, both players end up in a "new" game, whose starting position is the one chosen by the firts player : the process then repeats, with the players taking their turn one after the other, untill one of the players has no option left.

Remark 2.1.11. The Descending Game Condition (DGC) guarantees that any game will end at some point. Nevertheless, the number of moves until the end of the game usually cannot be uniformly bounded.

Remark 2.1.12. Technically, this definition does not tell us what games *are* : it only explains how to build a new game out from two sets of game already known. In fact, it is a recursive definition *without base* : it does not need one (just like transfinite induction does not need initialisation...) !

EXAMPLES 2.1.13. "Before" any set of games is created, the only set of games is the empty set. Therefore :

$$0 :\equiv \{\emptyset|\emptyset\}$$

is a game, and we now have two sets of games : $\emptyset$ and $\{0\}$. Therefore :

$$1 :\equiv \{\{0\}|\emptyset\} \qquad -1 :\equiv \{\emptyset|\{0\}\} \qquad * :\equiv \{\{0\}|\{0\}\}$$

are games.

Notation 2.1.14. In the rest of this document, if $G :\equiv \{L|R\}$ is a game with $L := \{G^{L_1}, G^{L_2}, \cdots\}$ and $R := \{G^{R_1}, G^{R_2}, \cdots\}$, then we might simplifiy the notation as follows :

$$G :\equiv \left\{G^{L_1}, G^{L_2}, \cdots \middle| G^{R_1}, G^{R_2}, \cdots\right\}$$

WARNING : we **do not** mean to indicate that L and R are countable or non-empty !

EXAMPLES 2.1.15. The four games decribes in the previous example can now be re-written as follows :

$$0 :\equiv \{\ |\ \} \qquad 1 :\equiv \{0|\ \} \qquad -1 :\equiv \{\ |0\} \qquad * :\equiv \{0|0\}$$

2 A fundamental proof method in game theory : Conway induction

Now that games are defined as proper mathematical objects, game theory aims to study their properties. To do so, the theorem below is one of the most powerful existing tools.

THEOREM 2.2.1 (Conway induction). *Let $\mathcal{P}$ be a property which games might have, such that any game G has property $\mathcal{P}$ whenever all left and right options of G have this property. Then every game has property $\mathcal{P}$.*

Proof. Let $\mathcal{P}$ be such a property. Suppose that there is a game G which does not satisfy $\mathcal{P}$. If all left and right options of G satisfied $\mathcal{P}$, then G would also, by hypothesis. So there is an option G' of G which does not satisfy $\mathcal{P}$. Repeating this argument inductively, we get an infinite sequence $G, G', G'', \cdots$ of games, each an option of its predecessor, which violates the Descending Game Condition.

□

Remark 2.2.2. This proof requires the Axiom of Choice.

Remark 2.2.3. Note that, just like transfinite induction, Conway induction does not need an explicit initialisation.

This proves that the Descending Game Condition implies Conway induction. It is interesting to notice that the converse is also true.

THEOREM 2.2.4. *Conway induction implies the Descending Game Condition.*

Proof. Suppose that Conway induction is true ans consider the property $\mathcal{P}(G)$: "There is no infinite chain of games $G, G', G'', \cdots$, starting with G so that every game is followed by one of its options". This property clearly is of the kind described by Conway induction, so it holds for every game.

□

Corollary 2.2.5. *Conway induction is equivalent to the Descending Game Condition.*

A generalised version of the above theorem is sometimes necessary.

Theorem 2.2.6 (Generalised Conway induction). *For any $n \in \mathbb{N}^*$, let $\mathcal{P}$ be a property which any n-tuple of games might have. Suppose that $\mathcal{P}(G_1, \cdots, G_i, \cdots, G_n)$ holds whenever, for all $i \in [\![1, n]\!]$ and for all $G_i' \in L_i \cup R_i$ (where $G_i \equiv \{L_i|R_i\}$), $\mathcal{P}(G_1, \cdots, G_i', \cdots, G_n)$ holds. Then $\mathcal{P}(G_1, \cdots, G_n)$ holds for every n-tuple of games.*

Proof. The proof given for theorem 1 on page 9, adapts easily. Let $n \in \mathbb{N}^*$ and suppose that there is a n-tuple of games $G_1, G_2, \cdots, G_n$ such that $\mathcal{P}(G_1, \cdots, G_n)$ is false. Then $\exists i \in [\![1, n]\!]$ and $\exists G_i' \in L_i \cup R_i$ (where $G_i \equiv \{L_i|R_i\}$) such that $\mathcal{P}(G_1, \cdots, G_i', \cdots, G_n)$ is false. Repeating this argument inductively, we get an infinite sequence $(G_{1,k}, \cdots, G_{n,k})_{k\in\mathbb{N}}$ such that, $\forall i \in [\![1, n]\!]$, $\forall k \in \mathbb{N}$, $G_{i,k+1} \in \{G_{i,k}\} \cup L_{i,k} \cup R_{i,k}$ (where $G_{i,k} \equiv \{L_{i,k}|R_{i,k}\}$). So there exists $j \in [\![1, n]\!]$ and $\Phi : \mathbb{N} \to \mathbb{N}$ a strictly increasing application such that $(G_{j,\Phi(k)})_{k\in\mathbb{N}}$ is an infinite sequence of games, each an option of its predecessor. This violates the Descending Game Condition.

□

Remark 2.2.7. This proof also requires the Axiom of Choice.

The following version of Conway induction will also be usefull and may simplify some proofs later on.

Theorem 2.2.8 (Targeted Conway induction). *Let $\mathcal{P}_\mathcal{C}$ be a hereditary property that games might have : for all game $G \equiv \{L|R\}$, $\mathcal{P}_\mathcal{C}(G) \Rightarrow \left(\forall G^L \in L, \forall G^R \in R, \mathcal{P}(G^L) \text{ and } \mathcal{P}(G^R)\right)$*

Let then :

$$\mathcal{C} := \{G \text{ games} \mid \mathcal{P}_\mathcal{C}(G) \text{ is true}\}$$

Finally, let $\mathcal{P}$ be a property wich games in $\mathcal{C}$ might have, such that any game $G \in \mathcal{C}$ has property $\mathcal{P}$ whenever all left and right options of G have property $\mathcal{P}$. Then $\mathcal{P}$ holds for every game $G \in \mathcal{C}$.

Proof. Let $\mathcal{C}$ be such a class of games and let $\mathcal{P}$ be such a property. Suppose that there is a game $G \in \mathcal{C}$ that does not satisfy $\mathcal{P}$. Then there is an option G' od G that does not satisfy $\mathcal{P}$, **and** $G' \in \mathcal{C}$ (by definition of $\mathcal{C}$). Repeating this argument inductively, we get an infinite sequence $G, G', G'', \ldots$ of games from the class $\mathcal{C}$, which violates the Descending Game Condition.

□

Example 2.2.9. As an example of how Conway induction works and how efficient it is, we are going to use it to prove the following statement.

Property 2.2.10. *The positions of a game form a set.*

Indeed, for any game $G \equiv \left\{G^L \middle| G^R\right\}$, let's consider the property :

$$\mathcal{P}(G) : \text{" } \textit{The positions of } G \textit{ form a set.}\text{"}$$

Then $\mathcal{P}(G)$ holds whenever $\mathcal{P}(G^L)$ and $\mathcal{P}(G^R)$ hold, as the positions of G are G and all the positions G^L and G^R. By Conway induction, then $\mathcal{P}(G)$ hols for any game G.

□

3 Classifying games

Game theory is mainly about predicting whether one of the players can enforce a win or not. This question thus needs to be formalised with a mathematical language. To do so, we define the following outcome classes.

Let G be a game.

(i) If the second player can enforce a win, no matter who they are, then G will be said to be *equal to zero* (or *a zero game*) : $G = 0$.

(ii) If the first player can enforce a win, no matter who they are, then G will be said to be *fuzzy to zero* : $G \parallel 0$.

(iii) If *Left* can enforce a win, no matter who starts, then G will be said to be *positive* : $G > 0$.

(iv) If *Right* can enforce a win, no matter who starts, then G will be said to be *negative* : $G < 0$.

From this classification on, we can also define :

(1) $G \geq 0$ if and only if $G > 0$ or $G = 0$: *Left* can enforce a win at least if he is the second player.

(2) $G \leq 0$ if and only if $G < 0$ or $G = 0$: *Right* can enforce a win at least if he is the second player.

(3) $G \rhd 0$ ("G is greater than or fuzzy to zero") if and only if $G > 0$ or $G \parallel 0$: *Left* can enforce a win at least if he is the first player.

