Construction of the exclusion process

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Contents

Assume given a matrix P of transition probability on the lattice $E = \mathbb{Z}^d$ (ie, nonnegative numbers that satify $\sum_{y \in E} p(x, y) = 1$ for each $x \in E$):

$$P = (p(x, y))_{x,y \in E}.$$

We assume that, for all $x, y \in E$, $p(x, y)$ is:

1. translation invariant: $p(x, y) = p(0, y - x)$;
2. finite range: there exists a finite set $B^p \subseteq E$ such that $p(0, x) = 0$ for all $x \notin B^p$.

Description of the exclusion process

We define the **state** of the system as a set of occupied and vacant sites. For all $x \in E$,

$$\eta(x) := \begin{cases} 1 \text{ if } x \text{ is occupied;} \\ 0 \text{ if } x \text{ is empty.} \end{cases}$$

Thus, we introduce the **configuration of the system** η on the **state space** $X = \{0,1\}^E$:

$$\eta := \{\eta(x) \mid x \in E\}.$$

Objective: Construct rigorously a Markov process $\eta_t = (\eta_t(x))_{x \in E}$ *that corresponds to the description given above.*

Let $E_p^2 := \{(x, y) \in E^2 \mid p(x, y) > 0\}$ be the set of pairs of sites between which jump attents can happen.

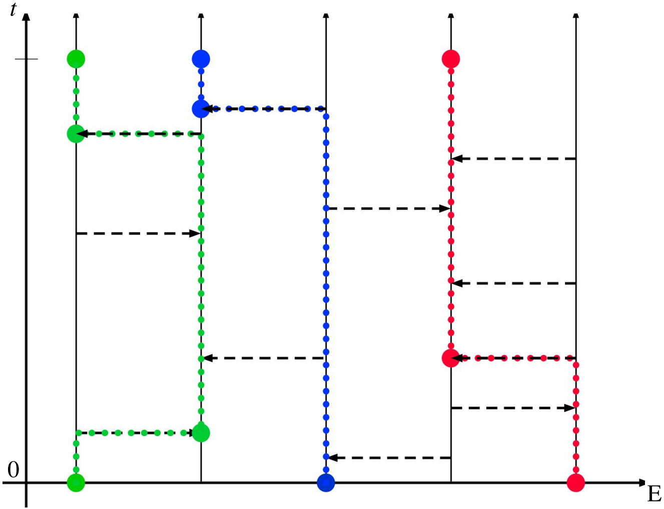
Let $(\Omega, \mathcal{H}, \mathbb{P})$ a probability space on which is defined a family $\{\mathcal{T}_{(x,y)} \mid (x, y) \in E_p^2\}$ of mutually independent Poisson processes on the time line $[0, +\infty[$.
The Poisson process $\mathcal{T}_{(x,y)}$ is homogeneous with rate $p(x, y)$ and the jump times of $\mathcal{T}_{(x,y)}$ are the random times at which a particle attempts to move from x to y.

Important hypothesis

The set of all times when a jump either in or out of x can happen has only finitely many jump times in every bounded interval $]0, T]$.

Graphical representation of the exclusion process

Assume given an initial state $\eta \in X$.



Problem of this construction

A problem arises:

If the initial state η has infinitely many particles, an inifinity of particles may attempt to jump in every small time interval $]0, \epsilon[$.

Example: To compute the value $\eta_t(0)$ *for some* $t > 0$*, we have to consider* $\eta_s(x),\ 0 \leq s \leq t$*, with all sites* x *that interacted with* 0 *during* $]0, t]$*. And so on...*

The percolation argument

The percolation argument guarantees that for a short deterministic time interval $[0, t_0]$, the set E can be decomposed into disjoint finite components that do not interact during $[0, t_0]$.
In each finite component, the evolution of the configuration η_t for $0 \leq t \leq t_0$ can be constructed because we consider only a finite number of time jumps.

For $0 \leq s < t$, let $\mathcal{G}_{s,t}$ the **undirected random graph** with **vertex set** E and **edge set** $\mathcal{E}_{s,t}$ defined by:

$$\mathcal{E}_{s,t} = \{\{x, y\} \in E^2 \mid \mathcal{T}_{(x,y)} \text{ or } \mathcal{T}_{(y,x)} \text{ has a jump time in }]s, t]\}.$$

Consequence: To compute the evolution $\eta_s(x)$ for $0 \leq s \leq t$, only the sites who are in the same connected component as x in the graph $\mathcal{G}_{0,t}$ are relevant.

The percolation argument

Lemma:

Each edge $\{x, y\}$ is present in $\mathcal{G}_{s,t}$ with probability $1 - e^{-(t-s)(p(x,y)+p(y,x))}$, independently of the other edges.

