Master degree thesis

Compactification of Teichmüller space via geodesic currents

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This thesis has been written as part of my master degree in Algebra and Geometry at Université de Rennes 1 during which I carried out a fifteen week internship on hyperbolic surfaces and Teichmüller spaces with Mr. Souto. The main goal of this internship was to understand Bonahon's interpretation [Bon88] of Thurston's compactification of Teichmüller spaces.

Introduction

Consider the following situation: S is a closed oriented surface of genus $g \geq 2$, this surface admits a hyperbolic metric. The set of all hyperbolic metrics over S can be quotiented by some equivalence relations. We will study the quotient by orientation preserving isometries homotopic to identity. This quotient space is the Teichmüller space $\mathcal{T}(S)$ of the surface S and can be endowed with a natural topology such that it homeomorphically identifies with $\mathbb{R}^{6g-6}$. Our goal in this paper is to study a compactification of this set.

Consider the hyperbolic plane $\mathbb{H}^2$: the usual way to compactify it is to add the boundary at infinity which consists of the equivalence classes of geodesics in $\mathbb{H}^2$. Similarly, Thurston's compactification of the Teichmüller space consists in constructing its boundary as the set of projective measured laminations $\mathbb{P}\mathcal{ML}(S)$ on S. We will follow Bonahon's construction of this compactification [Bon88] through the set of geodesic currents $\mathcal{C}(S)$. It consists in embedding the Teichmüller space of S and the set of projective measured laminations over S into the set of projective geodesic currents $\mathbb{P}\mathcal{C}(S)$.

5.2.1: Theorem (Bonahon [Bon88])**.** *A compactification of $\mathcal{T}(S)$ as a subset of the compact set $\mathbb{P}\mathcal{C}(S)$ is $\mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$.*

I will give next a section by section summary of the paper.

The first section will recall a few facts about geometry and topology of surfaces. In particular, we will see some properties of homeomorphisms and diffeomorphisms, or homotopy and isotopy, which we will require later. As we will focus on hyperbolic surfaces, in particular closed hyperbolic surfaces, we will recall some of their properties. The properties exposed in this section will be used throughout the document.

In the second section, we recall the definition of a Teichmüller space. Once we define the notion of length over $\mathcal{T}(S)$ and its topology, we will discuss two well-known theorems. The first, the existence of Fenchel-Nielson coordinates over $\mathcal{T}(S)$, induces a homeomorphism with $\mathbb{R}^{6g-6}$. The second theorem is the $9g-9$-theorem which says that the class into the Teichmüller space of a hyperbolic metric over S is uniquely determined by the length of $9g-9$ fixed curves.

2.5.1: 9g-9 Theorem. *There is a collection of simple closed curves $\alpha_1, ..., \alpha_{9g-9}$ on S such that the following map is a proper embedding.*

$$\begin{array}{rccc} \ell & : & \mathcal{T}(S) & \to & \mathbb{R}_+^{9g-9} \\ & & \rho & \mapsto & (\ell_\rho(\alpha_i))_{i=1,...,9g-9} \end{array}$$

Following the idea that we want to compactify $\mathcal{T}(S)$ this theorem will help us to understand divergent sequences in $\mathcal{T}(S)$.

In the next section, we will continue studying Gromov hyperbolic spaces, mainly referring to [GdlH90]. They will serve as an intermediate tool to study the action of the homeomorphisms of a surface on the geodesics of the universal cover. Indeed, we will study quasi-isometries over Gromov hyperbolic spaces. Naturally, the hyperbolic plane is Gromov hyperbolic, and so all results about quasi-isometries over hyperbolic spaces apply to $\mathbb{H}^2$. As a consequence we will be able to extend lifted homemophisms to $\partial\mathbb{H}^2$ and to the geodesics of the universal cover of S. The key result here is the Švarc-Milnor lemma which allows us to prove that lifted homeomorphisms from closed hyperbolic surfaces are quasi isometries.

3.1.5: Švarc-Milnor Lemma. *Let (X, d) be a proper simply connected geodesic metric space and G a group whose action on X is cocompact by isometries and properly discontinuous. G is finitely generated and for every $x_0 \in X$ the map $g \in G \mapsto g \cdot x_0 \in X$ is a quasi-isometry.*

We will give a proof of this theorem in section 3.1.