(4) $G \lhd 0$ ("G is smaller than or fuzzy to zero") if and only if $G < 0$ or $G \parallel 0$: *Right* can enforce a win at least if he is the first player.

Remark 2.3.1. WARNING : What is questioned here is the *possibility*, for one of the players, to enforce a win. Saying "*Left* can enforce a win" does not mean that they *will* win the game ! Saying "*Left* can enforce a win" just means that there exists a winning strategy for *Left*. But if they do not play *well* (i.e. if they do not follow this winning strategy) *Right* might be able to win ! Moreover, saying "there exists a winning strategy for *Left*" does not even mean that such an explicit strategy is known…

Notation 2.3.2. In the rest of this document, we will consider - as always in game theory - that both players play perfectly well : they always do the best choices for them (in particular, if there exists a winning strategy for them, they will follow it). Therefore, we might abuse "*Left* (resp. *Right*) can enforce a win" as "*Left* (resp. *Right*) wins".

Now let's formalise the notions previously introduced.

Using the previous notations, if $G \geq 0$ then *Left* wins as second player, which means that *Right* has no good opening move. A good opening move for Right would be a right option G^R of G such that *Right* would win as second player in G^R, i.e. a right option $G^R \leq 0$ of G. So if $G \geq 0$, then there is no right option $G^R \leq 0$ of G. The same line of reasoning brings that if $G \leq 0$, then there is no left option $G^L \geq 0$ of G.

This brings the formal definition below.

Definition 2.3.3 (Order of games)**.** *Let G be a game. Then :*

(1) $G \geq 0$ unless there is a right option $G^R \leq 0$ of G.

(2) $G \leq 0$ unless there is a left option $G^L \geq 0$ of G.

Remark 2.3.4. Once again, this is a recursive definition without base.

Remark 2.3.5. The Descending Game Condition guarantees that theses notions are well-defined (i.e. that the recursion caused by the definition ends at some point). Indeed, if there was a game G for which $G \geq 0$ or $G \leq 0$ were not well-defined, then there would be an option G' of G for which $G' \geq 0$ or $G' \leq 0$ were not well defined. Repeating the process, this would eventually give an infinite sequence of games, each an option of its predecessor, which would enventually violate the DGC.

Examples 2.3.6.

- As $0 \equiv \{|\}$, we have $0 \geq 0$ **and** $0 \leq 0$.
- As $1 :\equiv \{0|\}$, we have $1 \geq 0$.
- As $-1 :\equiv \{|0\}$, we have $-1 \leq 0$.
- As $* :\equiv \{0|0\}$ and both $0 \geq 0$ and $0 \leq 0$, we have **neither** $* \geq 0$ **nor** $* \leq 0$.

To link these notions with the different outcome classes intuitively defined here above, we also give the following definitions.

Definition 2.3.7 (Order of games, outcome classes)**.** *Let G be a game. Then :*

(i) $G = 0$ if $G \geq 0$ and $G \leq 0$.

(ii) $G > 0$ if $G \geq 0$ and not $G \leq 0$.

(iii) $G < 0$ if $G \leq 0$ and not $G \geq 0$.

(iv) $G \parallel 0$ if neither $G \leq 0$ nor $G \geq 0$.

and

(1) $G \rhd 0$ if not $G \leq 0$.

(2) $G \lhd 0$ if not $G \geq 0$.

EXAMPLES 2.3.8.

- According to the previous list of examples, we have $0 = 0$ and $* \parallel 0$.
- As $1 :\equiv \{0|\}$ and $0 \geq 0$, we have $1 > 0$.
- As $-1 :\equiv \{|0\}$ and $0 \leq 0$, we have $-1 < 0$.

Remark 2.3.9. The fundamental idea to keep in mind is that **"the more a game is positive, the more *Left* has a strategic advantage upon *Right*"** and conversely.

Remark 2.3.10. The following table illustrates the fact that those definitions enable to classify completely the games : every game G is part of exactly one of the four outcome classes $G = 0$; $G \parallel 0$; $G < 0$ or $G > 0$.

	$\exists G^L \geq 0$	$\nexists G^L \geq 0$
$\exists G^R \leq 0$	$G \parallel 0$	$G < 0$
$\nexists G^R \leq 0$	$G > 0$	$G = 0$

Which means, with words :

		If *Left* starts then...	
		Left wins.	*Right* wins
If *Right* starts, then...	*Right* wins.	$G \parallel 0$	$G < 0$
	Left wins.	$G > 0$	$G = 0$

Remark 2.3.11. WARNING : $\neg(G \geq 0) \iff (G \lhd 0)$ (and NOT : $\neg(G \geq 0) \iff (G < 0)$) !

4 Adding games

4.1 Definitions

What happens when you put two games (identical or not) next to each other and allow each player, when it is their turn, to play in one of the two games according to their choice, leaving the other game unchanged, untill one of the players has no option left in any of the two games ?

This situation enables to intoduce the concept of the *sum of games*.

What happens if you switch the sets of options for *Left* and *Right* in every position of the game[3] ?

This situation enables to intoduce the concept of the *negative of a game*.

These ideas lead to the following formal definitions.

Definition 2.4.1 (Sum of games). *Let* $G \equiv \left\{(G^{L_i})_{i\in I}\middle|(G^{R_j})_{j\in J}\right\}$ *and* $H \equiv \left\{(H^{L_{i'}})_{i'\in I'}\middle|(H^{R_{j'}})_{j'\in J'}\right\}$ *be two games (where* I, J, I' *and* J' *can be any index sets). Then the sum of* G *and* H *is defined as :*

$$G + H \equiv \left\{(G^{L_i} + H)_{i\in I}\ ,\ (G + H^{L_{i'}})_{i'\in I'}\ \middle|\ (G^{R_j} + H)_{j\in J}\ ,\ (G + H^{R_{j'}})_{j'\in J'}\right\}$$

That is the set of left options of $G + H$ *is :*

[3] For instance, in the Go game, the players would swith colors.

$$\left(\bigcup_{i\in I}(G^{L_i}+H)\right)\cup\left(\bigcup_{i'\in I'}(G+H^{L_{i'}})\right)$$

and the set of right options of $G+H$ is :

$$\left(\bigcup_{j\in J}(G^{R_j}+H)\right)\cup\left(\bigcup_{j'\in J'}(G+H^{R_{j'}})\right)$$

Remark 2.4.2. If G and/or H has no left (resp. right) option, some of the unions in this definition might be empty. Therefore, there might be no left (resp. right) option of $G+H$ at all, or they might all be of the form $G^{L_i}+H$ or all of the form $G+H^{L_{i'}}$ (resp. they might all be of the form $G^{R_j}+H$ or all of the form $G+H^{R_{j'}}$).

Example 2.4.3. Remind that we defined the game $1 :\equiv \{0|\}$ in the first section of this chapter. Therefore :

$$1+1\equiv\{0|\}+\{0|\}\equiv\{0+1;1+0|\}$$

Yet :

$$0+1\equiv\{|\}+\{0|\}\equiv\{0+0|\}\equiv\{(\{|\}+\{|\})|\}\equiv\{(\{|\})|\}\equiv\{0|\}\equiv 1$$

and a similar development gives :

$$1+0\equiv 1$$

So :

$$1+1\equiv\{1;1|\}\equiv\{1|\}:\equiv 2$$

Definition 2.4.4 (Negative of a game)**.** *Let $G\equiv\left\{(G^{L_i})_{i\in I}\middle|(G^{R_j})_{j\in J}\right\}$ be a game (where I can be any index set). Then the negative $-G$ of G is defined as :*

$$-G\equiv\left\{(-G^{R_j})_{j\in J}\middle|(-G^{L_i})_{i\in I}\right\}$$

Example 2.4.5. Remind that we also defined the game $-1 :\equiv \{|0\}$ in the first section of this chapter. We can now justify this notation, as -1 is truly the negative of 1 :

$$-1\equiv\{|-0\}\equiv\{|-(\{|\})\}\equiv\{|(\{|\})\}\equiv\{|0\}$$

Definition 2.4.6 (Substraction)**.** *Let G and H be two games. Then we define :*

$$G-H\equiv G+(-H)$$

Remark 2.4.7. Substracting a game H from a game G corresponds to putting the game G next to the negative of the game H (i.e. the game H in which the sets of options for *Left* and *Right* have been switched) and allowing each player, when it is their turn, to play in one of the two games according to their choice, leaving the other game unchanged.

Remark 2.4.8. All these are again inductive definitions without base.

