La formule de Stirling via le TCL

224, 262, (265), (241).

Ref : Garet-Kurtzman

Théorème : (Stirling) $n! \underset{+\infty}{\sim} \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

(1) Introduction

Démo : Soient (X_n) iid $\sim \mathcal{E}(1)$. On pose $S_n = \sum_{k=0}^{n} X_k$.

Alors $S_n \sim \Gamma(n+1, 1)$ $(= \mathcal{E}(n+1))$.

Or $\mathbb{E}(S_n) \underset{iid}{=} (n+1)\mathbb{E}(X_0) = n+1$. et $Var(S_n) \underset{iid}{=} (n+1) Var(X_0) = n+1$

Le TCL donne alors : $\frac{S_n - (n+1)}{\sqrt{n+1}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$

Alors $\frac{S_n - n}{\sqrt{n}} = \sqrt{\frac{n+1}{n}}\left(\frac{S_n-(n+1)}{\sqrt{n+1}} + \frac{n+1}{\sqrt{n+1}} - \frac{n}{\sqrt{n+1}}\right) = \underbrace{\sqrt{\frac{n+1}{n}}}_{\xrightarrow{\mathbb{P}} 1} \underbrace{\frac{S_n-(n+1)}{\sqrt{n+1}}}_{\xrightarrow{\mathcal{L}} \mathcal{N}(0,1)} + \underbrace{\frac{1}{\sqrt{n}}}_{\xrightarrow{\mathcal{L}} 0}$

Par Slutsky, $\sqrt{\frac{n+1}{n}} \frac{S_n-(n+1)}{\sqrt{n+1}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$

et par continuous mapping theorem, $\frac{S_n - n}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$.

(2) Lien entre densités

$N_n := \frac{S_n - n}{\sqrt{n}}$

On a $\mathbb{P}\left(\frac{S_n - n}{\sqrt{n}} \le x\right) = \mathbb{P}(S_n \le n + x\sqrt{n})$.

Ainsi $f_{N_n}(x) = \partial_x\left(\mathbb{P}(S_n \le n + x\sqrt{n})\right) = \sqrt{n}\, f_{S_n}(n + x\sqrt{n})$.

$$= \frac{\sqrt{n}\, e^{-(n+x\sqrt{n})}}{\Gamma(n+1)} (n + x\sqrt{n})^n \mathbb{1}_{\mathbb{R}_+^*}(n + x\sqrt{n})$$

$$= \frac{\sqrt{n}}{n!} e^{-(n+x\sqrt{n})} n^n \left(1 + \frac{x}{\sqrt{n}}\right)^n \mathbb{1}_{[-\sqrt{n}, +\infty[}(x)$$

$$g_n(x) := f_{N_n}(x) = \underbrace{\frac{\sqrt{2\pi n}}{n!}\left(\frac{n}{e}\right)^n}_{=: a_n} \underbrace{e^{-x\sqrt{n}}\left(1 + \frac{x}{\sqrt{n}}\right)^n \mathbb{1}_{[-\sqrt{n}, +\infty[}(x) \frac{1}{\sqrt{2\pi}}}_{=: h_n(x)}$$

$g_n(x) = a_n h_n(x)$.

3) Limite des intégrales

- $\frac{S_n - n}{\sqrt{n}} = N_n \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$ donc $\mathbb{P}(N_n \leq x) \underset{n\to+\infty}{\longrightarrow} \mathbb{P}(N \leq x)$

$N \sim \mathcal{N}(0,1)$.

Soit : $\int_0^1 g_n(x)\,dx = \mathbb{P}(N_n \in [0,1]) \underset{n\to+\infty}{\longrightarrow} \mathbb{P}(N \in [0,1])$

$= \int_0^1 \frac{e^{-x^2/2}\,dx}{\sqrt{2\pi}}$

- $$e^{-\sqrt{n}x}\left(1+\frac{x}{\sqrt{n}}\right)^n = e^{-\sqrt{n}x} e^{n\ln\left(1+\frac{x}{\sqrt{n}}\right)}$$
$$= e^{-\sqrt{n}x} e^{n\left(\frac{x}{\sqrt{n}} - \frac{x^2}{2n} + o\left(\frac{1}{n}\right)\right)}$$
$$= e^{-x^2/2 + o(1)} \longrightarrow e^{-x^2/2}$$

Pour $n \gg 1$, $e^{-\sqrt{n}x}\left(1+\frac{x}{\sqrt{n}}\right)^n \leq 2$ car $e^{-x^2/2} \leq 1$

et $2 \in L^1(0,1)$.

Donc par convergence dominée,

$$\lim_{n\to+\infty} \int_0^1 h_n(x)\,dx = \lim_{n\to+\infty} \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\sqrt{n}x}\left(1+\frac{x}{\sqrt{n}}\right)^n dx \overset{cvd}{\longrightarrow} \int_0^1 \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$$

Ainsi, $\underbrace{\int_0^1 g_n(x)\,dx}_{\to \int_0^1 \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx} = a_n \underbrace{\int_0^1 h_n(x)\,dx}_{\to \int_0^1 \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx}$ donc $a_n \to 1$.

Soit $\frac{\sqrt{2\pi n}}{n!}\left(\frac{n}{e}\right)^n \to 1$ ie $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.