Armed with this facts about Gromov hyperbolic spaces, in section four we can study the main objects we will need to compactify: measured laminations and currents. We will first study geodesic laminations. Geodesic laminations are compact subsets of S which are made of complete disjoint simple geodesics. This space can be naturally endowed with the topology induced by the Hausdorff distance and is independent from the metric over S. The object we will use in the compactification are measured laminations: they are geodesic laminations equipped with measures over transverse arcs. A positive multiple of a measure is also a measure so the set $\mathcal{ML}(S)$ of measured lamination can be projectivised to define $\mathbb{P}\mathcal{ML}(S)$. It is this set which will appear as the boundary of $\mathcal{T}(S)$.

The last thing we will need to understand Bonahon's compactification of $\mathcal{T}(S)$ is the notion of currents. After a brief study of geodesic currents we will see how it is possible to make $\mathcal{T}(S)$ and $\mathcal{ML}(S)$ subsets of $\mathcal{C}(S)$. A geodesic current for S is a measure over the set of geodesics $\mathcal{G}(\tilde{S})$ in the universal cover $\tilde{S}$ of S. It is easy to identify $\mathcal{ML}(S)$ as a subset of $\mathcal{C}(S)$ but somewhat more complicated for $\mathcal{T}(S)$. One first need to define the Liouville measure on $T^1(\mathbb{H}^2)$ and get an identification between currents and geodesic flip and geodesic flow invariant measures on $T^1(S)$. This identification will also lead to the compacity of $\mathbb{P}\mathcal{C}(S)$.

The last section aims to compactify $\mathcal{T}(S)$. For that purpose, we will use a tool introduced in [Bon88]: the intersection number.

5.1.2: Theorem. *The intersection number between free homotopy classes of closed curves admits a continuous symmetric bilinear extension* $i : \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}_+$.

This intersection number will prove that $\mathcal{T}(S)$ embeds in $\mathbb{P}\mathcal{C}(S)$ and will induce a characterisation of currents coming from measured laminations. With those elements we will be able to prove that a divergent sequence in $\mathcal{T}(S)$ converges to a projective measured lamination in $\mathbb{P}\mathcal{C}(S)$.

Contents

1 Topology and geometry of surfaces

The topic this thesis is on surfaces, so we begin by recalling some properties of their geometry and topology. In what follows, we suppose that the surfaces considered are connected and oriented.

1.1 Topological properties

Our study will be focused on compact connected oriented surfaces. It is well-known that such a surface is either a sphere or a g-torus with a finite number of disks removed, see [Kin93] for a proof. More precisely we have the Classification theorem.

Theorem 1.1.1 (Classification theorem)**.** *If S and S' are compact oriented and connected surfaces with the same number of boundary components and the same Euler characteristic then every bijection between $\pi_0(\partial S)$ and $\pi_0(\partial S')$ is induced by a homeomorphism $\Phi : S \to S'$ which can be moreover chosen orientation preserving or reversing.*

If S and S' are smooth then Φ can be chosen to be a diffeomorphism and if S and S' are triangulated then Φ can be chosen to be piece-wise linear.

Noting that cutting a surface along a simple non separating closed curve does not change the Euler characteristic, it appears that every two such curves γ and γ' in a compact connected oriented surface S can be mapped one to the other one by a homeomorphism of S.

Such an homeomorphism may be seen as a change of coordinates: every complex simple closed curve can be study through an "easy" one.

According to *Theorem 1.1.1*, a surface is topologically determined by it's Euler characteristic and the number $|\pi_0(\partial S)|$ of boundary components. There is a third constant, the genus $g(S)$, which is deeply related to $\chi(S)$ and $|\pi_0(\partial S)|$.

Definition 1.1.2. *If S is a compact oriented connected surface with $|\pi_0(\partial S)| = k$ (number of connected components of the boundary), then the genus of S is*

$$g(S) = \frac{1}{2}(2 - \chi(S) - k).$$

The genus has some geometrical interpretations:

- $g(S)$ is the number of simple closed curves along which you need to cut S to make it planar,
- $g(S)$ is the maximal number of simple closed curves which together don't separate S.

These properties can be proven using the fact that the Euler characteristic does not change when we cut a surface along a simple closed curve.

