Groupe de travail: ∞-categories.

Part 1: basic theory of ∞-categories

The aim of this first part of the reading group is to set the basic definitions and basic properties of ∞-categories, with as many details as possible, but avoiding model categories in the exposition.

- Exp. 1 and 2 present classical results on simplicial sets, which are useful for ∞-categories and for many other purposes.
- Exp. 3 introduces ∞-categories and their homotopy category, as a generalization of the notion of a category.
- Exp. 4 explains how to construct quasi-categories from simplical categories like **Top** and **sSet**. In particular it introduces the infinite category of spaces $N_\Delta(\mathbf{Kan})$.
- Exp. 5 explains the notion of (co)limits in ∞-categories as a generalization of the notion of (co)limits in categories.
- Exp. 6 explains why the ∞-category of spaces is the free quasi-category generated by a single object (the point).

Exp.1: Simplicial sets and topology (1)

Date: 1er Décembre 2023
Orateur: Marvin

1. The category **sSet**. [GJ, I.1], [Dw, Section 3], or [Fr, Section 3].

   Definition of simplicial sets and their morphisms. Limits and colimits of simplicial sets.

2. First geometric examples. [Dw, Section 3] (see also [Fr, section 2.3].)

   The category **OSC** of ordered simplicial complexes and the faithful functor Sing : $\mathbf{OSC} \to \mathbf{sSet}$. (The nondegenerate simplices of $\mathrm{Sing}(X)$ recover the simplicial complex X).

3. Simplicial sets vs topological spaces. [Dw, Section 3] [GJ, I.2] (see also [Fr, Section 4])

   Definitions: singular chain functor $\mathrm{S} : \mathbf{Top} \to \mathbf{sSet}$, realization functor: $|-| : \mathbf{sSet} \to \mathbf{Top}$.

   Properties: $|-|$ is left adjoint to S, $|X|$ is a CW complex.

The full subcategory **CGHaus** $\subset$ **Top** (see [May, Chap 5] or [Str] for the definition, this category contains CW-complexes.) Thm: $|-|$: **sSet** $\to$ **CGHaus** preserves finite limits. (note: don't spend too much time on CGHaus and finite limits...)

4. Kan complexes/fibrations. [GJ, I.3] (see also [Fr, Section 7]), [GJ, I.4]

   Definition of Kan complexes. Δ^n is not a Kan complex. The functor S sends spaces to Kan complexes.

   Definition of Kan fibrations. S sends Serre fibrations to Kan fibrations (see [Hat, Bre] for Serre fibrations and their uses in topology). Also $|-|$ send Kan fibrations to Serre fibrations [GJ, Thm 10.10] (just give this last statement, the proof relies on the concept of minimal fibrations, and there is no time to explain it in the reading group).

   Anodyne extensions. Kan fibrations have the right lifting property with respect to anodyne extensions.

Exp.2: Simplicial sets and topology (2)

Date: 8 Décembre 2023
Orateur: Nicola

1. Function complexes. [GJ, I.5]

   Definition, exponential law, function complexes and Kan complexes [GJ, Prop 5.2, Cor 5.3]

2. Homotopies. [GJ, I.6] (see also [Fr, Sections 8])

   Definition of homotopic maps. The functors S and $|-|$ preserve homotopies.

   $\pi_0(X)$ of Kan complexes. Homotopy is an equivalence relation on morphisms $X \to Y$ if Y is fibrant.

3. Weak equivalences.[GJ, I.7,I.11] (see also [Fr, Sections 8])

   Definition of simplicial homotopy groups, and weak equivalences (weak equivalences for general maps are defined below [GJ, Prop 11.1]). S and $|-|$ preserve weak equivalences.

4. The Homotopy category: an overview [GJ, I.11]

   Expand the discussion in [GJ, p. 63], state theorem 11.4. The idea of this overview is to explain the concrete results on **Top** and **sSet** without using the language of model categories (which has not been introduced).

   [For **Top**, the construction of Ho(**Top**) (including the fact that its Hom are *sets*) and the fact that it is equivalent to the category of CW-complexes and homotopy classes of maps follows from the results of [Hat, 4.1] (Whitehead theorem, CW approximation of spaces and of maps).

