Seminar on Wald-type optimal stopping for Brownian motion

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Contents

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying a Brownian motion $(B_t)_{t\geq 0}$. Its canonical filtration $(\mathcal{F}_t)_{t\geq 0}$ is supposed to satisfy the usual conditions: complete and right-continuous.

1 Wald's optimal stopping for Brownian motion

In this section, we are interested in the following optimal stopping problem: for a measurable map $G : \mathbb{R}^+ \to \mathbb{R}$, satisfying

$$\forall x \in \mathbb{R},\ G(|x|) \leq cx^2 + d, \tag{1}$$

for some $d \in \mathbb{R}$, $c > 0$, tempt to maximize the expectation $\mathbb{E}\left[G(|B_\tau|) - c\tau\right]$, over all integrable $(\mathcal{F}_t)$-stopping times. In the next section, we will see, as consequences, some estimates for expectation of randomly-stopped Brownian motion and maximal inequalities.

1.1 Particular case: $G : |x| \mapsto |x|^p$, $0 < p \leq 2$

1.1.1 An important case $G : |x| \mapsto x^2$

Theorem 1.1 (Wald's identity)**.** *For all integrable $(\mathcal{F}_t)$-stopping time τ,*

$$\mathbb{E}\left[B_\tau^2\right] = \mathbb{E}[\tau].$$

Proof. Let τ be an integrable $(\mathcal{F}_t)$-stopping time. Since $(B_t^2 - t)_{t\geq 0}$ is a martingale, $(B_{t\wedge\tau}^2 - t \wedge \tau)_{t\geq 0}$ is also a martingale as a stopped martingale, so

$$\forall t \geq 0,\ \mathbb{E}\left[B_{t\wedge\tau}^2\right] = \mathbb{E}\left[t \wedge \tau\right]. \tag{2}$$

Besides, $(B_{t\wedge\tau})_{t\geq 0}$ is a square-integrable martingale with continuous paths, thus, by Doob's inequality, for all $t \geq 0$,

$$\| \sup_{s\in[0,t]} |B_{s\wedge\tau}| \|_2 \leq 2\sqrt{\mathbb{E}\left[B_{t\wedge\tau}^2\right]} = 2\sqrt{\mathbb{E}[t \wedge \tau]} \leq 2\sqrt{\mathbb{E}[\tau]}.$$

By the monotone convergence theorem, we get $\mathbb{E}\left[\sup_{s\geq 0} B_{s\wedge\tau}^2\right] \leq 4\mathbb{E}[\tau] < +\infty$. Thus, $(B_{t\wedge\tau}^2)_{t\geq 0}$ is uniformly integrable, being dominated by $\sup_{s\geq 0} B_{s\wedge\tau}^2$, which is integrable. Hence it converges almost surely and in L^1. Since τ is finite a.s. (it is integrable), the almost sure limit is B_τ^2.
Then, taking the limit as t goes to $+\infty$ in (2), by convergence in L^1 for the left side, and monotone convergence theorem for the right side, we get $\mathbb{E}\left[B_\tau^2\right] = \mathbb{E}[\tau]$. □

Proposition 1.1. *Let $c > 0$, we have,*

$$\sup_\tau \mathbb{E}\left[B_\tau^2 - c\tau\right] = \begin{cases} +\infty & \text{if } c \in]0, 1[, \\ 0 & \text{elsewhere,} \end{cases}$$

where the supremum is taken over all integrable $(\mathcal{F}_t)$-stopping times.

Proof. Let τ be an integrable $(\mathcal{F}_t)$-stopping time. By Theorem 1.1, $\mathbb{E}\left[B_\tau^2 - c\tau\right] = (1 - c)\mathbb{E}[\tau]$. Three situations need to be considered:

- If $c \in]0, 1[$, with $\tau = n \in \mathbb{N}$, $\sup_\tau \mathbb{E}\left[B_\tau^2 - c\tau\right] \geq \sup_n (1 - c)n = +\infty$.
- If $c = 1$, $\sup_\tau \mathbb{E}\left[B_\tau^2 - c\tau\right] = 0$.
- If $c \in]1, +\infty[$, $(1 - c)\mathbb{E}[\tau] \leq 0$, the supremum is reached with $\tau = 0$.

□

1.1.2 Case $G : |x| \mapsto |x|^p$, $0 < p < 2$

We can then go further, taking any $p \in]0,2[$.

Theorem 1.2. *Let $0 < p < 2$ and $c > 0$, we have,*

$$\sup_\tau \mathbb{E}\left[|B_\tau|^p - c\tau\right] = \frac{2-p}{p}\left(\frac{p}{2c}\right)^{p/(2-p)},$$

where the supremum is taken over all integrable $(\mathcal{F}_t)$-stopping times.
The optimal stopping time is $\tau_{p,c} = \inf\left\{t \geq 0, |B_t| = \left(\frac{p}{2c}\right)^{1/(2-p)}\right\}$.

