Symmetric sums of squares over k-subset hypercubes

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Aim of the paper

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Produce succinct certificates for symmetric and d-sos polynomials over the hypercube $\mathcal{V}_{\mathsf{n}} = \{0,1\}^{\binom{n}{2}}$.

Notation

- Polynomials over $\mathcal{V}_{\mathrm{n}}$: $\mathbb{R}[\mathcal{V}_{\mathrm{n}}]$
- Action of $\mathfrak{S}_n$ on $\mathbb{R}[\mathcal{V}_{\mathrm{n}}]$: $\sigma \cdot x_{ij} := x_{\sigma(i)\sigma(j)}$
- Partitions of n: $\lambda \vdash n$
- Irreducible $\mathfrak{S}_n$-module indexed by λ: $\mathcal{S}_\lambda$
- $V := \mathbb{R}[\mathcal{V}_{\mathrm{n}}]_{\leq d}$ is a $\mathfrak{S}_n$-module, so $V = \bigoplus_{\lambda \vdash n} S_\lambda^{m_\lambda}$
- Row group of a tableau τ_λ of shape λ (subgroup of $\mathfrak{S}_n$ that leaves each row of τ_λ invariant): $\mathcal{R}_{\tau_\lambda}$
- **W_{τ_λ} is the subspace of $S_\lambda^{m_\lambda}$ consisting of all points fixed by $\mathcal{R}_{\tau_\lambda}$**

The main result

Theorem [Gatermann-Parrilo, 2004]

Suppose $p \in \mathbb{R}[\mathcal{V}_\mathrm{n}]$ is $\mathfrak{S}_n$-invariant and d-sos. For each partition $\lambda \vdash n$, fix a tableau τ_λ of shape λ and choose a vector space basis $\{b_1^{\tau_\lambda}, \dots, b_{m_\lambda}^{\tau_\lambda}\}$ for W_{τ_λ}. Then for each partition λ of n there exists a $m_\lambda \times m_\lambda$ psd matrix Q_λ such that

$$p = \sum_{\lambda \vdash n} tr(Q_\lambda Y^{\tau_\lambda}) \tag{1}$$

where $Y_{ij}^{\tau_\lambda} := sym(b_i^{\tau_\lambda} b_j^{\tau_\lambda})$.

Improvements

- Proving that one can bound the number of partitions in the sum independently of n
- Proving that one can relax the conditions on the living space of the $b_i^{\tau\lambda}$s.

1. A first result of Gatermann and Parrilo
2. Bounding the number of partitions
3. Finding spanning sets
4. Example in combinatorics

Bounding the number of partitions

Theorem

The dimension m_λ of W_{τ_λ} for any tableau of shape λ is zero unless $\lambda \geq_{\text{lex}} (n-2d, 1^{2d})$.

Bounding the number of partitions

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The dimension m_λ of W_{τ_λ} for any tableau of shape λ is zero unless $\lambda \geq_{\text{lex}} (n-2d, 1^{2d})$.

Proposition

The number of partitions λ such that m_λ is not zero is bounded above by $p(0) + p(1) + \ldots + p(2d)$ where $p(i)$ is the number of partitions of i.

1. A first result of Gatermann and Parrilo
2. Bounding the number of partitions
3. Finding spanning sets
   - A first result
   - Construction of polynomials by graph theory
   - Restricting to flag sos expressions
4. Example in combinatorics

We can look for other spanning spaces

Theorem

Suppose $p \in \mathbb{R}[\mathcal{V}_\mathrm{n}]$ is $\mathfrak{S}_n$-invariant and d-sos. For each partition $\lambda \vdash n$, fix a tableau τ_λ of shape λ and let $\{p_1^{\tau_\lambda}, \dots, p_{l_\lambda}^{\tau_\lambda}\}$ be a set of polynomials whose span contains W_{τ_λ}. Then for each partition λ of n there exists a $m_\lambda \times m_\lambda$ psd matrix Q_λ such that

$$p = \sum_{\lambda \vdash n} tr(Q_\lambda Y^{\tau_\lambda})$$

where $Y_{ij}^{\tau_\lambda} := sym(p_i^{\tau_\lambda} p_j^{\tau_\lambda})$.

