Symmetric sums of squares over k-subset hypercubes

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February 2, 2024

This article is about sums of squares (*sos*) over k-subsets hypercubes. This has a lot of applications in areas such as combinatorial optimization, decision problems and proof complexity.

Notation

We will consider the following 2-subset hypercube $\mathcal{V}_n := \{0,1\}^{\binom{n}{2}}$, and polynomial functions over it. To define them, let $\mathbb{R}[x] := \mathbb{R}[x_{ij} \mid 1 \leq i < j \leq n]$ and the ideal $\mathcal{I}_n := \langle x_{ij}^2 - x_{ij} \mid 1 \leq i < j \leq n \rangle$. The set of polynomial functions on $\mathcal{V}_n$, denoted by $\mathbb{R}[\mathcal{V}_n]$, is the quotient $\mathbb{R}[x]/\mathcal{I}_n$. Note that this set is in bijection with the *square-free* polynomials in $\mathbb{R}[x]$. Now we define a natural action of the symmetric group $\mathfrak{S}_n$ on $\mathbb{R}[\mathcal{V}_n]$: $\sigma \cdot x_{ij} := x_{\sigma(i)\sigma(j)}$. We will work on polynomials with degree at most d, denoted by $\mathbb{R}[\mathcal{V}_n]_{\leq d}$. We will also need some notation about the representations of the symmetric group $\mathfrak{S}_n$. The irreducible $\mathfrak{S}_n$-modules are indexed by the partitions λ of n, that we write $\lambda \vdash n$. Since $V := \mathbb{R}[\mathcal{V}_n]_{\leq d}$ is a $\mathfrak{S}_n$-module, it has an *isotypic* decomposition $V = \bigoplus_{\lambda \vdash n} V_\lambda$ with $V_\lambda = S_\lambda^{m_\lambda}$ where S_λ is the irreducible $\mathfrak{S}_n$-module associated to λ, and $m_\lambda \in \mathbb{N}$. If τ_λ is a tableau of shape λ, let $\mathfrak{R}_{\tau_\lambda}$ be the *row group* of τ_λ, that is the subgroup of $\mathfrak{S}_n$ that leaves each row of τ_λ invariant. Now we define W_{τ_λ} to be the subspace of V_λ consisting of all points fixed by $\mathfrak{R}_{\tau_\lambda}$.

1 Cornerstone of the paper: a result of Gatermann and Parrilo

The main point of the article is to present some improvements of a result of Gatermann and Parrilo.

1.1 The result

This result tells us the structure of symmetry-reduced sos-expressions for $\mathfrak{S}_n$-invariant d-sos polynomials, and shows that we can search for such sos expressions by solving a SDP of size $\sum_{\lambda \vdash n} m_\lambda$.

Theorem 1 (Gatermann-Parrilo, 2004)
Suppose $p \in \mathbb{R}[\mathcal{V}_n]$ is $\mathfrak{S}_n$-invariant and d-sos. For each partition $\lambda \vdash n$, fix a tableau τ_λ of shape λ and choose a vector space basis $\{b_1^{\tau_\lambda}, \dots, b_{m_\lambda}^{\tau_\lambda}\}$ for W_{τ_λ}. Then for each partition λ of n there exists a $m_\lambda \times m_\lambda$ psd matrix Q_λ such that

$$p = \sum_{\lambda \vdash n} tr(Q_\lambda Y^{\tau_\lambda}) \tag{1}$$

where $Y_{ij}^{\tau_\lambda} := \mathrm{sym}(b_i^{\tau_\lambda} b_j^{\tau_\lambda})$.

1.2 Improvements

The main aim of the article is to provide two improvements of this result:

- proving that one can bound the number of partitions in the sum independently of n, and that each m_λ is bounded above by a quantity independent of n;
- proving that one can relax the conditions on the living space of the $b_i^{\tau_\lambda}$s.

It also turns out that the methods used also have applications to combinatorial problems.