Remark 1.1. *$\tau_{p,c}$ is an integrable stopping time:* we show that the almost surely finite stopping time $T_x = \inf\{t \geq 0, |B_t| = x\} = \tau_x \wedge \tau_{-x}$, where $\tau_x = \inf\{t \geq 0, B_t = x\}$, is integrable. One will be able to conclude by taking $x = \left(\frac{p}{2c}\right)^{1/(2-p)}$.
Since, $T_x \wedge n$ is bounded, it is integrable. By Theorem 1.1, T_x being finite, we get by the monotone convergence theorem $\mathbb{E}\left[B^2_{T_x \wedge n}\right] = \mathbb{E}[T_x \wedge n] \underset{n\to+\infty}{\longrightarrow} \mathbb{E}[T_x]$.
Besides, $\mathbb{E}\left[B^2_{T_x \wedge n}\right] = x^2 \mathbb{P}(T_x \leq n) + \mathbb{E}\left[B_n^2 \mathbb{1}_{T_x > n}\right]$. Since T_x is finite a.s., $\mathbb{P}(T_x \leq n) \underset{n\to+\infty}{\longrightarrow} 1$, then, by dominated convergence theorem (using $|B_n^2 \mathbb{1}_{T_x>n}| \leq x^2$), $\mathbb{E}[B^2_{T_x\wedge n}] \underset{n\to+\infty}{\longrightarrow} x^2$. Thus $\mathbb{E}[T_x] = x^2 < +\infty$.

Proof. Call $V_\tau(p,c) = \mathbb{E}\left[|B_\tau|^p - c\tau\right]$, whenever τ is a stopping time. Let τ be an integrable $(\mathcal{F}_t)$-stopping time. By Theorem 1.1, we get

$$V_\tau(p,c) = \mathbb{E}\left[|B_\tau|^p - cB_\tau^2\right] = \int_{\mathbb{R}} (|x|^p - cx^2)dF_{B_\tau}(x),$$

where F_{B_τ} is the cumulative distribution function of B_τ. We maximize

$$\begin{array}{cccc} D_{p,c} & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & |x|^p - cx^2. \end{array}$$

It is an even function, so it suffices to maximize it on $\mathbb{R}^+$. $D_{p,c}$ is differentiable on $\mathbb{R}^+$ and $D'_{p,c} : x \mapsto px^{p-1} - 2cx$. Since $\lim_{x\to\infty} D_{p,c}(x) = -\infty$, $D_{p,c}$ reaches its maximum on $\mathbb{R}$ at $x = \pm\left(\frac{p}{2c}\right)^{1/(2-p)}$. As a consequence,

$$V_\tau(p,c) = \int_{\mathbb{R}} D_{p,c}(x)dF_{B_\tau}(x) \leq D_{p,c}\left(\left(\frac{p}{2c}\right)^{1/(2-p)}\right) = \frac{2-p}{p}\left(\frac{p}{2c}\right)^{p/(2-p)}.$$

But, using the fact that $B_{\tau_{p,c}} \in \left\{\pm\left(\frac{p}{2c}\right)^{1/(2-p)}\right\}$ a.s. and $D_{p,c}$ is even, we get

$$\begin{aligned} V_{\tau_{p,c}}(p,c) &= \mathbb{E}\left[\left|B_{\tau_{p,c}}\right|^p - cB^2_{\tau_{p,c}}\right] \\ &= D_{p,c}\left(-\left(\frac{p}{2c}\right)^{1/(2-p)}\right)\mathbb{P}\left(B_{\tau_{p,c}} = -\left(\frac{p}{2c}\right)^{1/(2-p)}\right) + D_{p,c}\left(\left(\frac{p}{2c}\right)^{1/(2-p)}\right)\mathbb{P}\left(B_{\tau_{p,c}} = \left(\frac{p}{2c}\right)^{1/(2-p)}\right) \\ &= D_{p,c}\left(\left(\frac{p}{2c}\right)^{1/(2-p)}\right) = \frac{2-p}{p}\left(\frac{p}{2c}\right)^{p/(2-p)}. \end{aligned}$$

□

Remark 1.2. If $p \in]2,+\infty[$, we can adapt the latter proof to find $\inf_\tau \mathbb{E}\left[|B_\tau|^p - c\tau\right]$, where the infimum is taken over all integrable $(\mathcal{F}_t)$-stopping times: we minimize

$$\begin{array}{cccc} \tilde{D}_{p,c} & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & |x|^p - cx^2. \end{array}$$

It is an even function, so it suffices to minimize it on $\mathbb{R}^+$. $\tilde{D}_{p,c}$ is differentiable on $\mathbb{R}^+$ and $\tilde{D}'_{p,c} : x \mapsto px^{p-1} - 2cx$. Since $\lim_{x\to\infty} \tilde{D}_{p,c}(x) = +\infty$, $\tilde{D}_{p,c}$ reaches a minimum on $\mathbb{R}$. Thus, $\tilde{D}_{p,c}$ reaches its minimum at $x = \pm\left(\frac{p}{2c}\right)^{1/(2-p)}$. As a consequence,

$$V_\tau(p,c) \geq \tilde{D}_{p,c}\left(\left(\frac{p}{2c}\right)^{1/(2-p)}\right) = \frac{2-p}{p}\left(\frac{p}{2c}\right)^{p/(2-p)}.$$

Thus

$$\inf_\tau \mathbb{E}[|B_\tau|^p - c\tau] \geq \frac{2-p}{p}\left(\frac{p}{2c}\right)^{p/(2-p)},$$

where the infimum is taken over all integrable $(\mathcal{F}_t)$-stopping times. And as in the preceding proof, the optimal stopping time is $\tau_{p,c}$.