Symmetrized monomials and hook partitions

Definition

For a partition λ and a tableau τ_λ of shape λ, we define the *symmetrization of monomials* :

$$\mathrm{sym}_{\tau_\lambda}(x^m) = \frac{1}{|\mathcal{R}_{\tau_\lambda}|} \sum_{\mathfrak{s} \in \mathcal{R}_{\tau_\lambda}} \mathfrak{s} \cdot x^m.$$

Hook partitions

Definition

Given τ_λ a tableau of shape $\lambda = (\lambda_1, \lambda_2, \ldots)$, we define *hook*$(\tau_\lambda)$ to be the tableau of shape $(\lambda_1, 1^{n-\lambda_1})$ were the first row is the same as in τ_λ and the labels in the tail are the remaining ones placed in increasing order.

Example with $\lambda = (5, 2, 2, 1)$

If $\tau_\lambda =$

7	4	9	2	6
10	8			
5	1			
3				

then *hook*$(\tau_\lambda) =$

7	4	9	2	6
1				
3				
5				
8				
10				

A first result

Theorem

For the tableau τ_λ, the vector space W_{τ_λ} is spanned by the polynomials $\mathrm{sym}_{hook(\tau_\lambda)}(x^m)$, as x^m varies over square-free monomials of degree at most d.

Combinatorial tools

Definition

Let $0 \leq t \leq f \leq n$.

- An *intersection type* T of size t is a simple graph T on t vertices labeled by distinct elements of $[t]$;

Combinatorial tools

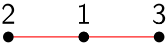
Definition

Let $0 \leq t \leq f \leq n$.

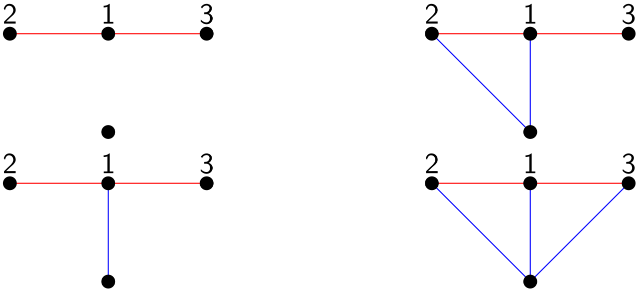
- An *intersection type* T of size t is a simple graph T on t vertices labeled by distinct elements of $[t]$;
- A T*-flag* F of size f is a simple graph on f vertices with t vertices labeled by distinct elements of $[t]$ which induce a copy of T in F.

We denote by $\mathcal{F}_T^f$ the set of all T-flag of size f, up to isomorphism.

Combinatorial tools

Example: If $T =$ 

and $f = 4$, then $\mathcal{F}_T^f$ has 8 elements such as



Combinatorial tools

Definition

For a $\Theta \in \mathrm{Inj}([t],[n])$, we define the set $\mathrm{Inj}_\Theta(V(F),[n])$ of injective functions $h : V(F) \to [n]$ that respect Θ :

$$h \in \mathrm{Inj}_\Theta(V(F),[n]) \Leftrightarrow h(v) = \Theta(i) \text{ for any } v \in V(F) \text{ labeled } i \in [t].$$

Construction of polynomials

Definition

For T, f and Θ fixed, we define for $F \in \mathcal{F}_T^f$:

$$g_F^{\Theta} := \sum_{h \in \mathrm{Inj}_{\Theta}(V(F),[n])} \prod_{\{i,j\} \in E(F)} x_{h(i),h(j)}.$$

Construction of polynomials

Definition

For T, f and Θ fixed, we define for $F \in \mathcal{F}^f_T$:

$$g_F^\Theta := \sum_{h \in \mathrm{Inj}_\Theta(V(F),[n])} \prod_{\{i,j\} \in E(F)} x_{h(i),h(j)}.$$