One can reformulate the classification theorem saying that two surfaces with same number of boundary components and same genus are homeomorphic. Hence, up to homeomorphism, the compact connected oriented surfaces are the following:

	$g = 0$	$g = 1$	$g = 2$	$g = 3$	...
$\lvert\pi_0(\partial S)\rvert = 0$	Sphere	Torus	2-Torus	3-Torus	...
$\lvert\pi_0(\partial S)\rvert=1$	$Sphere\diagdown disk$	$Torus\diagdown disk$	$2 - Torus\diagdown disk$	$3 - Torus\diagdown disk$	...
$\lvert\pi_0(\partial S)\rvert = 2$	$Sphere\diagdown 2\,disks$	$Torus\diagdown 2\,disks$	$2 - Torus\diagdown 2\,disks$	$3 - Torus\diagdown 2\,disks$	...
$\lvert\pi_0(\partial S)\rvert = 3$	$Sphere\diagdown 3\,disks$ = pair of pants	$Torus\diagdown 3\,disks$	$2 - Torus\diagdown 3\,disks$	$3 - Torus\diagdown 3\,disks$	...
...	...	...	...	...	...

For more details on topology of surfaces see [Kin93] chapters 4 and 5.

Besides the classification theorem, we will need in the following to understand the link between isotopy and homotopy, or homeomorphisms and diffeomorphisms, on surfaces. The following theorem is fundamental.

Theorem 1.1.3. *Let S be a compact oriented surface other than the annulus or the disk, for any $\varphi, \psi \in \mathrm{Diff}(S)$ the following are equivalent*

- *φ and ψ are homotopic,*
- *φ and ψ are isotopic,*
- *$\varphi\psi^{-1} \in \mathrm{Diff}_0(S)$.*

Fact. *The same equivalences as above are true for orientation preserving diffeomorphisms.*

Thus we will talk indifferently about isotopy or homotopy between diffeomorphisms. About being a homeomorphism or a diffeomorphism, we have the following theorem, for more details see [FM11] section 1.4.

Theorem 1.1.4. *Let S and S' be two surfaces as above, if $f : S \to S'$ is a proper homotopy equivalence then f is properly homotopic to a diffeomorphism.*

Especially, it applies to homeomorphisms between compact surfaces which are all homotopic to a diffeomorphism. Combinig the two preceding theorems we have

$$\begin{aligned}\mathrm{Diff}^+(S)/_{homotopy} &= \mathrm{Diff}^+(S)/_{isotopy} = \mathrm{Diff}^+(S)/_{\mathrm{Diff}_0^+(S)} \\ &= \mathrm{Homeo}^+(S)/_{homotopy} = \mathrm{Homeo}^+(S)/_{\mathrm{Homeo}_0^+(S)},\end{aligned}$$

where $\mathrm{Diff}_0^+(S)$ and $\mathrm{Homeo}_0^+(S)$ are the identity components of $\mathrm{Diff}^+(S)$ and $\mathrm{Homeo}^+(S)$. By this way we can define the mapping class group of a surface.

Definition 1.1.5. *The mapping class group of a closed oriented surface S is*

$$\mathrm{Map}^+(S) = \mathrm{Diff}^+(S)/_{homotopy}$$

and the extended mapping class group is

$$\mathrm{Map}(S) = \mathrm{Diff}(S)/_{homotopy}$$

Example. *For the torus we get $\mathrm{Map}(\mathbb{T}) = GL_2(\mathbb{Z})$ while $\mathrm{Map}^+(\mathbb{T}) = SL_2(\mathbb{Z})$.*

We will essentially work with closed surfaces so, we defined the mapping class group only in that case. One can define it for surfaces with boundary however some additional elements come into accounts, interested readers can refer to [FM11].

It is well-known that every map $\varphi : S \to S'$ between surfaces induces a homomorphism $\varphi_* : \pi_1(S) \to \pi_1(S')$ between fundamental groups. For some surfaces it is possible to go back from homomorphism between fundamental groups to maps between surfaces.

Definition 1.1.6. *Let X be a path-connected space and G a group, X is a Eilenberg-MacLane $K(G,1)$ space if $\pi_1(X) \cong G$ and $\tilde{X}$ is contractible.*

Example. *For example, the torus whose universal cover is $\mathbb{R}^2$ is a $K(\mathbb{Z}^2, 1)$ space. The universal cover of the sphere is the sphere which is not contractible then $\mathbb{S}^2$ is not Eilenberg-MacLane. We will see later that every closed oriented surface of genus $g \geq 2$ is Eilenberg-MacLane.*

Now, homomorphisms from $\pi_1(X)$ to a group G are induced by maps $X \to X'$ for any $K(G,1)$-space X'.