1.2 General case

With the same method as above, we have

Theorem 1.3. *Let $d \in \mathbb{R}$, $c > 0$ and let $G : \mathbb{R}^+ \to \mathbb{R}$ be a measurable map, satisfying the boundedness condition (1), then*

$$\sup_\tau \mathbb{E}\left[G(|B_\tau|) - c\tau\right] = \sup_{x\in\mathbb{R}}(G(|x|) - cx^2),$$

where the supremum is taken over all integrable $(\mathcal{F}_t)$-stopping times.
The optimal stopping time is the hitting time by the absolute value of Brownian motion $|B|$ of the set of all maximum points of the map $D_{G,c} : x \mapsto G(|x|) - cx^2$, when $D_{G,c}$ reaches a maximum on $\mathbb{R}$, i.e. $\tau_{G,c} = \inf\{t \geq 0, |B_t| = \text{argmax } D_{G,c}\}$.

Remark 1.3. We will see during the proof that if $D_{G,c}$ doesn't reach a maximum on $\mathbb{R}$, one can only find an optimal sequence of integrable $(\mathcal{F}_t)$-stopping times $(T_r)_r$: $\sup_\tau \mathbb{E}\left[G(|B_\tau|) - c\tau\right] = \lim_{r\to+\infty} \mathbb{E}\left[G(|B_{T_r}|) - cT_r\right]$.

Proof. As in the next section, introduce $V_\tau(G,c) = \mathbb{E}\left[G(|B_\tau|) - c\tau\right]$, whenever τ is a stopping time. Let τ be an integrable $(\mathcal{F}_t)$-stopping time. By Theorem 1.1, we get

$$V_\tau(G,c) = \mathbb{E}\left[G(|B_\tau|) - cB_\tau^2\right] = \int_{\mathbb{R}} (G(|x|) - cx^2) dF_{B_\tau}(x).$$

We maximize

$$\begin{array}{rccc} D_{G,c} & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & G(|x|) - cx^2. \end{array}$$

By the boudness condition (1), $D_{G,c}$ has an upper bound, we split into two cases:

- If $D_{G,c}$ reaches its maximum at $x_0 \in \mathbb{R}$, then $V_\tau(G,c) \leq D_{G,c}(x_0)$.
  By using the stopping time T_{x_0}, defined in Remark 1.1, since $D_{G,c}$ is even, doing as in the previous section, we get
  $$\sup_\tau \mathbb{E}\left[G(|B_\tau|) - c\tau\right] = D_{G,c}(x_0).$$
- If $D_{G,c}$ reaches its maximum on $\pm\infty$, then for all $x \in \mathbb{R}$, $D_{G,c}(x) \leq \lim_{x\to+\infty} D_{G,c}(x)$, thus $V_\tau(G,c) \leq \lim_{x\to+\infty} D_{G,c}(x)$.
  By using the stopping time T_r, defined in Remark 1.1, for $r > 0$, using the fact that $D_{G,c}$ is even, we get,
  $$\begin{aligned} V_{T_r}(G,c) &= \mathbb{E}\left[B_{T_r}^2 - cB_{T_r}^2\right] \\ &= D_{G,c}(-r)\mathbb{P}\left(B_{T_r} = -r\right) + D_{G,c}(r)\mathbb{P}\left(B_{T_r} = r\right) \\ &= D_{G,c}(r). \end{aligned}$$

Hence, $\forall r>0,\ \sup_\tau \mathbb{E}\left[G(|B_\tau|)-c\tau\right] \geq D_{G,c}(r)$. Taking the limit as r goes to $+\infty$, this yields to $\sup_\tau \mathbb{E}\left[G(|B_\tau|)-c\tau\right] \geq \lim\limits_{x\to+\infty} D_{G,c}(x)$. Thus, $\sup_\tau \mathbb{E}[G(|B_\tau|)-c\tau] = \lim\limits_{x\to+\infty} D_{G,c}(x)$.

□

Remark 1.4. If the boundedness condition (1) is not satisfied, then $\sup_{x\in\mathbb{R}}(G(|x|)-cx^2)$ could be infinite. The equality still holds by doing as in the second case of the proof.

Remark 1.5. Adapting the latter proof and using Remark 1.2, we get

$$\inf_\tau \mathbb{E}[G(|B_\tau|)-c\tau] = \inf_{x\in\mathbb{R}}(G(|x|)-cx^2),$$

where the infimum is taken over all integrable $(\mathcal{F}_t)$-stopping times. The optimal stopping time is the hitting time by the absolute value of Brownian motion $|B|$ of the set of all minimum points of the map $D_{G,c}: x\mapsto G(|x|)-cx^2$, when $D_{G,c}$ reaches a minimum on $\mathbb{R}$.

2 Some consequences

2.1 Estimates for expectation of stopped Brownian motion

Using Theorems 1.1, 1.2 and Remark 1.2, we get

Theorem 2.1. *For all integrable $(\mathcal{F}_t)$-stopping time τ,*

- *if $p\in]0,2[$,*
$$\mathbb{E}\left[|B_\tau|^p\right] \leq \mathbb{E}[\tau]^{p/2};$$
- *if $p=2$,*
$$\mathbb{E}\left[B_\tau^2\right] = \mathbb{E}[\tau];$$
- *if $p\in]2,+\infty[$,*
$$\mathbb{E}\left[|B_\tau|^p\right] \geq \mathbb{E}[\tau]^{p/2}.$$

Proof. Let τ be an integrable $(\mathcal{F}_t)$-stopping time.