4.2 Properties of addition

The following properties are intuitively true, and can quite easily be "translated" into words, if one keeps in mind the reasons why we imagined the notions of sum and negative of a game (playing two games at once, and switching roles). Nevertheless, they are a first step into mathematical abstraction in game theory : what they truly say is that the addition here above defined behaves like the classical addition on real numbers.

Property 2.4.9. *The sum of two games is a game.*

Proof. Let G and H be two games. Consider the property :

$$\mathcal{P}(G,H) : \text{"}G+H \text{ is a game"}$$

This property is of the kind defined in Generalised Conway induction. Indeed, by definition of the sum of two games, $\mathcal{P}(G,H)$ holds whenever all $\mathcal{P}(G^L,H)$, $\mathcal{P}(G,H^L)$, $\mathcal{P}(G^R,H)$ and $\mathcal{P}(G,H^R)$ hold, for any left (resp. right) option G^L (resp. G^R) of G and any left (resp. right) option H^L (resp. H^R) of H. Consequently, $\mathcal{P}(G,H)$ holds for any games G and H.

□

Property 2.4.10. *The negative of a game is a game.*

Proof. Let G be a game. Consider the property :

$$\mathcal{P}(G) : \text{"}-G \text{ is a game"}$$

Then, by definition of the negative of a game, $\mathcal{P}(G)$ holds whenever $\mathcal{P}(G^L)$ and $\mathcal{P}(G^R)$ hold for any left (reps. right) option G^L (resp. G^R) of G. So $\mathcal{P}$ is of the kind defined in Conway induction. Therefore, $\mathcal{P}(G)$ holds for any game G.

□

Another possible proof of this property is developped below. It illustrates the equivalence - proved in the second section of this chapter - between Conway induction and the Descending Game Condition.

Proof. Let G be a game. Suppose that $-G$ is not a game. Then there is an option of G whose negative is not a game either. Reapeting this process, we can build an infinite sequence og games, each an option of its predecessor, which violates the Descending Game Condition.

□

Corollary 2.4.11. *The difference of two games is a game.*

Property 2.4.12. *For any game G, $-(-G) \equiv G$*

Proof. This property is proved by Conway induction, by considering the property :

$$\mathcal{P}(G) : \text{"}-(-G) \equiv G\text{"}$$

$\square$

Property 2.4.13. *For any games G and H, $-(G + H) \equiv (-G) + (-H)$.*

Proof. This property is proved by Generalised Conway induction, by considering the property :

$$\mathcal{P}(G, H) : \text{"}-(G + H) \equiv (-G) + (-H)\text{"}$$

$\square$

Property 2.4.14. *Addition is associative : for all games G, H and K, then $(G+H)+K \equiv G + (H + K)$.*

Proof. This property is, once again, proved by Generalised Conway induction, by considering the property :

$$\mathcal{P}(G, H, K) : \text{"}(G + H) + K \equiv G + (H + K)\text{"}$$

$\square$

Property 2.4.15. *Addition is commutative : : for all games G and H, then $G+H \equiv H+G$.*

Proof. We use Generalised Conway induction, with the property :

$$\mathcal{P}(G, H) : \text{"}G + H \equiv H + G\text{"}$$

$\square$

Property 2.4.16. *Addition has $0 \equiv \{|\}$ as zero element.*

Proof. We use Conway induction, considering the property :

$$\mathcal{P}(G) : \text{"}G + 0 \equiv 0 + G \equiv G\text{"}$$

□

Remark 2.4.17. None of the above properties can be proved without Conway induction. Indeed, "classical" mathematical induction needs an induction base which does not exist with games, and none of those results can be proved directly, as they all depend on operations on *options* of games. For instance, for any game $G \equiv \left\{G^L \middle| G^R\right\}$:

$$G + 0 \equiv \left\{G^L \middle| G^R\right\} + \{|\} \equiv \left\{G^L + 0 \middle| G^R + 0\right\}$$

So to prove that $G + 0 \equiv G$, one *has to* prove that both $G^L + 0 \equiv G^L$ and $G^R + 0 \equiv G^R$: this is what Conway induction enables to formalise.

5 Comparing games

According to paragraph 4.2 (corrolary 11 on page 15), for any games G and H, the difference $G - H$ is a game. Therefore, $G - H$ can be classified as in the third section of this chapter. This idea is what will enable us to *compare* games.

As explained above, game theory is mainly about undestanding who can enforce a win, in a game. Therefore, we would like the notion of *equality of games* to reflect similar situations in two games. In other words, we would like two games to be equal when the player who has a strategic advantage upon his opponent is the same one in those two games. This is what brings the following definition.

Definition 2.5.1 (Equality of games). *Let G and H be two games. Then we define :*

$$G = H \text{ if and only if } G - H = 0.$$

THEOREM 2.5.2. *For any game G, we have the fundamental equality : $G = G$.*

Proof. To make this proof easier to read, we are going to consider games $G \equiv \left\{G^L \middle| G^R\right\}$ such that G has only one left option and one right option. The proof given below adapts immediately to prove the general statement (with generalised Conway induction instead of Conway induction).

$$G - G \equiv \left\{G^L - G; G - G^R \middle| G^R - G; G - G^L\right\}$$

Remind that $G - G = 0$ if and only if $(G - G) \leq 0$ and $(G - G) \geq 0$.

We are going to use Conway induction. Let's consider the property :

$$\mathcal{P}(G) : \text{"}(G - G) \geq 0 \text{ and } (G - G) \leq 0\text{"}.$$

Let's show that this property is of the kind used in Conway induction, i.e. let's show that $\mathcal{P}(G)$ holds whenever $\mathcal{P}(G^R)$ and $\mathcal{P}(G^L)$ hold.

Suppose that $\mathcal{P}(G^R)$ and $\mathcal{P}(G^L)$ hold. Thus, in particular, $(G^L - G^L) \leq 0$ and $(G^R - G^R) \leq 0$.

By definition, $G - G^R \geq 0$ unless there is a right option $(G - G^R)^R$ of $G - G^R$ such that $(G - G^R)^R \leq 0$. But as $(G^R - G^R)$ is a right option of $(G - G^R)$, and as we have just supposed that $(G^R - G^R) \leq 0$, then $\neg\left(G - G^R \geq 0\right)$ i.e. $\boxed{G - G^R \lhd 0}$.

Similarly, we obtain $\boxed{G^L - G \lhd 0}$ from $(G^L - G^L) \leq 0$.

Now remind that $G - G \leq 0$ unless $G - G^R \geq 0$ or $G^L - G \geq 0$. As we have proven that $G - G^R \lhd 0$ and $G^L - G \lhd 0$, then $\boxed{G - G \leq 0}$.

Moreover, as we supposed that $\mathcal{P}(G^R)$ and $\mathcal{P}(G^L)$ holded, thus we also have $(G^L - G^L) \geq 0$ and $(G^R - G^R) \geq 0$, which bring similarly $G - G^L \rhd 0$ and $G^R - G \rhd 0$. Consequently, $\boxed{G - G \geq 0}$.

So $\mathcal{P}(G)$ holds whenever $\mathcal{P}(G^L)$ and $\mathcal{P}(G^R)$ hold. So $\mathcal{P}$ is of the kind used in Conway induction, which proves the theorem.

□

Remark 2.5.3. "$G - G = 0$" is in fact a very intuitive statement. Indeed, "$G - G = 0$" means that the second player has a winning strategy in the game $G - G$, which is the game G put next to the same game in which the sets of options for each player have been switched. And indeed, in this case, if the first player makes any move in G, then the second player has the same move available in $-G$: when their turn comes, they will always be able to copy the first player. The same holds if the first player moves in $-G$, because $-(-G) \equiv G$. Therefore, the second player can never run out of moves before the first does, and the Normal Play Convention guarantees that the second player can enforce a win.

Now remind that we have classified the games such that "the more a game is positive, the more *Left* has a strategic advantage upon *Right*" and conversely. Then intuitively we would like a game G to be greater than another game H when *Left* has a greater advantage uppon *Right* in G than in H, and conversely we would like G to be smaller than H when *Right* has a greater advantage uppon *Left* in G than in H. Besides, we would like two games to be fuzzy to each other when the strategic advantages of the players of the two games cannot be compared properly. This is what the definition below formalises.

Definition 2.5.4. *Let G and H be two games. Then :*

(i) $G > H$ *if and only if* $G - H > 0$

(ii) $G < H$ *if and only if* $G - H < 0$

(iii) $G \parallel H$ *if and only if* $G - H \parallel 0$

6 The GROUP of games

6.1 Addition and outcome classes

The following properties show that addition behaves "as expected" when combined with the definitions of section 5. Most of those results do not seem very exciting, and they are more difficult to think about with intuition on games than the previous ones, but they are necessary in order to obtain the interesting GROUP structure on games, explained in the following paragraph.