Theorem 1.1.7. *If X is connected and X' is $K(G,1)$ for some G then every group homomorphism $\rho : \pi_1(X) \to G$ is of the form φ_* and $\varphi : X \to X'$ is unique up to homotopy.*

Moreover, if ρ is an isomorphism then φ is a equivalence homotopy.

For more details about $K(G,1)$-spaces see [Hat02] section 1.B. The previous theorems can be applied to the study of the mapping class group of a closed surface, the following corollary is proved in [FM11].

Corollary 1.1.8. *Let S be a closed surface of genus $g \geq 1$, there is an isomorphism between* $\mathrm{Map}(S)$ *and* $\mathrm{Out}(\pi_1(S))$*.*

1.2 Hyperbolic surfaces

We will mainly be interested in hyperbolic surfaces. See [BP91] for details on their properties, here we briefly recall some of them.

Definition 1.2.1. *A Riemannian surface X is said to be hyperbolic if it is complete, has totally geodesic boundary, and is locally isometric to $\mathbb{H}^2$.*

A metric ρ on a surface S is said to be hyperbolic if the Riemannian surface $X = (S, \rho)$ is hyperbolic.

Equivalently, a hyperbolic surface X is a complete Riemannian 2-manifold which admits an atlas $\{\phi_U : U \to V\}$ where the U are opens subsets of X which cover X, the V are open subsets of $\mathbb{H}^2$ and the ϕ_U are isometries.

It is also equivalent to take a smooth 2-manifold S having an atlas with values in $\mathbb{H}^2$ such that the transition maps $\phi_{U'}^{-1} \circ \phi_U$ are restrictions of global isometries.

Always equivalently, a hyperbolic surface is a complete Riemannian surface with constant curvature -1.

Theorem 1.2.2. *If S is a hyperbolic complete simply connected surface then S is isometric to* $\mathbb{H}^2$*.*

Up to isometry, there is a unique simply connected hyperbolic surface which is $\mathbb{H}^2$. If X is a closed hyperbolic surface we even know that $X = \tilde{X}/\pi_1(X)$, where $\tilde{X}$ is the universal cover. We can endow the universal cover with the pull back metric, we obtain a simply connected hyperbolic surface and the action of $\pi_1(X)$ on $\tilde{X}$ by deck transformations is now by orientation preserving isometries. According to *Theorem 1.2.2*, $\tilde{X}$ is isometric to $\mathbb{H}^2$. As a consequence, $\pi_1(X)$ identifies with a subgroup of the isometries of $\mathbb{H}^2$ and every closed hyperbolic surface identifies with $\mathbb{H}^2/\Gamma$ where Γ is a subgroup of $PSL_2(\mathbb{R})$.

We have a precise characterisation of hyperbolic closed Riemannian surfaces.

Theorem 1.2.3. *Let X be a closed Riemannian surface, X is hyperbolic if and only if it is diffeomorphically isometric to $\mathbb{H}^2/\Gamma$, where Γ is a torsion free subgroup of $PSL_2(\mathbb{R})$ whose action is discreet and free.*

Remark. *One can note that since $\mathbb{H}^2$ is contractible, every closed hyperbolic surface is an Eilenberg-MacLane space.*

We would like to know which surfaces are hyperbolic. For the g-torus with $g \geq 2$ it is possible to build it by gluing two by two the sides of a $4g$-gone. As $\mathbb{H}^2$ admits a tilling made of regular $4g$-gones we have a hyperbolic structure on the g-torus.

For the closed surfaces of genus 1 and 0 we will use *Theorem 1.2.3*. A closed surface satisfying this theorem has a universal cover diffeomorphically isometric to $\mathbb{H}^2$ and its fundamental group is Γ. As the sphere is simply connected, it is its own universal cover so the sphere does not admit hyperbolic metric. For the torus, one can prove that there is no copy of $\mathbb{Z}^2$ in $PSL_2(\mathbb{R})$ which is torsion free and whose action on $\mathbb{H}^2$ is discreet and free, as a consequence the torus as no hyperbolic structure. Following those observations there is a characterization of compact hyperbolic surfaces.

Proposition 1.2.4. *Let S be a compact oriented surface, S admits a hyperbolic metric if and only if $\chi(S) < 0$.*

If S has no boundary we find what we saw above, indeed, $\chi(S) < 0$ means $g(S) > 1$.