- By Theorem 1.2, we have, for all $c>0$,
$$\mathbb{E}\left[|B_\tau|^p\right] \leq c\mathbb{E}[\tau] + \frac{2-p}{p}\left(\frac{p}{2c}\right)^{p/(2-p)}.$$
Then
$$\mathbb{E}\left[|B_\tau|^p\right] \leq \inf_{c>0}\left(c\mathbb{E}[\tau] + \frac{2-p}{p}\left(\frac{p}{2c}\right)^{p/(2-p)}\right).$$
Let $f: c\mapsto c\mathbb{E}[\tau] + \frac{2-p}{p}\left(\frac{p}{2c}\right)^{p/(2-p)}$. f is differentiable on $\mathbb{R}_+^*$, $f': c\mapsto \mathbb{E}[\tau] - \left(\frac{p}{2c}\right)^{2/(2-p)}$. Thus $f'(c)\geq 0 \iff c\geq \frac{p}{2}\mathbb{E}[\tau]^{(p-2)/2}$. So f reaches a minimum at $\frac{p}{2}\mathbb{E}[\tau]^{(p-2)/2}$ and $f\left(\frac{p}{2}\mathbb{E}[\tau]^{(p-2)/2}\right) = \mathbb{E}[\tau]^{p/2}$. It shows that
$$\mathbb{E}[|B_\tau|^p] \leq \mathbb{E}[\tau]^{p/2}.$$
Remark 2.1. This comes also from Theorem 1.1 and Jensen's inequality, using $x\mapsto x^{p/2}$ which is concave.
- We have already proved this in Theorem 1.1.

- As stated in Remark 1.2, for all $c > 0$,

$$\mathbb{E}\left[|B_\tau|^p - c\tau\right] \geq \frac{2-p}{p}\left(\frac{p}{2c}\right)^{p/(2-p)},$$

then

$$\mathbb{E}\left[|B_\tau|^p\right] \geq \sup_{c>0}\left(c\mathbb{E}[\tau] + \frac{2-p}{p}\left(\frac{p}{2c}\right)^{p/(2-p)}\right).$$

Define the same f as above. Now, we have

$$f'(c) \geq 0 \iff c \leq \frac{p}{2}\mathbb{E}[\tau]^{(p-2)/2}.$$

so f reaches a maximum and we conclude as in the first point that

$$\mathbb{E}[|B_\tau|^p] \geq \mathbb{E}[\tau]^{p/2}.$$

Remark 2.2. This comes also from Theorem 1.1 and Jensen's inequality, using $x \mapsto x^{p/2}$ which is convex.

□

With the same method as above, using Theorem 1.3 and Remark 1.5, we get

Theorem 2.2. *Let $G : \mathbb{R}^+ \to \mathbb{R}$ be a measurable map, then for all integrable $(\mathcal{F}_t)$-stopping time τ,*

$$\sup_{c>0}\left(c\mathbb{E}[\tau] + \inf_{x\in\mathbb{R}}(G(|x|) - cx^2)\right) \leq \mathbb{E}\left[G(|B_\tau|)\right] \leq \inf_{c>0}\left(c\mathbb{E}[\tau] + \sup_{x\in\mathbb{R}}(G(|x|) - cx^2)\right).$$

Remark 2.3.

- If τ is not integrable, the upper bound equals $+\infty$ so the right-inequality is trivial.
- As seen in Remark 1.4, we do not need the boundedness condition (1).

Remark 2.4. Under the hypothesis of the theorem, if $H : x \mapsto G(\sqrt{x})$ is concave, then by Jensen's inequality and Wald's identity, $\mathbb{E}\left[G\left(|B_\tau|\right)\right] \leq G\left(\sqrt{\mathbb{E}[\tau]}\right)$. Using that the concave-biconjugate $\tilde{\tilde{H}}$ of H is a concave function which is greater than H, one can find directly the right inequality of Theorem 2.2. "Concave" being changed into "convex" and "greater" into "lower", one can find the left inequality. See [GP] for details.

Thanks to the change of time theorem (Theorem A.1), we can extend Theorem 2.2 to local martingales:

Theorem 2.3. *Let M be a continuous local martingale starting at 0 and let $G : \mathbb{R}^+ \to \mathbb{R}$ be a measurable function. Then for any $t > 0$ for which $\mathbb{E}[\langle M, M\rangle_t] < +\infty$, we have*

$$\sup_{c>0}\left(c\mathbb{E}[\langle M, M\rangle_t] + \inf_{x\in\mathbb{R}}(G(|x|) - cx^2)\right) \leq \mathbb{E}[G(|M_t|)] \leq \inf_{c>0}\left(c\mathbb{E}[\langle M, M\rangle_t] + \sup_{x\in\mathbb{R}}(G(|x|) - cx^2)\right).$$

Proof. Using the Dambis-Dubins-Schwarz's Brownian motion β for M, we have, for $t > 0$ for which $\mathbb{E}[\langle M, M\rangle_t] < +\infty$,

$$\begin{aligned}\mathbb{E}[G(|M_t|)] &= \tilde{\mathbb{E}}[G(|M_t|)] = \tilde{\mathbb{E}}\left[G\left(\left|\beta_{\langle M,M\rangle_t}\right|\right)\right] \\ &\leq \inf_{c>0}\left(c\tilde{\mathbb{E}}[\langle M, M\rangle_t] + \sup_{x\in\mathbb{R}}(G(|x|) - cx^2)\right) = \inf_{c>0}\left(c\mathbb{E}[\langle M, M\rangle_t] + \sup_{x\in\mathbb{R}}(G(|x|) - cx^2)\right),\end{aligned}$$

using Theorem 2.2.
One can do the same for the other inequality. □

Optimality in the bound:

In Theorem 2.2, the inequalities are sharp:

Theorem 2.4. *Let $G : \mathbb{R}^+ \to \mathbb{R}$ be a measurable map. Suppose that there exists $c_0 > 0$ such that $D_{G,c_0} : x \mapsto G(|x|) - c_0 x^2$ reaches a maximum over $\mathbb{R}$, then*

$$\sup_\tau \left(\mathbb{E}\left[G(|B_\tau|)\right] - \inf_{c>0} \left(c\mathbb{E}[\tau] + \sup_{x\in\mathbb{R}}(G(|x|) - cx^2) \right) \right) = 0,$$

where the supremum is taken over all integrable $(\mathcal{F}_t)$-stopping times.
Suppose that there exists $c_0 > 0$ such that $D_{G,c_0} : x \mapsto G(|x|) - c_0 x^2$ reaches a minimum over $\mathbb{R}$, then

$$\inf_\tau \left(\mathbb{E}\left[G(|B_\tau|)\right] - \sup_{c>0} \left(c\mathbb{E}[\tau] + \inf_{x\in\mathbb{R}}(G(|x|) - cx^2) \right) \right) = 0,$$

where the infimum is taken over all integrable $(\mathcal{F}_t)$-stopping times.

Proof.

- We denote $a_{G,c}$ a point where $D_{G,c}$ reaches its maximum (possibly infinite) over $\bar{\mathbb{R}}$. Call $\sigma_c = \inf\{t \geq 0,\ |B_t| = a_{G,c}\}$. By hypothesis, $a_{G,c_0} \in \mathbb{R}$ and σ_{c_0} is an integrable stopping time. Thus we have
$$\begin{aligned}
0 &= \mathbb{E}\left[G\left(\left|B_{\sigma_{c_0}}\right|\right) - c_0\sigma_{c_0}\right] - D_{G,c_0}(a_{G,c_0}) \\
&= \mathbb{E}\left[G\left(\left|B_{\sigma_{c_0}}\right|\right) - c_0\sigma_{c_0}\right] - \sup_{x\in\mathbb{R}}\left(G\left(|x|\right) - c_0 x^2\right) \\
&\leq \sup_{c>0}\left(\mathbb{E}\left[G\left(\left|B_{\sigma_{c_0}}\right|\right) - c\sigma_{c_0}\right] - \sup_{x\in\mathbb{R}}\left(G\left(|x|\right) - cx^2\right)\right) \\
&\leq \sup_\tau \sup_{c>0}\left(\mathbb{E}\left[G\left(|B_\tau|\right) - c\tau\right] - \sup_{x\in\mathbb{R}}\left(G\left(|x|\right) - cx^2\right)\right) \\
&\leq \sup_\tau \left(\mathbb{E}\left[G\left(|B_\tau|\right)\right] + \sup_{c>0}\left(-c\mathbb{E}[\tau] - \sup_{x\in\mathbb{R}}\left(G\left(|x|\right) - cx^2\right)\right)\right) \\
&\leq \sup_\tau \left(\mathbb{E}\left[G\left(|B_\tau|\right)\right] - \inf_{c>0}\left(c\mathbb{E}[\tau] + \sup_{x\in\mathbb{R}}\left(G\left(|x|\right) - cx^2\right)\right)\right),
\end{aligned}$$
where the supremum is taken over all integrable $(\mathcal{F}_t)$-stopping times. The other inequality comes from Theorem 2.2.

- One can adapt the first point in order to get the other part of the theorem.

□

We deduce from this that the inequalities in Theorem 2.1 are sharp:

Corollary 2.5.

- *If $p \in]0, 2[$,*
$$\sup_\tau \left(\mathbb{E}\left[|B_\tau|^p\right] - \mathbb{E}[\tau]^{p/2}\right) = 0,$$
where the supremum is taken over all integrable $(\mathcal{F}_t)$-stopping times.

- *If $p = 2$, for all integrable $(\mathcal{F}_t)$-stopping time τ,*
$$\mathbb{E}\left[B_\tau^2\right] = \mathbb{E}[\tau].$$

- *If $p \in]2, +\infty[$,*
$$\inf_\tau \left(\mathbb{E}\left[|B_\tau|^p\right] - \mathbb{E}[\tau]^{p/2}\right) = 0,$$
where the infimum is taken over all integrable $(\mathcal{F}_t)$-stopping times.

Proof. Using $G : x \mapsto |x|^p$, for $p \in \mathbb{R}$ and Theorem 2.4:

- If $p \in]0,2[$, as stated in the proof of Theorem 1.2, for all $c > 0$, $D_{p,c} : x \mapsto |x|^p - cx^2$ reaches a maximum over $\mathbb{R}$. The infimum over $c > 0$ has been computed in the proof of Theorem 2.1.
- If $p = 2$, we have already proved this in Theorem 1.1.
- If $p \in]2,+\infty[$, as stated in Remark 1.2, for all $c > 0$, $D_{p,c} : x \mapsto |x|^p - cx^2$ reaches a minimum over $\mathbb{R}$. The supremum over $c > 0$ has been computed in the proof of Theorem 2.1.

□

2.2 Dubins-Jacka-Schwarz-Shepp-Shiryaev maximal inequalities for randomly stopped Brownian motion

Proposition 2.1. *If τ is an integrable $(\mathcal{F}_t)$-stopping time, then*

$$\mathbb{E}\left[\max_{0\le t\le\tau} B_t\right] \le \sqrt{\mathbb{E}[\tau]}.$$

This is a sharp inequality.