Lemma 2.6.1. *For any game G, $G \geq 0 \Rightarrow -G \leq 0$ and $G \leq 0 \Rightarrow -G \geq 0$.*

Proof. We use Conway induction with the property :

$$\mathcal{P}(G) : \text{"}(G \geq 0 \Rightarrow -G \leq 0) \textbf{ and } (G \leq 0 \Rightarrow -G \geq 0)\text{"}.$$

□

Remark 2.6.2. In the previous proof, the two statements have to be proved simultaneously, using one single property for Conway induction (one could not prove one of the statements without the other).

Lemma 2.6.3. *Let G be a game. Then $G = 0 \Rightarrow -G = 0$*

Proof. Let G be a game such that $G = 0$, i.e. such that $G \geq 0$ and $G \leq 0$. Suppose that $\neg(-G = 0)$. Then either $\neg(-G \leq 0)$ or $\neg(-G \geq 0)$, i.e. $-G$ must have either a left option $(-G)^L \geq 0$ or a right option $(-G)^R \leq 0$.

Let's consider that $-G$ has a left option $(-G)^L \geq 0$ (the proof is similar in the other case). Then, by definition of $-G$, G has a right option G^R such that $(-G)^L \equiv -G^R$ which gives $-G^R \geq 0$. Therefore, according to lemma 1 on page 18, $G^R \leq 0$ which gives $\neg(G \geq 0)$, which violates the initial hypothesis.

□

Lemma 2.6.4. *Let G and H be two games.*

(i) If $G \geq 0$ and $H \geq 0$, then $G + H \geq 0$.

(ii) If $G \geq 0$ and $H \rhd 0$, then $G + H \rhd 0$.

(iii) If $G \leq 0$ and $H \leq 0$, then $G + H \leq 0$.

(iv) If $G \leq 0$ and $H \lhd 0$, then $G + H \lhd 0$.

Proof. We use Generalised Conway induction, with the properties :

(i) $\mathcal{P}(G, H)$: "If $G \geq 0$ and $H \geq 0$, then $G + H \geq 0$."

(ii) $\mathcal{P}(G, H)$: "If $G \geq 0$ and $H \rhd 0$, then $G + H \rhd 0$."

(iii) $\mathcal{P}(G, H)$: "If $G \leq 0$ and $H \leq 0$, then $G + H \leq 0$."

(iv) $\mathcal{P}(G, H)$: "If $G \leq 0$ and $H \lhd 0$, then $G + H \lhd 0$."

which can be rephrased as :

(i) $\mathcal{P}(G, H)$: "If there is no right option $G^R \leq 0$ of G and no right option $H^R \leq 0$ of H, then there is no right option $(G + H)^R \leq 0$ of $G + H$."

(ii) $\mathcal{P}(G, H)$: "If there is no right option $G^R \leq 0$ of G and if there is a left option $H^L \geq 0$ of H, then there is a left option $(G + H)^L \geq 0$ of $G + H$."

(iii) $\mathcal{P}(G, H)$: "If there is no left option $G^L \geq 0$ of G and no left option $H^L \geq 0$ of H, then there is no left option $(G + H)^L \geq 0$ of $G + H$."

(iv) $\mathcal{P}(G, H)$: "If there is no left option $G^L \geq 0$ of G and if there is a right option $H^R \leq 0$ of H, then there is a right option $(G+H)^R \leq 0$ of $G+H$."

□

Property 2.6.5. *Let H be a game such that $H = 0$, and let G be another game.*

(i) $G = 0$ if and only if $G + H = 0$.

(ii) $G > 0$ if and only if $G + H > 0$.

(iii) $G < 0$ if and only if $G + H < 0$.

(iv) $G \parallel 0$ if and only if $G + H \parallel 0$.

Proof. We only developp, as an example, the proof of the second statement. The other statements can be proved similarly, manipulating the definitions and previous properties.

Let $H = 0$ be a game and let G be another game.

Suppose that $G > 0$: this is equivalent to $G \geq 0$ and $G \rhd 0$. Then, according to lemma 4 on page 19 (as $H = 0 \Rightarrow H \geq 0$), $G + H \geq 0$ and $G + H \rhd 0$, i.e. $G + H > 0$: the "only if" direction is proved.

Conversely, suppose that $G + H > 0$: this is equivalent to $G + H \geq 0$ and $G + H \rhd 0$. Suppose that $G \lhd 0$. Then, according to lemma 4 on page 19 (as $H = 0 \Rightarrow H \leq 0$), we would have $G + H \lhd 0$ which violates the hypothesis. So $\boxed{G \geq 0}$. Now, suppose that $G \leq 0$. Then according to the same lemma (and as $H = 0 \Rightarrow H \leq 0$) we would have $G + H \leq 0$ which would violate again the hypothesis. So $\boxed{\neg(G \leq 0)}$. So $\boxed{G > 0}$: the "if" direction is proved.

□

Remark 2.6.6. This property can be rephrased as "two games that differ only from a zero game are in the same outcome class".

THEOREM 2.6.7. *Equal games are in the same outcome class.*

Proof. Let G and H be two games such that $G = H$. Then $G - H = 0$ and, according to property 5 on page 20, $G + (H - H)$ is in the same outcome class as G on the one hand (because $H - H = 0$), and on the other hand $G + (H - H) \equiv H + (G - H)$ (by commutativity) is in the same outcome class as H. Therefore, G and H are in the same outcome class.

□

EXAMPLE 2.6.8. Cf. example 3 on page 14, $2 \equiv 1 + 1$. Then, according to theorem 2 on page 17, $1 - 1 = 0$ and therefore :

$$2 - 1 \equiv 1 + 1 - 1 = 1 + 0$$

as addition is associative. Moreover, according to lemma 5 on page 20, $1+0$ is in the same outcome class as 1 and therefore, according to the previous theorem, $2-1$ is also in the same outcome class as 1.

Moreover, according to example 8 on page 13, $1 > 0$. Then $2-1 > 0$ and by definition, this gives $2 > 1$.

Corollary 2.6.9. *Addition respects the order : let G, H and K be three games, then :*

(i) $G = H$ *if and only if* $G+K = H+K$

(ii) $G > H$ *if and only if* $G+K > H+K$

(iii) $G < H$ *if and only if* $G+K < H+K$

(iv) $G \parallel H$ *if and only if* $G+K \parallel H+K$

Proof. Once again, we only developp, as an example, the proof of the fourth statement. The other statements can be proved similarly, using the property 5 on page 20.

Let G, H and K be three games. Then, by definition, $G \parallel H$ if and only if $G-H \parallel 0$, and according to property 5 on page 20, as $K-K=0$, $G-H \parallel 0$ if and only if $G-H+(K-K) \parallel 0$, i.e. (by definition and commutativity) $G+K \parallel H+K$.

□

THEOREM 2.6.10. *The relation = is an equivalence relation.*

Proof.

- Reflexivity : For any game G, $G = G$
- Transitivity : For any games G, H and K such that $G = H$ and $H = K$, then $G-H+K = H-H+K = K$ and as $K-H = 0$ then $G-H+K = G$. Therefore, $G = K$.
- Symmetry : For any games G and H, $G = H \iff (G-H) = 0$ and according to lemma 3 on page 19, we have $(G-H) = 0 \Rightarrow -(G-H) = 0$ and $-(G-H) = H-G$ according to property 13 on page 16 (and because addition is commutative). Therefore, $G = H \Rightarrow H = G$.

□

Remark 2.6.11. Theorem 10 on page 21 combined with corrolary 7 on page 20 show that replacing any game by an equivalent (i.e. equal) one does not change the outcome. We will not developp the proof here, but this statement holds even when replacing any *option* of a game by an equivalent game.

THEOREM 2.6.12. *The relations $\geq$ and $\leq$ are order relations.*

Proof. We only developp the proof for the relation $\geq$: the proof is similar for $\leq$

- Reflexivity : For any game G, $G = G$ so $G \geq G$.
- Transitivity : For any games G, H and K such that $G \geq H$ and $H \geq K$, then $G - H + K \geq H - H + K = K$ and as $K - H = 0$ then $G \geq K$.
- Antisymmetry : For any games G and H, if $G \geq H$ and $H \geq G$, then $G = H$.