2 Bounding the number or partitions

The following results shows that we can restrict our attention on some specific partitions.

Theorem 2
The dimension m_λ of W_{τ_λ} for any tableau of shape λ is zero unless $\lambda \geq_{\text{lex}} (n-2d, 1^{2d})$.

It allows us to bound the number of partitions needed in the sum of (1) independently on n.

Proposition 1
The number of partitions λ such that m_λ is not zero is bounded above by $p(0)+p(1)+\ldots+p(2d)$ where $p(i)$ is the number of partitions of i.

3 Finding spanning sets

3.1 More general spanning sets

The following theorem allows us to look for other spanning sets than bases for the W_{τ_λ}.

Theorem 3
Suppose $p \in \mathbb{R}[\mathcal{V}_n]$ is $\mathfrak{S}_n$-invariant and d-sos. For each partition $\lambda \vdash n$, fix a tableau τ_λ of shape λ and let $\{p_1^{\tau_\lambda}, \ldots, p_{l_\lambda}^{\tau_\lambda}\}$ be a set of polynomials whose span contains W_{τ_λ}. Then for each partition λ of n there exists a $m_\lambda \times m_\lambda$ psd matrix Q_λ such that

$$p = \sum_{\lambda \vdash n} tr(Q_\lambda Y^{\tau_\lambda})$$

where $Y_{ij}^{\tau_\lambda} := \text{sym}(p_i^{\tau_\lambda} p_j^{\tau_\lambda})$.

Now our concern is to find such spanning sets that yield succinct d-sos expressions for $\mathfrak{S}_n$-invariant polynomials. Moreover, we want these polynomials to be easier to enumerate, in order to have a general expression.

The idea is to substitute a tableau τ_λ by its *hook*, i.e we keep the first row of length λ_1 and we put $n-\lambda_1$ rows of length 1 underneath, and then to take the *symmetrizations of monomials* ($\text{sym}_{\tau_\lambda}(x^m) = \frac{1}{|\mathfrak{R}_{\tau_\lambda}|} \sum_{\mathfrak{s} \in \mathcal{R}_{\tau_\lambda}} \mathfrak{s} \cdot x^m$) by these tableaux to get the desired polynomials.

3.2 Construction of polynomials with flags

From the previous spanning sets, we will now construct two more polynomials families. The main point with these polynomials is that they can be computed thanks to graph theory and flags.

Definition
Let $0 \leq t \leq f \leq n$.

- An *intersection type* T of size t is a simple graph T on t vertices labeled by distinct elements of $[t]$;
- A *T-flag* F of size f is a simple graph on f vertices with t vertices labeled by distinct elements of $[t]$ which induce a copy of T in F. We denote by $\mathcal{F}_T^f$ the set of all T-flag of size f, up to isomorphism.

One can now consider, for $\Theta \in \text{Inj}([t],[n])$, the set $\text{Inj}_\Theta(V(F),[n])$ of injective functions $h : V(F) \to [n]$ that respect Θ, i.e. $h(v) = \Theta(i)$ for any vertex $v \in V(F)$ labeled i. These definitions lead us to our first interesting family of polynomials.

Definition
For T, f and Θ fixed, we define for $F \in \mathcal{F}_T^f$:

$$g_F^\Theta := \sum_{h \in \text{Inj}_\Theta(V(F),[n])} \prod_{\{i,j\} \in E(F)} x_{h(i),h(j)}.$$

Remark 1:
Since our graphs are simple, g_F^Θ is square-free. Moreover, it is possible to rewrite these polynomials in terms of the previous symmetrizations :

$$g_F^\Theta = \frac{(n-t)!}{(n-f)!} \text{sym}_{hook(\tau_\lambda)}(x^m)$$

where $x^m = \prod_{\{i,j\} \in E(F)} x_{h(i),h(j)}$ for any $h \in \text{Inj}_\Theta([f],[n])$.