Proof. Let us write for $t \ge 0$, $S_t = \max_{0\le s\le t} B_s$.

- Let $c > 0$. We first define for $t \ge 0$, $Z_t = c\left((S_t - B_t)^2 - t\right) + \frac{1}{4c}$. It is a martingale:
  - for all $t \ge 0$, Z_t is $\mathcal{F}_t$-measurable.
  - for all $t \ge 0$, $\mathbb{E}\left[|Z_t|\right] \le c\mathbb{E}[(S_t - B_t)^2] + ct + \frac{1}{4c} = c\mathbb{E}[B_t^2] + ct + \frac{1}{4c} = 2ct + \frac{1}{4c} < +\infty$, using the fact that $(S_t - B_t)$ has the same law as $|B_t|$ (see Proposition A.1).
  - Let $0 \le s \le t$,

$$\begin{aligned}\mathbb{E}[Z_t - Z_s|\mathcal{F}_s] &= c\mathbb{E}\left[(S_t - B_t)^2 - (S_s - B_s)^2 - t + s\right]\\ &= c\mathbb{E}\left[B_t^2 - B_s^2 - t + s\right] = 0,\end{aligned}$$

  using again the fact that $(S_t - B_t)$ has the same law as $|B_t|$.

  Let σ be a bounded $(\mathcal{F}_t)$-stopping time, since $\mathbb{E}[B_\sigma] = 0$ (see Proposition A.2), we get

$$\mathbb{E}[S_\sigma - c\sigma] = \mathbb{E}[S_\sigma - B_\sigma - c\sigma] \le \mathbb{E}[Z_\sigma] = \mathbb{E}[Z_0] = \frac{1}{4c},$$

  using :

  - $\forall x \in \mathbb{R},\ \forall t \ge 0,\ x - ct \le c(x^2 - t) + \frac{1}{4c}$,
  - and the Doob's optional stopping theorem for martingale with a bounded stopping time.

  Thus $\mathbb{E}[S_\sigma] \le \inf_{c>0}\left(\frac{1}{4c} + c\mathbb{E}[\sigma]\right) = \sqrt{\mathbb{E}[\sigma]}$.
  Let now τ be an integrable $(\mathcal{F}_t)$-stopping time. Applying what we have just shown to the stopping time $\tau \wedge t$, for $t \ge 0$, we get

$$\forall t \ge 0,\ \mathbb{E}[S_{\tau\wedge t}] \le \sqrt{\mathbb{E}[\tau \wedge t]}.$$

  We conclude by the monotone convergence theorem, since $(S_t)_{t\ge0}$ is non decreasing.

- Let $a \in \mathbb{R}$. We take $\tau = \inf\{t \geq 0, S_t - B_t = a\}$ which is equal in law to $T_a = \inf\{t \geq 0, |B_t| = a\}$, by Proposition A.1. Then, using the integrability of τ and Proposition A.2, we get $\mathbb{E}[S_\tau] = a + \mathbb{E}[B_\tau] = a$. Since $\mathbb{E}[\tau] = \mathbb{E}[T_a] = a^2$ (see Remark 1.1), we have the equality.

□

Remark 2.5. We can extend this inequality to any continuous local martingale M starting at 0, using β, the Dambis-Dubins-Schwarz's Brownian motion of M (see Theorem A.1). Let $t \geq 0$ such that $\mathbb{E}[\langle M, M\rangle_t] < +\infty$, then,

$$\mathbb{E}\left[\max_{0\leq s\leq t} M_s\right] = \mathbb{E}\left[\max_{0\leq s\leq t} \beta_{\langle M,M\rangle_s}\right] = \mathbb{E}\left[\max_{0\leq s\leq \langle M,M\rangle_t} \beta_s\right] \leq \sqrt{\mathbb{E}[\langle M, M\rangle_t]}.$$

Proposition 2.2. *If τ is an integrable $(\mathcal{F}_t)$-stopping time, then*

$$\mathbb{E}\left[\max_{0\leq t\leq \tau} |B_t|\right] \leq \sqrt{2}\sqrt{\mathbb{E}[\tau]}.$$

This is a sharp inequality.

Proof.

- Let τ be an integrable $(\mathcal{F}_t)$-stopping time. For $t \geq 0$, define $M_t = \mathbb{E}\left[|B_\tau| - \mathbb{E}\left[|B_\tau|\right] | \mathcal{F}_{t\wedge\tau}\right]$. This is a martingale:
  - for all $t \geq 0$, M_t is $\mathcal{F}_{t\wedge\tau}$-measurable so it is $\mathcal{F}_t$-measurable.
  - for all $t \geq 0$,

$$\begin{aligned}\mathbb{E}\left[|M_t|\right] &= \mathbb{E}\left[|\mathbb{E}\left[|B_\tau| - \mathbb{E}\left[|B_\tau|\right] |\mathcal{F}_{t\wedge\tau}\right]|\right] \leq \mathbb{E}\left[\mathbb{E}\left[||B_\tau| - \mathbb{E}\left[|B_\tau|\right]| |\mathcal{F}_{t\wedge\tau}\right]\right] = \mathbb{E}\left[||B_\tau| - \mathbb{E}\left[|B_\tau|\right]|\right] \\ &\leq 2\mathbb{E}\left[|B_\tau|\right] \leq 2\sqrt{\mathbb{E}\left[B_\tau^2\right]} \leq 2\sqrt{\mathbb{E}[\tau]} < +\infty,\end{aligned}$$