For **sSet**, there are analogous results to construct Ho(**sSet**) and prove that it is equivalent to the category of Kan complexes and homotopy classes of maps. These results can be retrieved from the properties of topological spaces, using that $\eta_X : X \to \mathrm{S}|X|$ is a weak equivalence, and a homotopy equivalence if X is a Kan complex.

Alernatively, a reference for the simplicial Whitehead theorem is Lurie's website `https://kerodon.net/tag/00WU`.]

Exp.3: Simplicial sets and categories

Date: 8 Décembre 2023
Orateur: Jérôme

1. The nerve of a category [Dw, Section 5] and [Cis, Section 1.4]

   Definition of the nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$. Natural transformations of functors give homotopies of morphisms of simplicial sets. The nerve is fully faithful.

   A simplicial set is the nerve of a category iff it satisfies the unique filling condition of inner horns [Cis, Prop 1.4.11 and 1.4.13].

2. ∞-categories [Cis, Section 1.5]

   Definition of an ∞-category, functors between ∞-categories (=morphism of simplicial sets). Examples: Kan complexes, (nerves of) categories.

   Basic notions inside an ∞-category: objects, maps, composition, homotopies, equivalences in an ∞-category.

3. The homotopy category of an ∞-category

   Description of the homotopy category [Cis, Thm 1.6.6]

Exp.4: Simplicial categories versus ∞-categories

Date: 19 Janvier 2024
Orateur: Rubèn

1. Simplicial categories

   Definition of simplicial category (=category enriched over **sSet**), simplicial functor. Category $\mathbf{Cat_s}$ of simplicial categories.

   Examples: **sSet**, **Top** (the latter can be viewed as a simplicial category by applying S on morphism spaces), or any full subcategory of them.

2. The coherent nerve (=simplicial nerve) [Gro, Section 1.2] and [HTT, Section 1.1.5]

   Definition of the simplicial thickenings $C[\Delta^n]$, and the coherent nerve [Gro, Def. 1.26], [HTT, 1.1.5]:

   $$N_\Delta : \mathbf{Cat_s} \to \mathbf{sSet} \ .$$

   The coherent nerve sends categories enriched in Kan complexes to ∞-categories [HTT, Prop 1.1.5.10], and it has a left adjoint $C[-]$[1].

   Definition of the homotopy category of a simplicial category $\mathcal{C}$ enriched in Kan complexes (= apply π_0 on Hom). The homotopy category of $\mathcal{C}$ equals the homotopy category of $N_\Delta(\mathcal{C})$ (exercise).

   Application: the ∞-category of spaces is defined as $N_\Delta(\mathbf{Kan})$ where **Kan** is the full subcategory of **sSet** on Kan complexes. Its homotopy category is $\mathrm{Ho}(\mathbf{sSet})$.

3. Simplicial Homs [HTT, Sections 1.2.2 and 2.2.2]

   Definition of the simplicial mapping space $\mathrm{Hom}^R_{\mathcal{C}}(x,y)$ between two objects of x, y of an ∞-category $\mathcal{C}$. It is a Kan complex [HTT, Prop 1.2.2.3].

   Prop: if $\mathcal{C}$ is a category enriched in Kan complexes, then $\mathrm{Hom}^R_{\mathcal{C}}(x,y)$ is (weakly) homotopy equivalent to $\mathrm{Hom}_{\mathcal{C}}(x,y)$.

   [This results from the fact that for a suitable cosimplicial object $Q^\bullet$ in **sSet**, there is an isomorphism $\mathrm{Hom}^R_{N_\Delta(\mathcal{C})}(x,y) \simeq \mathrm{Sing}_{Q^\bullet}\mathrm{Hom}_{\mathcal{C}}(x,y)$ (see [HTT, Prop 2.2.2.13]) and for every Kan complex X, there is a weak homotopy equivalence: $\mathrm{Sing}_{Q^\bullet}(X) \leftarrow |\mathrm{Sing}_{Q^\bullet}(X)|_{Q^\bullet} \to X$ by [HTT, Prop 2.2.2.7 and Cor. 2.2.2.10] - which can be applied to $X = \mathrm{Hom}_{\mathcal{C}}(x,y)$.]