Proof.

$$\begin{aligned}
\mathbb{P}(\{x, y\} \in \mathcal{G}_{s,t}) &= \mathbb{P}(\{x, y\} \in \mathcal{E}_{s,t}) \\
&= \mathbb{P}(\mathcal{T}_{(x,y)} \text{or } \mathcal{T}_{(y,x)} \text{ has a jump time in }]s, t]) \\
&= 1 - \mathbb{P}(\mathcal{T}_{(x,y)} \text{ and } \mathcal{T}_{(y,x)} \text{ have not a jump time in }]s, t]) \\
&= 1 - \mathbb{P}(\mathcal{T}_{(x,y)} \text{ has not a jump time in }]s, t])\times \\
&\qquad \mathbb{P}(\mathcal{T}_{(y,x)} \text{ has not a jump time in }]s, t]) \\
&= 1 - \mathbb{P}(\text{the waiting time of } \mathcal{T}_{(x,y)} \text{ is bigger than } t - s)\times \\
&\qquad \mathbb{P}(\text{the waiting time of } \mathcal{T}_{(y,x)} \text{ is bigger than } t - s) \\
&= 1 - e^{-(t-s)(p(x,y)+p(y,x))}
\end{aligned}$$

because waiting times are exponentially distributed. □

Proposition: Percolation argument

If t_0 is small enough, the random graph $\mathcal{G}_{0,t_0}$ has almost surely only finite connected components.

Proof.

Notations: $B_* = B^p \cup (-B^p)$; $R = \max_{x \in B_*} |x|_\infty$; $k_* = \mathrm{Card}(B_*)$.

Remark: $\mathcal{T}_{(x,y)}$ contains jump times implies that $|y-x|_\infty \leq R$.

Let show that for t_0 fixed small enough, the connected component containing 0 is finite a.s.

We will use:

✎ *The **finite range** assumption:* there exists a finite set $B^p \subseteq E$ such that $p(0,x) = 0$ for all $x \notin B^p$.

✎ *The **lemma**:* $\mathbb{P}(\{x,y\} \in \mathcal{G}_{s,t}) = 1 - e^{-(t-s)(p(x,y)+p(y,x))}$.

✎ *The assumption of **translation invariance**:* $p(x,y) = p(0, y-x)$ for all $x, y \in E$.

Construction of the exclusion process

To construct the process η_t for $0 \leq t \leq t_0$, we use the percolation argument and the "Important hypothesis".

Step 1: Construction with finitely many jump times on the time interval $[0, t_0]$ for a particular connected component.

Jump times: $0 < \tau_1 < \tau_2 < ... < \tau_n$.
Let $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ such that $\tau_k \in \mathcal{T}_{(x_k, y_k)}$.

We have:

$$\eta_t = \eta_0 \quad \text{for } 0 \leq t \leq \tau_1;$$

$$\begin{cases} \eta_{\tau_1} := \eta^{x_1, y_1}_{\tau_1^-} & \text{if } \eta(x_1) = 1 \text{ and } \eta(y_1) = 0 \\ \eta_{\tau_1} = \eta_{\tau_1^-} & \text{if } \eta(x_1) = 0 \text{ or } \eta(y_1) = 1 \end{cases}$$

and so on...

Construction of the exclusion process

To construct the process η_t for $0 \leq t \leq t_0$, we use the percolation argument and the "Important hypothesis".

Step 1: Construction with finitely many jump times on the time interval $[0, t_0]$ for a particular connected component.

Step 2: We repeat the construction for each connected component.

Step 3: Construction for all time $0 \leq t < \infty$.

Conclusion: The evolution η_t can be constructed for all time $(0 \leq t < \infty)$, for all initial configuration η and for all set of jump time processes $\left(\mathcal{T}_{(x,y)}\right)_{(x,y) \in E_p^2}$.

Theorem 1:

$\eta_t = (\eta_t(x))_{x \in E}$ defined by this construction is a Markov process.

Let Ω_0 the **set of paths** ω that satisfy "Important hypotesis" and for which the random graphs $\mathcal{G}_{kt_0,(k+1)t_0}$ have finite connected components for all $k \in \mathbb{N}$.

Theorem 2:

For all $(\eta, \omega) \in X \times \Omega_0$, the function $t \longmapsto \eta_t^\eta(\omega)$ is right-continuous and has left limits for all $t \geq 0$.

Theorem 3:

The path $t \longmapsto \eta_t^\eta(\omega)$ is continuous in $(\eta, \omega) \in X \times \Omega_0$.

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