□

6.2 GROUP structure

As we have proved that equality was an equivalence relation on games and that equal games were in the same outcome class, then it seems natural to work on equivalence classes for that relation of equality. Moreover, all the previous results lead to a very interesting conclusion providing an algebraic structure on such equivalence classes.

> THEOREM 2.6.13. *The equivalence classes formed by equal games form an additive abelian GROUP in which the zero element is represented by any game $G = 0$.*

Proof. First, we have to notice that addition and negation are both compatible with the equivalence relation of equality. Indeed, let G, G', H and H' be four games such that $G = G'$ and $H = H'$. Then $G - G' = 0$ and $H - H' = 0$, hence $G - G' + H - H' = 0$ i.e. $G + H = G' + H'$, as needed for addition. Moreover, $0 = G - G' \equiv -(-G) + (-G')$, hence $(-G') - (-G) = 0$ i.e. $-G' = -G$, as needed for negation.

Furthermore, addition is associative (according to property 14 on page 16) and commutative (according to property 15 on page 16).

Finally, according to theorem 2 on page 17, for any game G, the equivalence class of $-G$ is the inverse equivalence class of the class of G.

□

7 Simplifying games

As equality of games is an equivalence relation, there are many ways of writing down a game that belongs in a particular equivalence class. Some of them are easier to manipulate than others. Therefore, it can be interesting to figure out how to "simplify" games. This is what this section focuses on.

To understand the following results, it is important to keep in mind, once again, that "the more a game is positive, the more *Left* has a strategic advantage upon *Right*" and conversely, and that "two fuzzy games cannot be compared properly". Moreover, one must also remind that game theory is about understanding whether one of the players can *enforce* a win or not : we have to think about games as if the two players always played the best option for them !

7.1 Preliminary remark

The folllowing lemma will be helpful in the rest of this paper.

Lemma 2.7.1. *For any game G and for any left option G^L and any right option G^R of G, we have :* $G^L \lhd G \lhd G^R$

Proof. Let G be a game and let G^L resp. G^R be a left resp. a right option of G. Then 0 is a right option of $G^L - G$ and $0 \leq 0$, so $G^L \lhd G$. Similarly, 0 is a right option of $G - G^R$ and $0 \leq 0$, so $G \lhd G^R$.

□

7.2 Gift horses

The fundamental idea of the following definition and lemma is that offering *Left* new options that are less interesting for them than the ones he already has in a game will not change the outcome, and similarly for *Right*.

Definition 2.7.2 (Gift horses)**.** *Let G and H be two games. If $H \lhd G$, then H is a left gift horse for G, and if $H \rhd G$, then H is a right gift horse for G.*

Remark 2.7.3. A game H is a *left gift horse* for another game G if *Left* as a better strategic advantage upon *Right* in G than in H.

Lemma 2.7.4 (Gift Horse Principle)**.** *Let $G \equiv \left\{(G^{L_i})_{i\in I}\middle|(G^{R_j})_{j\in J}\right\}$ be a game, and let $(H^{L_{i'}})_{i'\in I'}$ be a set of left gift horses for G and $(H^{R_{j'}})_{j'\in J'}$ be a set of right gift horses for G (where I, J, I' and J' can be any index sets). Then :*

$$G = \left\{(G^{L_i})_{i\in I} \cup (H^{L_{i'}})_{i'\in I'}\middle|(G^{R_j})_{j\in J} \cup (H^{R_{j'}})_{j'\in J'}\right\}$$

Proof. Keeping the notations given in the lemma, let :

$$G' :\equiv \left\{(G^{L_i})_{i\in I} \cup (H^{L_{i'}})_{i'\in I'}\middle|(G^{R_j})_{j\in J} \cup (H^{R_{j'}})_{j'\in J'}\right\}$$

and let's show that $G' = G$ i.e. $G' - G = 0$

The set of right options of $G' - G$ is :

$$\left(\bigcup_{j\in J} G^{R_j} - G\right) \cup \left(\bigcup_{j'\in J'} H^{R_{j'}} - G\right) \cup \left(\bigcup_{i\in I} G' - G^{L_i}\right)$$

$\forall j \in J$, $\boxed{\left(G^{R_j} - G\right) \rhd 0}$, because $G - G = 0$ implies that there is no right option $(G-G)^R \leq 0$ of $G - G$.

Moreover, $\forall j' \in J'$, $\boxed{\left(H^{R_{j'}} - G\right) \rhd 0}$, by asumption (as $H^{R_{j'}}$ is a right gift horse for G).

Finally, $\forall i \in I$, $\boxed{\left(G' - G^{L_i}\right) \rhd 0}$ as, otherwise, there would exist $i \in I$ such that $\left(G' - G^{L_i}\right) \leq 0$, which is impossible as $G^{L_i} - G^{L_i} = 0$ is a positive left option of $G' - G^{L_i}$.

Therefore, there is no right option $(G' - G)^R \leq 0$ of $G' - G$, i.e. $\boxed{G' - G \geq 0}$.

We get $\boxed{G' - G \leq 0}$ similarly.

So $G' = G$.

□

7.3 Dominated options

The Gift Horse Principle gives us a way to *add* options to a game without changing the outcome. We now are going to use a symmetric process to *delete* options from a game. Intuitively, if a left option is less interesting for *Left* than another one, then *Left* will not use it, and it can therefore be deleted. Of course, it works similarly for *Right*.

Definition 2.7.5 (Dominated option)**.** *Let G be a game. Then :*

- *A left option G^L of G is dominated by another left option $G^{L'}$ of G if $G^L \leq G^{L'}$.*
- *A right option G^R of G is dominated by another right option $G^{R'}$ of G if $G^R \geq G^{R'}$.*

Theorem 2.7.6 (Deleting dominated options). *Let G be a game, and let G^L and G^R be resp. a left and a right options of G. Then removing from G some or all left option(s) of G dominated by G^L (but keeping G^L) and some or all right option(s) of G dominated by G^R (but keeping G^R) does not change the outcome of G.*

Proof. Let G be a game, and let G^L be a left option of G. Let G' be the game obtained from G by removing some or all lef options that are dominated by G^L (but keeping G^L). Then all the deleted options are left gift horses for G', therefore one can add these options as left options for G', thereby obtaining G, without changing the value of the game : $G = G'$. The same holds for right options.

□

Examples 2.7.7.

- $\{0, 1|\} = \{1|\} \equiv 2$
- $\{0, 1, 2|\} = \{2|\} :\equiv 3$
- $\{0, 1, 2, 3|\} = \{3|\} :\equiv 4$
- $\{|0, -1\} = \{|-1\} :\equiv -2$
- $\{0, 1|2, 3\} = \{1|2\}$

Remark 2.7.8. In the three first examples above, we recover Von Neumann's definition of natural numbers.

Remark 2.7.9. Warning : it is possible that all options are dominated, but this does not mean that all options can be removed ! As an example, consider $\omega :\equiv \{0, 1, 2, 3, \ldots|\}$.

7.4 Reversible options

Another way of simplifying a game G is based on the idea that, if there is a left option G^L of G and a right option G^{LR} of G^L such that $G^{LR} \leq G$, then G^L can be replaced by *all* left options of

G^{LR} without changing the value of the game. Indeed, such a replacement does not help *Left*, as $G^{LR} \leq G$ (therefore the "best" left option of G^{LR} will be at most as interesting for *Left* as G^L, sonce G^L is a left option of G), but it does not hurt *Left* either, as if *Left* wants to play G^L, then they must expect the answer G^{LR} from *Right* (and they will then have to play in the set of left options of G^{LR}). This is what the following definition and theorem formalise.

Definition 2.7.10 (Reversible option)**.** *Let G be a game. A left option G^L of G is called reversible (through G^{LR}) if G^L has a right option $G^{LR} \leq G$. Similarly, a right option G^R of G is called reversible (through G^{RL}) if G^R has a left option $G^{RL} \geq G$.*

THEOREM 2.7.11 (Bypassing reversible options). *Let $G \equiv \left\{H, \left(G^{L_i}\right)_{i\in I} \middle| \left(G^{R_j}\right)_{j\in J}\right\}$ be a game (where I and J can be any index sets). If the left option H of G is reversible through H^R, then :*

$$G = \left\{\left(H^{RL_{i'}}\right)_{i'\in I'}, \left(G^{L_i}\right)_{i\in I} \middle| \left(G^{R_j}\right)_{j\in J}\right\}$$

where $\left(H^{RL_{i'}}\right)_{i'\in I'}$ is the set of left options of H^R.

A similar statement holds for right options.

