Ergodic BSDEs and Large time behaviour of solutions of finite horizon BSDEs

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In the following, W is a d-dimensional Brownian motion over a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, whose natural filtration is $(\mathcal{F}_t)_{t\geq 0}$.

What does it look like?

$$\forall x \in \mathbb{R}^d, \forall 0 \leq t \leq T < \infty, \; Y_t^x = Y_T^x + \int_t^T \left[\psi\left(X_s^x, Z_s^x\right) - \lambda\right] \mathrm{d}s - \int_t^T Z_s^x \mathrm{d}W_s,$$

where:

- X^x satisfies the SDE: $X_t^x = x + \int_0^t \Xi\left(X_s^x\right) \mathrm{d}s + \int_0^t \sigma\left(X_s^x\right) \mathrm{d}W_s$;
- $\Xi : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathrm{GL}_d(\mathbb{R})$ and $\psi : \mathbb{R}^d \times \left(\mathbb{R}^d\right)^* \to \mathbb{R}$.

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The unknowns are $\left(Y_t^x\right)_{t\geq 0}$, $\left(Z_t^x\right)_{t\geq 0}$ and the real number λ.

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Assumptions:

- Ξ is Lipschitz continuous and weakly dissipative (*i.e.* $\langle \Xi(x), x \rangle \leq \eta_1 - \eta_2 |x|^2$);
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- $\psi(\bullet, 0)$ is bounded and $\psi(x, z)$ is Lipschitz continuous *w.r.t.* $\left(x, z\sigma(x)^{-1}\right)$ (with constant K);
- $\sqrt{r_2} K \|\sigma^{-1}\|_\infty + \frac{r_2}{2} < \eta_2$ and $\Lambda < \frac{1}{\sqrt{2}\,\|\sigma^{-1}\|_\infty}$.

How do we solve them?

Auxiliary BSDE of infinite horizon: for $x \in \mathbb{R}^d$ and $0 \leq t \leq T < \infty$,

$$Y_t^{\alpha,x} = Y_T^{\alpha,x} + \int_t^T \left[\psi\left(X_s^x, Z_s^{\alpha,x}\right) - \alpha Y_s^{\alpha,x}\right] \mathrm{d}s - \int_t^T Z_s^{\alpha,x} \mathrm{d}W_s,$$

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Theorem (Briand, Hu '98)

This BSDE has a unique solution $\left(Y^{\alpha,x}, Z^{\alpha,x}\right)$ with $Y^{\alpha,x}$ bounded continuous and $Z^{\alpha,x} \in \mathcal{M}^2_{loc}\left(0, \infty, \mathbb{R}^d\right)$.

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For a subsequence, when $\alpha \longrightarrow 0$:

$$v^\alpha(x) - v^\alpha(0) \longrightarrow \overline{v}(x), \qquad \alpha v^\alpha(0) \longrightarrow \overline{\lambda}, \qquad \zeta^\alpha(x) \longrightarrow \overline{\zeta}(x).$$

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Define $\overline{Y}^x = \overline{v}\left(X^x\right)$ and $\overline{Z}^x = \overline{\zeta}\left(X^x\right)$, we get:

$$\overline{Y}_t^x = \overline{Y}_T^x + \int_t^T \left[\psi\left(X_s^x, \overline{Z}_s^x\right) - \overline{\lambda}\right] \mathrm{d}s - \int_t^T \overline{Z}_s^x \mathrm{d}W_s.$$

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Theorem (Hu, L. '17)

We consider the EBSDE (under the previous assumptions)

$$Y_t^x = Y_T^x + \int_t^T \left[\psi\left(X_s^x, Z_s^x\right) - \lambda\right] ds - \int_t^T Z_s^x dW_s.$$

It has a solution $\left(\overline{v}\left(X^x\right), \overline{\zeta}\left(X^x\right), \overline{\lambda}\right)$ *where* $\overline{v}$ *is locally Lipschitz, has polynomial growth,* $\overline{v}(0) = 0$*, and* $\overline{\zeta}$ *is measurable.*
This solution is unique among the triples $\left(v\left(X^x\right), \zeta\left(X^x\right), \lambda\right)$ *where* v *is continuous, has polynomial growth,* $v(0) = 0$ *and* ζ *is measurable.*

Large time behaviour of solutions of finite horizon BSDEs

Consider the BSDE, for $0 \leq t \leq T$:

$$Y_t^{T,x} = g\left(X_T^x\right) + \int_t^T \psi\left(X_s^x, Z_s^{T,x}\right) \mathrm{d}s - \int_t^T Z_s^{T,x} \, \mathrm{d}W_s,$$

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- 2^{nd} *behaviour: there exists* $L \in \mathbb{R}$, *such that:*

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- 3^{rd} *behaviour: for any* $\delta > 0$ *small enough,*

$$\forall x \in \mathbb{R}^d, \forall T > 0,\ \left|Y_0^{T,x} - \overline{\lambda}T - \overline{Y}_0^x - L\right| \leq C_\delta\left(1 + |x|^{\mu+\delta}\right) e^{-\nu\delta T}.$$

Some references

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- Hu, Madec, Richou. A probabilistic approach to large time behavior of mild solutions of Hamilton-Jacobi-Bellman equations in infinite dimension. *SIAM Journal on Control and Optimization*, 53(1):378-398, 2015.

Thanks for your attention!