Exp.5: Limits and colimits

Date: 26 Janvier 2024
Orateur: Matthew

1. Opposites ∞-category and various mapping spaces. [HTT, 1.2]

   Definition of the opposite X^{op} of a simplicial set, $N(\mathcal{C}^{\mathrm{op}}) = N(\mathcal{C})^{\mathrm{op}}$, the opposite of an ∞-category is an ∞-category [HTT, 1.2.1]. Notice that X^{op} and X have homeomorphic realizations, hence are weakly homotopy equivalent.

[1] If this has not been done in a previous talk it will be useful to present the presheaf point of view [Cis, Thm 1.1.10] or [HTT, begining of section 2.2.2 p. 75].

Definition of the simplicial mapping space $\mathrm{Hom}^L_{\mathcal{C}}(x,y)$ in an ∞-category $\mathcal{C}$. Isomorphism $\mathrm{Hom}^R_{\mathcal{C}^{\mathrm{op}}}(y,x)^{\mathrm{op}} = \mathrm{Hom}^L_{\mathcal{C}}(x,y)$. Weak equivalence between $\mathrm{Hom}^L_{\mathcal{C}}(x,y)$ and $\mathrm{Hom}^R_{\mathcal{C}}(x,y)$ [HTT, 1.2.2]. (There is no time to prove the weak equivalence, but it is possible to define the zig-zag with $\mathrm{Hom}_{\mathcal{C}}(x,y)$ in the middle).

2. <u>Final and initial objects</u> [Gro, 2.4]

   (Because of duality $-^{\mathrm{op}}$, we can restrict to final objects)

   Definition of final objects (using [Gro, Prop 2.23 (ii) and (iii)]), uniqueness of final objects (corollary 2.24) This notion extends the categorical notion: [Gro, Lm 2.26]

3. <u>Recollections of general (co)limits in categories</u> [Gro, 2.2, 2.3]

   Recall (co)limits be reformulated as (co)terminal objects using slice categories. The link between slices and joins.

4. <u>General limits in ∞-categories</u> [Gro, 2.2, 2.3, 2.5]

   Joins of simplicial sets. The nerve preserves joins. The join of ∞-categories is an ∞-category. (see [Gro, 2.2] or `https://ncatlab.org/nlab/show/join+of+simplicial+sets`)

   Slice construction for ∞-categories. The nerve preserves slices [Gro, Lm 2.20].

   Definition of (co)limits in ∞-categories [Gro, 2.5]. This notion extends the notion of (co)limits in categories

Exp.6: Functors between ∞-categories

Date: 2 Février 2024
Orateur: Théo

1. <u>Functor categories</u>. [Gro, 2.1]

   The ∞-category $\mathrm{Fun}(K,\mathcal{C})$, K simplicial set, $\mathcal{C}$ ∞-category. The isomorphism $\mathrm{NFun}(\mathcal{C},\mathcal{D}) \simeq \mathrm{Fun}(\mathrm{N}(\mathcal{C}),\mathrm{N}(\mathcal{D}))$.

   Definition of functors preserving colimits, the ∞-category $\mathrm{Fun}^L(\mathcal{C},\mathcal{D})$, for $\mathcal{C},\mathcal{D}$ ∞-categories.

2. <u>Presheaves.</u>[Gro, Sections 3.1, 3.3]

   Classical setting: explain [Gro, Thm 3.2]. Example with $A = \Delta$.

   Presheaves in the ∞-categorical context: construction of the Yoneda embedding, existence of colimits in presheaves, theorem [Gro, 3.16]. Details can be found in [HTT, Section 5.1].

   Application: $\mathrm{N}_\Delta(\mathbf{Kan})$ is freely generated by Δ^0, [Gro, Cor 3.17]

Exp. bonus: (Co)limits in ∞-categories vs homotopy (co)limits

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