    using Jensen's inequality and Wald's identity.
  - Let $0 \leq s \leq t$. We have, since M_t is $\mathcal{F}_\tau$ measurable,

$$\begin{aligned}\mathbb{E}[M_t|\mathcal{F}_s] &= \mathbb{E}[\mathbb{E}[M_t|\mathcal{F}_\tau]|\mathcal{F}_s] = \mathbb{E}[M_t|\mathcal{F}_{s\wedge\tau}] = \mathbb{E}\left[\mathbb{E}\left[|B_\tau| - \mathbb{E}\left[|B_\tau|\right] |\mathcal{F}_{t\wedge\tau}\right] |\mathcal{F}_{s\wedge\tau}\right] \\ &= \mathbb{E}\left[|B_\tau| - \mathbb{E}\left[|B_\tau|\right] |\mathcal{F}_{s\wedge\tau}\right] = M_s.\end{aligned}$$

  It admits a modification which is right-continuous. But, using Jensen's inequality, for all $t \geq 0$, we get

$$\begin{aligned}\mathbb{E}[M_t^2] &\leq \mathbb{E}\left[\mathbb{E}\left[\left(|B_\tau| - \mathbb{E}\left[|B_\tau|\right]\right)^2 |\mathcal{F}_{t\wedge\tau}\right]\right] = \mathbb{E}\left[\left(|B_\tau| - \mathbb{E}\left[|B_\tau|\right]\right)^2\right] \\ &\leq \mathbb{E}\left[B_\tau^2\right] - \mathbb{E}\left[|B_\tau|\right]^2 \leq \mathbb{E}[\tau],\end{aligned}$$

  by Wald's identity. The right-continuous martingale $(M_t)_{t\geq 0}$ is then bounded in L^2, hence $\mathbb{E}[\langle M, M\rangle_\infty] < +\infty$ and $M^2 - \langle M, M\rangle$ is an uniformly integrable martingale (see [RZ, Chapter IV, Propostion 1.23 p.108]). Then $M^2 - \langle M, M\rangle$ converges a.s. and in L^1 to $M_\infty^2 - \langle M, M\rangle_\infty$. The martingale property and the L^1 convergence yields to $\mathbb{E}[\langle M, M\rangle_\infty] = \mathbb{E}[M_\infty^2]$. For all $t \geq 0$, $\mathbb{E}[\langle M, M\rangle_t] \leq \mathbb{E}[\langle M, M\rangle_\infty] < +\infty$, so by the remark above

$$\mathbb{E}\left[\max_{0\leq s\leq t} M_s\right] \leq \sqrt{\mathbb{E}[\langle M, M\rangle_t]} \leq \sqrt{\mathbb{E}[\langle M, M\rangle_\infty]} = \sqrt{\mathbb{E}[M_\infty^2]}.$$

  Then by the monotone convergence theorem and using the uniform bound of the second moment of M, we get

$$\mathbb{E}\left[\max_{s\geq 0} M_s\right] \leq \sqrt{\mathbb{E}[M_\infty^2]} \leq \sqrt{\mathbb{E}\left(|B_\tau| - \mathbb{E}\left[|B_\tau|\right]\right)^2}.$$

Since, $(B_{t\wedge\tau})_{t\geq 0}$ is a martingale closed by B_τ,

$$\mathbb{E}\left[\max_{0\leq t\leq\tau}|B_t|\right] = \mathbb{E}\left[\max_{t\geq 0}|B_{t\wedge\tau}|\right] \leq \mathbb{E}\left[\max_{t\geq 0}\mathbb{E}\left[|B_\tau|\,|\mathcal{F}_{t\wedge\tau}\right]\right].$$

Thus we have

$$\begin{aligned}\mathbb{E}\left[\max_{0\leq t\leq\tau}|B_t|\right] &\leq \mathbb{E}\left[\max_{t\geq 0}M_t\right] + \mathbb{E}[|B_\tau|] \leq \sqrt{\mathbb{E}\left(|B_\tau| - \mathbb{E}\left[|B_\tau|\right]\right)^2} + \mathbb{E}[|B_\tau|] \\ &\leq \sqrt{\mathbb{E}[\tau] - \mathbb{E}[|B_\tau|]^2} + \mathbb{E}[|B_\tau|],\end{aligned}$$

using Wald's identity. But $g : x \mapsto \sqrt{\mathbb{E}[\tau] - x^2} + x$ defined on $\left[0, \sqrt{\mathbb{E}[\tau]}\right]$ reaches its maximum at $\sqrt{\mathbb{E}[\tau]/2}$ so by Proposition 2.1,

$$\mathbb{E}\left[\max_{0\leq t\leq\tau}|B_t|\right] \leq \sqrt{2}\sqrt{\mathbb{E}[\tau]}.$$

- Take $\tau_2 = \inf\{t \geq 0, \max_{0\leq s\leq t}|B_s| - |B_t| = a\}$ for $a > 0$, one can show that it gives the equality. See [DSS].