Proof. To make this proof easier to read, we are going to consider a game $G \equiv \left\{H, G^L \middle| G^R\right\}$ with only one right and two left options, among which H reversible through H^R, which we will also assume to have only one left option H^{RL}. The proof of the general statement is perfectly similar.

Let then :

$$G = \left\{H, G^L \middle| G^R\right\}$$

be such a game, and let :

$$G' = \left\{H^{RL}, G^L \middle| G^R\right\}$$

and :

$$G'' = \left\{H^{RL}, H, G^L \middle| G^R\right\}$$

We are going to show that H is a left gift horse for G'. Indeed, H^{RL} is a left option of G', therefore (according to lemma 1 on page 23) $\mathbf{H^{RL} \lhd G'}$ on the one hand. On the other hand, $H^R \leq G$ (because the left option H of G is reversible through H^R), and as G^R is a right option of G, then (according again to lemma 1 on page 23) $G \lhd G^R$. So $\mathbf{H^R \lhd G^R}$. These statements together imply that $\mathbf{H^R \leq G'}$ [4]. Finally, since H^R is a right option of H, then $H \lhd H^R \leq G'$. So $\boxed{H \text{ is a left gift horse for } G'}$, hence $\boxed{G' = G''}$.

On the other hand, $H^{RL} \lhd H^R \leq G$ (the first inequality holds because H^{RL} is a left option of H^R, and the second has just been proved above), so $\boxed{H^{RL} \text{ is a left gift horse for } G}$, hence $\boxed{G = G''}$.

Consequently, $\boxed{G = G'}$.

[4] Indeed, $H^R \leq G'$ unless there is a left option $\left(H^R - G'\right)^L \geq 0$ of $H^R - G'$, and the only left options of $H^R - G'$ are $H^{RL} - G'$ and $H^R - G^R$.

□

Remark 2.7.12. One aspect of reversible options might be surprising: if a left option G^L of a game G is reversible through G^{LR}, then *Left* may bypass the move to G^L **and** *Right*'s answer to G^{LR}, moving directly to some left option of G^{LR}. But what if *Right* had better chosen another right option $(G^{LR})'$ of G^L, instead of G^{LR} ? Could *Right* be deprived of their better move by such a bypass ? For the answer, consider the example : $\{|1\} = \{|100\} = 0$. Although *Right* might prefer that her only move was 1 rather than 100, the first player to move will *always* lose, which is all that we are interested in ! Back to the general statement, depriving right from a better move towards a position $(G^{LR})'$ instead of G^{LR} would change the outcom only *Left* could not win in G ($G \leq 0$) but could win when jumping directly to G^{LR} ($G^{LR} \geq 0$). However, $G^{LR} \leq G \leq 0$, contradicting the second assumption.

CHAPTER 3

LET THERE BE NUMBERS ! - A PARTICULAR KIND OF GAMES

We now have a class of mathematical objects - games - that can be added, compared, and on which we can define equivalence classes which form an abelian additive GROUP. Some of them have been denoted as relative numbers (0,1,-1,2,...), because their definition is reminiscent of the way the set $\mathbb{Z}$ is built out from set theory axioms (as reminded in chapter 1). Those games behave like numbers, and in particular they are easy to *compare* - which is interesting, in game theory, to understand which player has the greatest advantage in what game.

In this chapter, we introduce the definition of *surreal numbers*, which formalises this idea of "games that can be compared easily" : the fundamental idea of their definition is that surreal numbers have to be totally ordered. In other words, two surreal numbers should never be fuzzy to each other. In fact, their definition is reminiscent of the construction of the set $\mathbb{R}$ of real numbers with Dedekind cuts (also see chapter 1), as will be explained below.

It is more difficult to read this chapter with the "intuitive" definition of games in mind, as the concept of surreal number is more abstract than the one of game. In order to picture what surreal numbers are, one would better keep in mind the analogy with real numbers.

1 Surreal numbers

1.1 Definition

Intuitively, as numbers are a particular kind of games, we would like to impose that, while playing in a number, one remains in a number : if x is a number, then all left and right options of x have to be numbers. Moreover, as explained above, we want numbers never to be fuzzy to each other. As we already know that, for any game G and for any left option G^L and any right option G^R of G, $G^L \lhd G \lhd G^R$ (see lemma 1 on page 23), then we have to impose that, for any number x and for any left option x^L and any right option x^R of x, $x^L < x < x^R$. All this leads to the following definition.

Definition 3.1.1 (Surreal number). *Let x be a game. Then x is a surreal number if all left and right options of x are surreal numbers, and if, for all left option x^L and all right option x^R of x, then $x^L < x^R$.*

Remark 3.1.2. This definition enables to use Conway induction directely on numbers, using theorem 8 on page 10.

Notation 3.1.3. In the rest of this document, "surreal number" might be abbreviated in "number".

Notation 3.1.4. We will use lowercase letters x, y, z, ... to denote numbers.

Examples 3.1.5.

- $0 \equiv \{|\}$ is a number.
- $1 \equiv \{0|\}$ is a number.
- $-1 \equiv \{|0\}$ is a number.
- $\omega \equiv \{0, 1, 2, 3, \ldots|\}$ is a number.
- $\frac{1}{2} :\equiv \{1|2\}$ is a number.
- $* \equiv \{0|0\}$ is a game but NOT a number !

Remark 3.1.6. The last exemple above shows that all games are not numbers !

1.2 Basic properties

The definition of numbers enables to use Conway induction directely on numbers.

Theorem 3.1.7 (Conway induction on numbers). *Let $\mathcal{P}$ be a property which **numbers** might have, such that every number x has property $\mathcal{P}$ whenever all left and right options of x have this property. Then every number has property $\mathcal{P}$.*

Proof. Let x be a number, then all left and right options of x are numbers. Therefore, this theorem is a direct consequence of theorem 8 on page 10, with $\mathcal{P}_{\mathcal{C}}(G) :=$ "G is a number" for all game G, and therefore $\mathcal{C} := \{G \text{ game} \mid G \text{ is a number}\}$.

□

As expected, we have the following properties.

Property 3.1.8. *Let x be a number. Then for any left option x^L and right option x^R of x, we have $x^L < x < x^R$.*

Proof. We use Conway induction on numbers with the property :

$$\mathcal{P}(x) : \text{"For all left option } x^L \text{ and all right option } x^R \text{ of } x\text{, then } x^L < x < x^R\text{"}$$

Once again, to make the proof easier to read, we are going to consider a number $x \equiv \{x^L | x^R\}$ with only one left and one right option.

Suppose that $\mathcal{P}(x^L)$ and $\mathcal{P}(x^R)$ hold, and let's show that $\mathcal{P}(x)$ then holds.

Let's first show that $x^L < x$. The left options of $x^L - x$ are $x^L - x^R$ or $x^{LL} - x$, where x^{LL} is a left option of x^L. Since x is a number, then $x^L - x^R < 0$. By induction hypothesis, $x^{LL} < x^L$, and according to 1 on page 23, $x^L \lhd x$. Therefore, $x^{LL} - x = (x^{LL} - x^L) + (x^L - x) \lhd 0$. Consequently, $x^L \leq x$. Similarly, we get $x \leq x^R$. As $x^R \lhd x \lhd x^R$ (see lemma 1 on page 23), then we have as wanted $x^L < x < x^R$.

Conclusion : if $\mathcal{P}(x^r)$ and $\mathcal{P}(x^L)$ hold, then $\mathcal{P}(x)$ holds. Then Conway induction on numbers enables to conclude.

□

Property 3.1.9. *The negative of a number is a number.*

Proof. We are going to use Conway induction on numbers with the property :

$$\mathcal{P}(x) : \text{"}-x \text{ is a number"}$$

To make the proof easier to read, let's consider a number $x \equiv \{x^L | x^R\}$ with only one right and one left option. Suppose $-x^L$ and $-x^R$ are numbers. Then $-x^R < -x^L$, so $-x \equiv \{-x^R | -x^L\}$ is a game. Conway induction on numbers then enables to conclude.

□

Similarly, we can prove the following theorem.

Property 3.1.10. *The sum of two numbers is a number.*

As an immediate consequence of those properties, we get the following theorem.

Theorem 3.1.11. *The equivalence classes formed by equal numbers form an abelian SUBGROUP of games.*

The next theorem concludes this short list of basic properties of numbers.

Theorem 3.1.12. *Let x and y be two numbers, then either $x < y$, either $x > y$ or $x = y$.*

Proof. As the sum of numbers and the negative of a number are still numbers (see above), it is sufficient to prove that there is no game $z \parallel 0$. Suppose that there exists such a game $z \parallel 0$. Then this would imply the existence of z^L a left option of z and z^R a right option of z such that $z^L \rhd z \rhd z^R$, which is impossible according to property 8 on page 28.