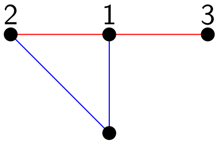
Definition

For a flag $F \in \mathcal{F}^f_{\geq T}$ we define

$$d_F^\Theta := \sum_{\substack{F' \in \mathcal{F}^f_{\geq T} \\ F' \geq F}} (-1)^{|E(F')|-|E(F)|} g_{F'}^\Theta.$$

Construction of polynomials

Example: Let F be the following T-flag:



and Θ be such that $\Theta(1) = i$, $\Theta(2) = j$, $\Theta(3) = k$. Then

$$g_F^{\Theta} = x_{ij}x_{ik} \sum_{l \in [n] \setminus \{i,j,k\}} x_{il}x_{jl}.$$

Certificate theorem

Theorem

For the tableau τ_λ, the vector space W_{τ_λ} is spanned on one hand by the polynomials $g_F^{\Theta_{\tau_\lambda}}$ for $F \in \mathcal{F}_T^{2d}$ where $|T| = n - \lambda_1$ and on the other hand by the polynomials $d_F^{\Theta_{\tau_\lambda}}$ for $F \in \mathcal{F}_T^{2d}$ where $|T| = n - \lambda_1$.

Conclusion

Theorem

Suppose that p is symmetric and s-sos. For each partition $\lambda \geq_{\text{lex}} (n-2d, 1^{2d})$, fix a tableau τ_λ of shape λ. Then there exists psd matrices R_λ such that

$$p = \sum_{\lambda \geq_{\text{lex}} (n-2d, 1^{2d})} tr(R_\lambda Z^{\tau_\lambda})$$

where $Z^{\tau_\lambda} := \text{sym}(d_{\tau_\lambda} d_{\tau_\lambda}^T)$ and d_{τ_λ} is the vector of polynomials $d_F^{\Theta_{\tau_\lambda}}$ such that $F \in \mathcal{F}_T^{2d}$ with $|T| = n - \lambda_1$.

A substitution to symmetrized polynomial

Definition

Let $\Theta_0 \in \mathrm{Inj}([f],[n])$ and $F, F' \in \mathcal{F}_T^f$ where $|T| = t$. We define

$$\mathbb{E}_{\Theta_0}[d_F^{\Theta_0} d_{F'}^{\Theta_0}] = \frac{1}{|\mathrm{Inj}([f],[n])|} \sum_{\Theta \in \mathrm{Inj}([f],[n])} d_F^{\Theta} d_{F'}^{\Theta}$$

A substitution to symmetrized polynomial

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Definition

Let $\mathbf{d}^{\Theta,T,f} = (d_F^{\Theta})_{F \in \mathcal{F}_T^f}$ be the vector of flag polynomials for a fixed intersection type T, flag size f, and labeling Θ. A *flag sos* is a sos expression of the form

$$\sum_{T,f} tr\left(R_{T,f}\mathbb{E}_{\Theta}[\mathbf{d}^{\Theta,T,f}\ {}^t\mathbf{d}^{\Theta,T,f}]\right).$$

Main theorem

Theorem

If p is a $\mathfrak{S}_n$-invariant and d-sos polynomial, then p is also $2d$-flag sos.

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Combinatorial result

Theorem [Kővari-Sós-Turán]

Let G be a n-vertices graph not containing a C_4 (the cycle on four vertices). Then the number of edges of G is at most $\frac{1}{2}n^{3/2} + O(n)$.

Combinatorial result

Theorem [Kővari-Sós-Turán]

Let G be a n-vertices graph not containing a C_4 (the cycle on four vertices). Then the number of edges of G is at most $\frac{1}{2}n^{3/2} + O(n)$.

Proof:

$$s = \sum_{1 \leq i < j \leq n} x_{ij}$$

$$\mathcal{I} = \langle x_{ij}^2 - x_{ij} \ \forall 1 \leq i < j \leq n, \ x_{ij}x_{jk}x_{kl}x_{li} \ \forall i, j, k, l \rangle.$$

We prove that $n + \frac{2}{n-1}s - \frac{2}{\binom{n}{2}}s^2$ is 2-sos modulo $\mathcal{I}$.