□

3 On Doob's maximal inequalities for Brownian motion

Let τ be an integrable $(\mathcal{F}_t)$-stopping time. The Doob's maximal inequality states the sharp inequality

$$\mathbb{E}\left[\max_{0\leq t\leq\tau}B_t^2\right] \leq 4\mathbb{E}\left[B_\tau^2\right].$$

One can wonder if there exists a similar sharp inequality for Brownian motion started at any point $x \in \mathbb{R}^+$. Considering the optimal stopping problem,

$$V(x,s) = \sup_\tau \mathbb{E}_{x,s}\left[S_\tau - c\tau\right],$$

where the expectation is taken with respect to the probability measure under which $(S_t)_{t\geq 0} = (\max_{0\leq r\leq t}B_r^2 \vee s)_{t\geq 0}$ starts at s and $(B_t)_{t\geq 0}$ starts at x, one can show that

$$\mathbb{E}_x\left[\max_{0\leq t\leq\tau}B_t^2\right] \leq 4\mathbb{E}_x\left[B_\tau^2\right] - 2x^2,$$

which is a sharp inequality. It can be extend to any power $p > 1$. See [GP2] for details.

A Appendix

Theorem A.1 (Change of time). *If M is a local martingale starting at 0, with $\langle M,M\rangle_\infty = +\infty$ a.s., then there exists a Brownian motion $(\beta_t)_t$ such that $M_t = \beta_{\langle M,M\rangle_t}$, for all $t \geq 0$. β is called the* Dambis-Dubins-Schwarz's Brownian motion of *M.*

Proof. See [RZ, Chapter V, Theorem 1.6 p.181]. □

Remark A.1. Up to enlarge the probability space, we can remove the condition on the bracket in the previous theorem:
$(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_t, \tilde{\mathbb{P}})$ is an *enlargement* of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ if there exists $\pi : \tilde{\Omega} \to \Omega$ such that $\forall t \geq 0$, $\pi^{-1}(\mathcal{F}_t) \subset \tilde{\mathcal{F}}_t$ and $\pi(\tilde{\mathbb{P}}) = \mathbb{P}$. A process X defined on Ω can be viewed as a process on $\tilde{\Omega}$ with $X(\tilde{\omega}) = X(\omega)$, when $\omega = \pi(\tilde{\omega})$.

Remark A.2. For a random variable X defined on Ω,

$$\tilde{\mathbb{E}}[X] = \int_{\tilde{\Omega}} X(\pi(\tilde{\omega}))\tilde{\mathbb{P}}(d\tilde{\omega}) = \int_{\Omega} X(\omega)\mathbb{P}(d\omega) = \mathbb{E}[X].$$

Proposition A.1. *For all $t \geq 0$, $S_t - B_t \stackrel{\mathcal{L}}{=} |B_t|$, where $S_t = \max_{0\leq s\leq t} B_s$.*

Remark A.3. Thanks to P. Lévy, we have more: $(S_t - B_t)_{t\geq 0}$ has the same law as $(|B_t|)_{t\geq 0}$. See [KS, Chapter III, Theorem 6.17 p.210].

Proof. Let $t \geq 0$. By the reflexion principle, the density of (S_t, B_t) is given by

$$f_t : (a,b) \mapsto \frac{2(2a-b)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2a-b)^2}{2t}\right) \mathbb{1}_{a>0,b<a}.$$

Let $u \geq 0$, with the change of variables $(a,b) \mapsto (a, a-b)$, we get

$$\begin{aligned}
\mathbb{P}(S_t - B_t \geq u) &= \int_{\mathbb{R}^2} \mathbb{1}_{a-b\geq u} \frac{2(2a-b)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2a-b)^2}{2t}\right) \mathbb{1}_{a>0,b<a} da db \\
&= \int_{\mathbb{R}^2} \mathbb{1}_{c\geq u} \frac{2(a+c)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(a+c)^2}{2t}\right) \mathbb{1}_{a>0} da dc \\
&= \int_{\mathbb{R}} \mathbb{1}_{c\geq u} \frac{2}{\sqrt{2\pi t^3}} \exp\left(-\frac{c^2}{2t}\right) dc = 2\mathbb{P}\left(B_t \geq u\right) = \mathbb{P}\left(|B_t| \geq u\right).
\end{aligned}$$

□

Proposition A.2. *For all integrable $(\mathcal{F}_t)$-stopping time τ, $\mathbb{E}[B_\tau] = 0$.*

Proof. Since $(B_t^2 - t)_{t\geq 0}$ is a martingale, $(B_{t\wedge\tau}^2 - t\wedge\tau)_{t\geq 0}$ is a martingale as a stopped martingale. It implies that $\mathbb{E}\left[B_{t\wedge\tau}^2 - t\wedge\tau\right] = \mathbb{E}\left[B_{0\wedge\tau}^2 - 0\wedge\tau\right] = 0$, hence

$$\sup_{t\geq 0} \mathbb{E}\left[B_{t\wedge\tau}^2\right] = \sup_{t\geq 0} \mathbb{E}[t\wedge\tau] \leq \mathbb{E}[\tau] < +\infty.$$

The stopped martingale $(B_{t\wedge\tau})_{t\geq 0}$ is then uniformly integrable, and since τ is a.s. finite (because it is integrable), it converges almost surely and in L^1 to $\lim_{t\to+\infty} B_{t\wedge\tau} = B_\tau$.
But, by the martingale property, $\mathbb{E}\left[B_{t\wedge\tau}\right] = 0$ and by L^1-convergence $\mathbb{E}[B_{t\wedge\tau}] \underset{t\to+\infty}{\longrightarrow} \mathbb{E}[B_\tau]$. We conclude by the uniqueness of the limit. □

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