Example. *As a consequence, the closed surfaces which admits hyperbolic metrics are the closed surfaces of genus at least 2. For the surfaces with boundary, we will be interested in the pair of pants, it has genus 0 and 3 boundary components, hence its Euler characteristic is -1 and the pair of pants is hyperbolic.*

2 Teichmüller spaces

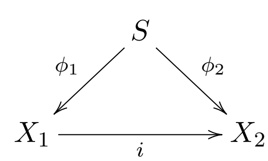
We have seen above that every closed surface of genus $g \geq 2$ admits at least a hyperbolic metric. The goal here is to study some properties of the set of all hyperbolic metrics on a surface. We will consider two equivalence relations over the set of hyperbolic metrics on S. They are linked by the action of the mapping class group but differ in that one is easier to study than the other. That's why we will focus on the Teichmüller spaces. Most of the results of this section are available on [FM11] chapter 10 and [BP91] section B.4. We will denote surfaces without fixed metric by S and Riemannian surfaces by X.

2.1 Teichmüller and moduli spaces

Choosing a surface S which admits a hyperbolic metric we introduce here two equivalence relations on the set of hyperbolic metrics on S.

Definition 2.1.1. *Let S be a closed oriented surface such that $g(S) \geq 2$. A marked hyperbolic structure on S is a pair (X, ϕ) where X is a Riemannian surface with a complete hyperbolic metric and $\phi : S \to X$ is an orientation preserving diffeomorphism.*

Two marked hyperbolic structures are equivalent $(X_1, \phi_1) \sim_{\mathcal{T}} (X_2, \phi_2)$ if there exists an orientation preserving isometry $i : X_1 \to X_2$ such that the following diagram commutes up to homotopy.



Remark. *One can define it for surfaces with boundary, in that case the marked surface X is asked to have totally geodesic boundary.*

Consider the following application

$$\begin{array}{ccc} \{(X, \phi) \text{ marked hyperbolic structure on } S\} & \to & \{\rho \text{ hyperbolic metric on } S\} \\ (X, \phi) & \mapsto & \phi_*^{-1}\rho_X \end{array}$$

where ρ_X is the metric on X.

If (X_1,ϕ_1) and (X_2,ϕ_2) are equivalent then $\phi_2^{-1}\circ i\circ\phi_1 : (S,\phi_{1*}^{-1}\rho_{X_1}) \to (S,\phi_{2*}^{-1}\rho_{X_2})$ is an Orientation Preserving Isometry Homotopic to the Identity (OPIHI).

The existence of an orientation preserving isometry homotopic to identity defines an equivalence relation on the set of hyperbolic metrics on S and we have the following application :

$$\begin{array}{ccc} \{(X,\phi) \text{ marked hyperbolic structure on } S\}/\sim_{\mathcal{T}} & \to & \{\rho \text{ hyperbolic metric on } S\}/\text{OPIHI} \\ (X,\phi) & \mapsto & \phi_*^{-1}\rho_X. \end{array}$$

It is a bijection an the reverse application is given by:

$$\begin{array}{ccc} \{\rho \text{ hyperbolic metric on } S\}/\text{OPIHI} & \to & \{(X,\phi) \text{ marked hyperbolic structure on } S\}/\sim_{\mathcal{T}} \\ \rho & \mapsto & id : S \to (S,\rho) \end{array}$$

As a map $\varphi : (S,\rho_1) \to (S,\rho_2)$ is an isometry if an only if $\varphi_*\rho_1 = \rho_2$ then

$$\{\rho \text{ hyperbolic metric on } S\}/\text{OPIHI} = \{\rho \text{ hyperbolic metric on } S\}/\text{Diff}_0^+(S)$$

where $\text{Diff}_0^+(S)$ acts via push forward.

Definition 2.1.2. *The Teichmüller space of a closed oriented surface S with $g(S) \geq 2$ is defined by*

$$\begin{array}{rcl} \mathcal{T}(S) & = & \{(X,\phi) \textit{ marked hyperbolic structure on } S\}/\sim_{\mathcal{T}} \\ & = & \{\rho \textit{ hyperbolic metric on } S\}/\textit{orientation preserving isometry homotopic to id} \\ & = & \{\rho \textit{ hyperbolic metric on } S\}/\text{Diff}_0^+(S) \end{array}$$

As discussed above, a closed oriented surface S is topologically characterized by its Euler characteristic $\chi(S)$ or its genus $g(S)$. Suppose that S is a closed surface with $g(S) \geq 2$ and set $\mathcal{S} = \{X \text{ hyperbolic oriented surfaces of genus } g(S)\}$, there is the following equivalence relation on $\mathcal{S}$:

$$X_1 \sim_{\mathcal{M}} X_2 \iff \exists\, \varphi : X_1 \to X_2 \text{ an orientation preserving isometry .}$$

Arguing as in the case of the Teichmüller space, the set $\mathcal{S}/\sim_{\mathcal{M}}$ is in bijection with the set of hyperbolic metrics on S up to the action of $\text{Diff}^+(S)$ by push forward.