Now we introduce another family of polynomials than span W_{τ_λ}. For that, for a fixed intersection type T, we construct a natural order over the set $\mathcal{F}^f_{\geq T}$ of T'-flags such that T is a subgraph of T', denoted by $\leq$.

Definition
For a flag $F \in \mathcal{F}^f_{\geq T}$ we define

$$d^\Theta_F := \sum_{\substack{F' \in \mathcal{F}^f_{\geq T} \\ F' \geq F}} (-1)^{|E(F')|-|E(F)|} g^\Theta_{F'}.$$

Theorem 4
For the tableau τ_λ, the vector space W_{τ_λ} is spanned on one hand by the polynomials $g_F^{\Theta_{\tau_\lambda}}$ for $F \in \mathcal{F}^{2d}_T$ where $|T| = n - \lambda_1$ and on the other hand by the polynomials $d_F^{\Theta_{\tau_\lambda}}$ for $F \in \mathcal{F}^{2d}_T$ where $|T| = n - \lambda_1$.

According to Theorems 3 and 4, we can express a certificate for a $\mathfrak{S}_n$-invariant and d-sos polynomial with those flag-based polynomials. Moreover, thanks to the restricted set of partitions we are considering and the properties of intersection types, one can prove that the size of the psd matrices obtained in the certificate does not depend on n.

3.3 Restricting to flag sos expressions

However, in the literature, more is known about more restricted flag expressions. Thus we will show that those ones, which we refer to as *flag sos* expressions, suffice for our certificates.

Definition
Let $\Theta_0 \in \mathrm{Inj}([f],[n])$ and $F, F' \in \mathcal{F}^f_T$ where $|T| = t$. We define

$$\mathbb{E}_{\Theta_0}[d^{\Theta_0}_F d^{\Theta_0}_{F'}] = \frac{1}{|\mathrm{Inj}([f],[n])|} \sum_{\Theta \in \mathrm{Inj}([f],[n])} d^\Theta_F d^\Theta_{F'}$$

Definition
Let $\mathbf{d}^{\Theta,T,f} = (d^\Theta_F)_{F \in \mathcal{F}^f_T}$ be the vector of flag polynomials for a fixed intersection type T, flag size f, and labeling Θ. A *flag sos* is a sos expression of the form

$$\sum_{T,f} tr\left(R_{T,f} \mathbb{E}_\Theta[\mathbf{d}^{\Theta,T,f}\ {}^t\mathbf{d}^{\Theta,T,f}]\right).$$

The key argument for the following theorem is that for two flags F, F' with different intersection type of size t, the product $d^\Theta_F d^\Theta_{F'}$ is zero.

Theorem 5
If p is a $\mathfrak{S}_n$-invariant and d-sos polynomial, then p is also $2d$-flag sos.

4 Example in combinatorics

Here is an example where the method developed has applications to a combinatorial problem.

Theorem 6 (Kővari-Sós-Turán)
Let G be a n-vertices graph not containing a C_4 (the cycle on four vertices). Then the number of edges of G is at most $\frac{1}{2}n^{3/2} + O(n)$.

Proof: We just give an overview of the proof. We consider

$$s = \sum_{1 \leq i < j \leq n} x_{ij}$$

and

$$\mathcal{I} = \langle x_{ij}^2 - x_{ij}\ \forall 1 \leq i < j \leq n,\ x_{ij}x_{jk}x_{kl}x_{li}\ \forall i,j,k,l \rangle.$$

Note that for a graph G, $s(\mathbb{1}_G)$ is the number of edges of G, and the variety of $\mathcal{I}$ consists exactly of the characteristic vectors of graphs avoiding 4-cycles. Thus, to prove the theorem it suffices to show that $n + \frac{2}{n-1}s - \frac{2}{\binom{n}{2}}s^2$ is 2-sos modulo $\mathcal{I}$. Indeed it will imply that $s(\mathbb{1}_G) = |E(G)| \leq \frac{n+\sqrt{4n^3-3n^2}}{4}$ for all C_4-free graph G. The method developed enables to provide such a certificate. □