□

Remark 3.1.13. The above last theorem, combined witht the fact that equals games are in the same outcome class (see theorem 7 on page 20), shows that the equivalence classes formed by equal numbers form a **totally ordered** class.

2 Multiplying numbers

2.1 Definition

Remember that we would like surreal numbers to behave like real numbers. In particular, although we can already *add* them (as they are games), we would also like to be able to *multiply* them : this is what this section is about.

We are going to build up a definition of multiplication by analysis and synthesis, based on the properties we would like the multiplication for surreal numbers to have.

Notation 3.2.1. From now on, the product of two numbers x and y will be independantly denoted as $x \times y$ or xy.

As with addition, we would like the product of two numbers to remain a number. Moreover, as with multiplication of real numbers, we would like our multiplication to be distributive uppon addition, and we would also like it to behave as expected with comparison : the product of two positive numbers has to remain positive, the product of two negative numbers has to be positive too, and the product of two numbers that have opposite signs has to be negative.

Let's try and picture what such requirements imply. Suppose that mutliplication of surreal numbers has already been defined and that it verifies all the properties here above listed. To set ideas down, let $x :\equiv \left\{x^L \middle| x^R\right\}$ and $y :\equiv \left\{y^L \middle| y^R\right\}$ be two numbers. Then we have :

$$\begin{cases} x^L < x < x^R \\ y^L < y < y^R \end{cases} \tag{3.1}$$

As we want xy to remain a number, then we have to impose :

$$(xy)^L < xy < (xy)^R \tag{3.2}$$

for any left option $(xy)^L$ and any right option $(xy)^R$ of xy.

The relations given in 3.1, combined with the properties of multiplication here above described (distributivity, behavior with comparison), give :

$$\text{(i)} \qquad \begin{cases} (x - x^L) > 0 \\ (y - y^L) > 0 \end{cases} \quad \Longrightarrow \quad (x - x^L)(y - y^L) > 0 \quad \Longrightarrow \quad \boxed{xy > xy^L + x^L y - x^L y^L}$$

$$\text{(ii)} \qquad \begin{cases} (x - x^R) < 0 \\ (y - y^R) < 0 \end{cases} \quad \Longrightarrow \quad (x - x^R)(y - y^R) > 0 \quad \Longrightarrow \quad \boxed{xy > xy^R + x^R y - x^R y^R}$$

$$\text{(iii)} \qquad \begin{cases} (x - x^L) > 0 \\ (y - y^R) < 0 \end{cases} \quad \Longrightarrow \quad (x - x^L)(y - y^R) < 0 \quad \Longrightarrow \quad \boxed{xy < xy^R + x^L y - x^L y^R}$$

$$\text{(iv)} \qquad \begin{cases} (x - x^R) < 0 \\ (y - y^L) > 0 \end{cases} \quad \Longrightarrow \quad (x - x^R)(y - y^L) < 0 \quad \Longrightarrow \quad \boxed{xy < xy^L + x^R y - x^R y^L}$$

Those results combined with 3.2 offer two possible forms for the left options of xy, and two possible forms for the right options of xy. In fact, the following general definition, based on these calculations, is precise enough to guarantee all the properties needed for our multiplication, as will be explained further.

Definition 3.2.2 (Product of numbers)**.** *Let* $x \equiv \left\{(x^{L_i})_{i\in I}\middle|(x^{R_j})_{j\in J}\right\}$ *and* $y \equiv \left\{(y^{L_{i'}})_{i'\in I'}\middle|(y^{R_{j'}})_{j'\in J'}\right\}$ *be two numbers (where* I*,* J*,* I' *and* J' *can be any index sets). Then the product of* x *and* y *is defined as :*

$$xy \equiv \{L|R\}$$

where L *and* R *are sets of numbers defined by :*

$$L = \left(\bigcup_{i\in I}\bigcup_{i'\in I'} xy^{L_{i'}} + x^{L_i}y - x^{L_i}y^{L_{i'}}\right) \cup \left(\bigcup_{j\in J}\bigcup_{j'\in J'} xy^{R_{j'}} + x^{R_j}y - x^{R_j}y^{R_{j'}}\right)$$

and

$$R = \left(\bigcup_{i\in I}\bigcup_{j'\in J'} xy^{R_{j'}} + x^{L_i}y - x^{L_i}y^{R_{j'}}\right) \cup \left(\bigcup_{i'\in I'}\bigcup_{j\in J} xy^{L_{i'}} + x^{R_j}y - x^{R_j}y^{L_{i'}}\right)$$

Remark 3.2.3. In this definition, note that the "=" symbol is to be understood as a set theoretic symbol : it is not an equality of *games*, but an equality of *sets* !

2.2 Properties of multiplication

The following results show that our muliplication of surreal numbers behaves as expected (i.e. similarly to the one of real numbers) when considered as an operation *on the equivalence classes formed by equal numbers.*

The proofs of the two following properties are very similar to the many already developped in this document, and therefore won't be given here. Nevetheless, one can refer to [Con79, p. 19, Theorem 7] for detailed proofs of these properties.

Property 3.2.4. *Let* x *and* y *be two numbers, then :*

$$xy \equiv yx \qquad 0 \times x \equiv 0 \qquad 1 \times x \equiv x \qquad (-x)y \equiv y(-x) \equiv -xy$$

Remark 3.2.5. The previous property gives *identities* of games (not equalities) !

Property 3.2.6. *Let* x*,* y *and* z *be numbers, then :*

$$(x+y)z = xz + yz \qquad \text{and} \qquad (xy)z = x(yz)$$

Remark 3.2.7. The previous property gives *equalities* : multiplication is not strictly speaking associative and distributive uppon addition, when considered as an operation on *numbers* themselves (but it is when considered as an operation on *equivalence classes* of numbers) !

The three following properties have to be proved simultaneously, using Conway induction. Although this proof is interesting, it is quite laborious and therefore will not be detailed here : see [SS05, Theorem 3.8, p. 16] for details.

Property 3.2.8. *The product of two numbers is a number.*

Property 3.2.9. *Let x, y and z be three numbers such that $x = y$. Then $xz = yz$.*

Property 3.2.10. *Let x_1, x_2, y_1 and y_2 be four numbers such that $x_1 < x_2$ and $y_1 < y_2$. Then $x_1y_2 + x_2y_1 < x_1y_1 + x_2y_2$.*

In particular, for any games y, x_1 and x_2, if $y > 0$, then $x_1 < x_2$ implies $x_1y < x_2y$.

3 Dividing numbers

Now that we can add, substract and multiply numbers, we would like to be able to *divide* them. In other words, we would like to be able to find the multiplicative inverse of a number. In fact, what can be done is to define the multiplicative inverse of an *equivalence class* of numbers : let x be a number, we want to find y such that $xy = 1$. This is what this section deals with.

Notation 3.3.1. We might abuse "multiplicative inverse of the equivalence classe of x in "multiplicative inverse of x". Such an inverse will be denoted as $\frac{1}{x}$.

First, let's notice that finding such a multiplicative inverse for any number $x > 0$ is enough. Indeed, lemma 1 on page 18 gives easily that if $y < 0$ is a negative number, then $-y > 0$ is a positive number. Then, if x is the multiplicative inverse of $-y$, then $(-x)$ is the multiplicative inverse of y (cf. property 4 on page 31).

Let's then focus on positive numbers. The following lemma is going to simplify some proofs.

Lemma 3.3.2. *Let $x > 0$ be a number. Then there exists a number y without negative option such that $y = x$.*

Proof. Let $x > 0$ be a number. Let's add the left gift horse 0 to the set of left options of x (as allowed by lemma 4 on page 23), and then delete all left options dominated by 0 (as allowed by theorem 6 on page 24). The number obtained thanks to this process is equal to x and has no negative left option, which means no negative option at all (as all right options of a number are bigger than any of his left options).

□

The following definition is an inductive one : the fundamental idea that lies behind it is that we can find the multiplicative inverse of a number as long as we know the multiplicative inverse of all its left and right options. For a detailed proof of this theorem, see [SS05, Theorem 3.10, p. 18].