Definition 2.1.3. *The moduli space of S is the set*

$$\begin{array}{rcl} \mathcal{M}(S) & = & \mathcal{S}/\sim_{\mathcal{M}} \\ & = & \{\rho \textit{ hyperbolic metric on } S\}/\text{Diff}^+(S). \end{array}$$

To clarify the relation between $\mathcal{T}(S)$ and $\mathcal{M}(S)$ note that we can define an action of $\text{Map}^+(S)$ on $\mathcal{T}(S) = \{\rho \text{ hyperbolic metric on } S\}/\text{Diff}_0^+(S)$ given by

$$\forall\, [\psi]_{map} \in \text{Map}^+(S) \, and \, [\rho]_{\mathcal{T}} \in \mathcal{T}(S) \; : \; [\psi]_{map} \cdot [\rho]_{\mathcal{T}} = [\psi_*\rho]_{\mathcal{T}},$$

where $[\psi]_{map} = \{\varphi \circ \psi \;:\; \varphi \in \text{Diff}_0^+(S)\}$ and $[\rho]_{\mathcal{T}} = \{\varphi_*\rho \;:\; \varphi \in \text{Diff}_0^+(S)\}$.

We verify that the above formula is well defined as a group action :

- if ψ_1 and ψ_2 are two diffeomorphisms then $(\psi_1 \circ \psi_2)_* = \psi_{1*} \circ \psi_{2*}$ and $[\psi_1 \circ \psi_2]_{map} \cdot [\rho]_{\mathcal{T}} = [\psi_1]_{map} \cdot ([\psi_2]_{map} \cdot [\rho]_{\mathcal{T}})$,

- if $\psi \in \text{Diff}_0^+(S)$ then $[\psi]_{map} = 1_{\text{Map}^+(S)}$ and $[\psi]_{map} \cdot [\rho]_{\mathcal{T}} = [\psi_*\rho]_{\mathcal{T}} = [\rho]_{\mathcal{T}}$ by definition of the Teichmüller space of S,
- consider $\phi \in \text{Diff}^+(S)$, $\varphi \in \text{Diff}_0^+(S)$ and ρ a hyperbolic metric on S,

  $(\varphi \circ \psi)_*\rho = \varphi_*(\psi_*\rho)$ so $[\varphi \circ \psi]_{map} \cdot [\rho]_{\mathcal{T}} = [\psi]_{map} \cdot [\rho]_{\mathcal{T}}$ and the definition does not depend on the representative of $[\psi]_{map}$,

  $\psi_*(\varphi_*\rho) = (\psi \circ \varphi)_*\rho$ where φ is isotopic to identity, thus $\psi\circ\varphi$ is isotopic to ψ and $\psi\circ\varphi\circ\psi^{-1}$ to identity (all with orientation preserved) and there is $\phi \in \text{Diff}_0^+(S)$ such that $\psi \circ \varphi \circ \psi^{-1} = \phi$ finally $(\psi \circ \varphi)_*\rho = \phi_*(\psi_*\rho)$ and $[\psi]_{map} \cdot [\varphi_*\rho]_{\mathcal{T}} = [\psi]_{map} \cdot [\rho]_{\mathcal{T}}$ and the definition does not depend on the representative of $[\rho]_{\mathcal{T}}$.

Remark. *If we consider the elements of $\mathcal{T}(S)$ as classes of hyperbolic marked structures on S, then the action is given by $[\psi]_{map} \cdot [(X, \phi)]_{\mathcal{T}} = [(X, \phi \circ \psi^{-1})]_{\mathcal{T}}$.*

Proposition 2.1.4. *With the above action*

$$\mathcal{T}(S) / \text{Map}^+(S) = \mathcal{M}(S)$$

Proof. Consider $\mathcal{T}(S)$ and $\mathcal{M}(S)$ as quotients of the set of hyperbolic metrics on S.

If the following applications are well defined then they are inverses of one other.

$$\begin{array}{ccc} \mathcal{M}(S) & \to & \mathcal{T}(S)/\text{Map}^+(S) \\ [\rho]_{\mathcal{M}} & \mapsto & [[\rho]_{\mathcal{T}}]_{map} \end{array} \qquad \begin{array}{ccc} \mathcal{T}(S)/\text{Map}^+(S) & \to & \mathcal{M}(S) \\ [[\rho]_{\mathcal{T}}]_{map} & \mapsto & [\rho]_{\mathcal{M}} \end{array}$$

- If ρ and ρ' represent the same element of $\mathcal{M}(S)$ then there is φ in $\text{Diff}^+(S)$ such that $\rho = \varphi_*\rho$.

  However $[[\rho]_{\mathcal{T}}]_{map} = \{[\psi]_{map} \cdot [\rho]_{\mathcal{T}} \mid [\psi]_{map} \in \text{Map}^+(S)\} = \{[\psi_*\rho]_{\mathcal{T}} \mid \psi \in \text{Diff}^+(S)\}$ so $[\rho']_{\mathcal{T}} \in [[\rho]_{\mathcal{T}}]_{map}$ and finally $[[\rho']_{\mathcal{T}}]_{map} = [[\rho]_{\mathcal{T}}]_{map}$ and the first application is well defined.
- Now, if $[[\rho']_{\mathcal{T}}]_{map} = [[\rho]_{\mathcal{T}}]_{map}$ then there is $[\psi]_{map} \in \text{Map}^+(S)$ such that $[\psi]_{map} \cdot [\rho]_{\mathcal{T}} = [\rho']_{\mathcal{T}}$. It means that $\exists\, \psi \in \text{Diff}^+(S)$: $[\psi_*\rho]_{\mathcal{T}} = [\rho']_{\mathcal{T}}$ and $\exists\, \psi \in \text{Diff}^+(S)$, $\varphi \in \text{Diff}_0^+(S)$: $\varphi_*(\psi_*\rho) = \rho'$. However $\varphi_*(\psi_*\rho) = (\varphi \circ \psi)_*\rho$ where $\varphi \circ \psi \in \text{Diff}^+(S)$ thus $[\rho]_{\mathcal{M}} = [\rho']_{\mathcal{M}}$ and the reverse application is well defined.

□

2.2 Length function

We have defined two quotient spaces, the moduli space and the Teichmüller space. The moduli space is easier to imagine as two hyperbolic metrics over S are equivalent in $\mathcal{M}(S)$ if they are isometric, however we will work with the Teichmüller space. It is often easier to work with $\mathcal{T}(S)$ rather than with $\mathcal{M}(S)$. For example, length functions are defined on $\mathcal{T}(S)$ and not on $\mathcal{M}(S)$.

Let S be a closed oriented surface of genus at least 2 and ρ a hyperbolic metric on S. If α is a simple closed curve in S and $[\alpha]$ its free homotopy class then $[\alpha]$ contains a unique ρ-geodesic α_ρ. We can define $\ell_\rho([\alpha])$ as the length $\ell_\rho(\alpha_\rho)$ of the unique geodesic representative α_ρ of $[\alpha]$ on (S, ρ).

Theorem 2.2.1. *Let $\mathfrak{T}$ be an element of $\mathcal{T}(S)$ and $\mathcal{S}(S)$ the set of free isotopy classes of simple closed curves in S. The length function $\ell_{\mathfrak{T}}$ is well defined.*

$$\begin{array}{rcccl} l_{\mathfrak{T}} & : & \mathcal{S}(S) & \to & \mathbb{R}^+ \\ & & [\alpha] & \mapsto & \ell_\rho([\alpha]) \quad \textit{if } \mathfrak{T} = [\rho] \end{array}$$

Proof. We want to show that the definition of $\ell_{\mathfrak{X}}$ does not depend on the representing of $\mathfrak{X}$.

Let ρ and ρ' be two metrics on S which represent the same element of $\mathcal{T}(S)$: there is an orientation preserving isometry isotopic to identity $\varphi : (S,\rho) \to (S,\rho')$ and $\rho' = \varphi_*\rho$. Let $H : [0,1] \times S \to S$ be an isotopy such that $H_0 = id$ and $H_1 = \varphi$. As a consequence, if α is a simple closed curve in S the map $H.(\alpha(\cdot))$ is a free isotopy between α and $\varphi(\alpha)$ and φ preserves $[\alpha]$.