Theorem 3.3.3 (Mutliplicative inverse). *Let* $x \equiv \left\{ \left(x^{R_i}\right)_{i\in I} \middle| \left(x^{L_j}\right)_{j\in J} \right\}$ *be a positive number without negative option, as in the previous lemma (where* I *and* J *can be any index sets). Then the number* y *defined as follows is such that* $xy = 1$*.*

$$y \equiv \left\{ \bigcup_{n\in\mathbb{N}} Y_n^L \middle| \bigcup_{n\in\mathbb{N}} Y_n^R \right\}$$

where :

$$\begin{cases} Y_0^L = \{0\} \\ Y_0^R = \emptyset \end{cases}$$

and $\forall n \in \mathbb{N}$ *:*

$$\begin{cases} Y_{n+1}^L = Y_n^L \cup \left(\bigcup_{j\in J} \bigcup_{y^L \in Y_n^L} \frac{1+(x^{R_j}-x)y^L}{x^{R_j}} \right) \cup \left(\bigcup_{i\in I, x^{L_i}\neq 0} \bigcup_{y^R\in Y_n^R} \frac{1+(x^{L_i}-x)y^R}{x^{L_i}} \right) \\ Y_{n+1}^R = Y_n^R \cup \left(\bigcup_{i\in I, x^{L_i}\neq 0} \bigcup_{y^L\in Y_n^L} \frac{1+(x^{L_i}-x)y^L}{x^{L_i}} \right) \cup \left(\bigcup_{j\in J} \bigcup_{y^R\in Y_n^R} \frac{1+(x^{R_j}-x)y^R}{x^{R_j}} \right) \end{cases}$$

Remark 3.3.4. Warning : in this theorem, the symbol "=" in "$xy = 1$" represents an equality of *games*, BUT in the definition of $\left(Y_n^L\right)_{n\in\mathbb{N}}$ and $\left(Y_n^R\right)_{n\in\mathbb{N}}$ it represents equalities of *sets* !

Examples 3.3.5.

- The multiplicative inverse of $2 :\equiv \{1|\}$ is $\frac{1}{2} \equiv \{0, -1|1\} = \{0|1\}$ (the last equality is obtained by deleting the left dominated option -1).
- The multiplicative inverse of $3 :\equiv \{2|\}$ is : $\frac{1}{3} :\equiv \left\{0, \frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \cdots \middle| \frac{1}{2}, \frac{3}{8}, \frac{11}{32}, \cdots \right\}$.

Remark 3.3.6. The last example shows that dividing numbers is much more complex than adding them or even multiplying them : the inverse of a very simple number with a finite number of positions (3) can have infinitely many positions ! But all those positions are generated from the ones of the initial number.

4 The FIELD of numbers

Theorem 3.4.1. *The equivalence classes formed by equal numbers form a totaly ordered FIELD, in which the zero element for addition is represented by any number* $x = 0$ *and the neutral element for multiplication is represented by any number* $y = 1$*.*

Proof. First, the equivalence classes formed by equal numbers form an abelian (SUB)GROUP of games, according to theorem 11 on page 29.

Moreover, multiplication and division are compatible with the equivalence relation of equality : see property 9 on page 32.

Furthermore, multiplication is associative (according to property 6 on page 31), commutative (according to property 4 on page 31), and distributice uppon addition (according to property 6 on page 31), when considered as an operation on equivalence classes.

Finally, for all number x, there is a number y such that $xy = yx = 1$ (see theorem 3 on page 33), and 1 is neutral for the operation of multiplication (see propety 4 on page 4).

Consequently, the equivalence classes formed by equal numbers form a FIELD. This FIELD is totally ordered according to theorem 12 on page 29.

□

5 The CLASS of numbers

In this section, we give some elements that enable to prove an interesting structure property that surreal numbers have : they form a proper CLASS (i.e. **not** a SET).

To do so, it is useful to give a precise definition of some particular kinds of numbers that have already been used in many examples in this document : relative numbers, real numbers and ordinal numbers. Of course, those names are not a coincidence : the definitions of these particular *games* are reminiscent of the "standard" definitions given in chapter 1.

Definition 3.5.1 (Natural numbers)**.** *The set $\mathbb{N}$ of natural numbers, **in the meaning of games**, is the smallest set (in the meaning of inclusion) verifying the two following properties.*

1. $0 \equiv \{|\} \in \mathbb{N}$.
2. $\forall n \in \mathbb{N}$, $n^+ :\equiv \{n|\} \in \mathbb{N}$.

Definition 3.5.2 (Relative numbers)**.** *The set $\mathbb{Z}$ of relative numbers, **in the meaning of games**, is the set $\mathbb{Z} := \mathbb{N} \cup (-\mathbb{N})$, i.e. the set $\mathbb{Z} := \{n \mid n \in \mathbb{N} \text{ or } -n \in \mathbb{N}\}$.*

Remark 3.5.3. It is easy to check that the following definition is equivalent :

Definition 3.5.4 (Relative numbers)**.** *The set $\mathbb{Z}$ of relative numbers, **in the meaning of games**, is the smallest set (in the meaning of inclusion) verifying the two following properties.*

1. $0 \equiv \{|\} \in \mathbb{N}$.
2. $\forall n \in \mathbb{N}$, $n^+ :\equiv \{n|\} \in \mathbb{N}$ *and* $n^- :\equiv \{|n\} \in \mathbb{N}$.

Of course, these definitions are very similar to the ones given in chapter 1. In fact, one can easily prove the following property.

Property 3.5.5.

(i) The set of natural numbers in the meaning of games is isomorphic to the set of natural numbers in the "standard" meaning given in chapter 1.

(ii) The set of relative numbers in the meaning of games is isomorphic to the set of relative numbers in the "standard" meaning given in chapter 1.

Definition 3.5.6 (Real numbers)**.** *Let x be a game. Then x is a real number if there exists a game $n \in \mathbb{N}$ such that $-n < x < n$, and :*

$$x = \left\{ \left(x - \frac{1}{k}\right)_{k \in \mathbb{N}^*} \middle| \left(x + \frac{1}{k}\right)_{k \in \mathbb{N}^*} \right\}$$

Remark 3.5.7. Warning : In the previous definition, k ranges over natural numbers in the meaning of the definition given on page 3, but n is a natural number in the meaning of games, i.e. in the meaning of definition 1 on page 34

As the following property is a bit tedious to prove properly, no proof of it will be detailed here. Nevertheless, one can keep in mind that the fundamental idea of the proof is to show that the definition of real numbers in the meaning of games is nearly the same as the one of "standard" real numbers given in chapter 1. More precisely, the definition of the left and right sets of options of a real number is reminiscent of Dedekind cuts, and this is what enables to prove this property.

Property 3.5.8. *The set of real numbers in the meaning of games is isomorphic to the set of real numbers in the "standard" meaning given in chapter 1.*

Definition 3.5.9 (Ordinal numbers)**.** *Let α be a game. Then α is said to be an ordinal number* ***in the meaning of games*** *of α can be written $\alpha = \{L|\}$ where L is a set of ordinal numbers.*

Notation 3.5.10. We will use Greek lowercase letters $\alpha, \beta, \gamma, \ldots$ to denots ordinal numbers.

Here again, the following property is realy tedious to prove properly, and therefore no proof of it will be detailed here. Nevertheless, the fundamental idea of the proof is to notice that ordinal numbers in the meaning of games are constructed with a transfinite induction process, which gives an embedding of "standard" ordinal numbers in ordinal numbers in the meaning of games, and that this process does not remain stagnant at some point (because otherwise, one could build an infinite sequence of games that would violate the Decending Game Condition). Therefore, as according to property 2 on page 5, "standard" ordinal numbers form a proper class, the result comes easily.

Property 3.5.11. *Ordinal numbers (in the meaning of games) form a proper CLASS.*

Lemma 3.5.12. *The class of surreal numbers contains ordinal numbers.*

Proof. We use targetted Conway Induction on ordinal numbers, with the property :

$$\mathcal{P}(\alpha) : \text{"}\alpha \text{ is a surreal number."}$$

for any ordinal number α.

□

Theorem 3.5.13. *The class of surreal numbers is a proper CLASS.*

Proof. As the class of surreal numbers contains the ordinal numbers, then if surreal numbers formed a SET, then ordinal numbers would form a SUB-SET of numbers which is impossible.

□

BIBLIOGRAPHY

[Con79] John Horton Conway. *On Numbers and Games.* Academic Press Inc. (London), 1979.

[Kri98] Jean-Louis Krivine. *Théorie des ensembles.* Cassini, 1998.

[Rudon] Walter Rudin. *Principles of Mathematical Analysis.* McGraw-Hill, 1976 (3^{rd} edition).

[SS05] Dierk Schleicher and Michael Stoll. An introduction to Conway's games and numbers. *Moscow Math Journal*, September 2005. arXiv:math/0410026v2 [math.CO].