Moreover, φ is an isometry so it maps geodesics to geodesics and preserves the lengths: $\varphi(\alpha_\rho) = \alpha_{\rho'}$ and $\ell_\rho(\alpha_\rho) = \ell_{\rho'}(\alpha_{\rho'})$ thus

$$\ell_{\rho'}([\alpha]) = \ell_{\rho'}(\alpha_{\rho'}) = \ell_\rho(\alpha_\rho) = \ell_\rho([\alpha]).$$

We have shown that $\ell_{\mathfrak{X}}$ is well defined. □

Remark. *If we consider elements of $\mathcal{T}(S)$ as hyperbolic marked structures then $\ell_{\mathfrak{X}}$ becomes*

$$\begin{array}{rcccl} l_{\mathfrak{X}} & : & \mathcal{S}(S) & \to & \mathbb{R}^+ \\ & & [\alpha] & \mapsto & \ell_{\rho_X}([\Phi(\alpha)]) \quad \textit{if } \mathfrak{X} = [(X,\Phi)]. \end{array}$$

As shown above, the application is well defined because the isometry φ is isotopic to identity. For $\mathcal{M}(S)$ the quotient is by isometries but they are not necessarily isotopic to identity, thus isotopy classes are not preserved. There is no length function on $\mathcal{M}(S)$. Length function will have a key role in the next results.

2.3 Topology of $\mathcal{T}(S)$

To equip $\mathcal{T}(S)$ with a topology we are going to identify $\mathcal{T}(S)$ with a space with a well-known topology.

Theorem 2.3.1. *If $g \geq 2$ is the genus of S, then $\mathcal{T}(S)$ is in bijection with $DF(\pi_1(S), PSL_2(\mathbb{R}))/PGL_2(\mathbb{R})$ where $DF(\pi_1(S), PSL_2(R))$ is the set of discrete and faithful representations $\rho : \pi_1(S) \to PSL_2(\mathbb{R})$ and $PGL_2(\mathbb{R})$ acts on this set by conjugation.*

Remark. *$PSL_2(\mathbb{R})$ is the group of orientation preserving isometries of $\mathbb{H}^2$ and $PGL_2(\mathbb{R})$ the group of isometries.*

Proof. Constructing a discrete faithful representation from (X,Φ) a marked hyperbolic structure: $\Phi : S \to X$ is a diffeomorphism thus $\Phi_* : \pi_1(S) \to \pi_1(X)$ is an isomorphism, moreover $\pi_1(X)$ could be identified with the group of deck transformations which is discrete and whose action on $\tilde{X}$ is free, properly discountinuous and by orientation preserving isometries. Take $\eta : \mathbb{H}^2 \to \tilde{X}$ an isometry comming from the hyperbolic structure on $\tilde{X}$, we have the following discrete and faithful action

$$\begin{array}{rcccl} \rho & : & \pi_1(S) & \to & PSL_2(\mathbb{R}) \\ & & [\alpha] & \mapsto & \eta^{-1} \circ \Phi_*([\alpha]) \circ \eta \end{array}$$

We want to prove that that if we change the isometry η or the representative (X,Φ) then the new representation is conjugated in $PGL_2(\mathbb{R})$ to ρ.

- If η' is another isometry between $\mathbb{H}^2$ and $\tilde{X}$ then $\eta^{-1} \circ \eta' = \nu$ is an isometry of $\mathbb{H}^2$ thus $\nu \in PGL_2(\mathbb{R})$ and $\rho' = \nu^{-1}\rho\nu$,

- if (X',Φ') represents the same element of $\mathcal{T}(S)$ as (X,Φ) then there is an isometry $i : X \to X'$ such that $i \circ \Phi$ is homotopic to Φ'. Thus $i_* \circ \Phi_* = \Phi'_*$ and the definition of ρ does not change.

Take $\rho : \pi_1(S) \to PSL_2(\mathbb{R})$ a faithful discrete action: as $\rho(\pi_1(S))$ is discrete in $PSL_2(\mathbb{R})$ its action on $\mathbb{H}^2$ is properly discontinuous. If this action is not free then there is $[\alpha] \in \pi_1(S)$ non-trivial such that $\rho([\alpha])$ has a fixed point z_0 in $\mathbb{H}^2$. Therefore $\rho([\alpha]) = f$ is a non-trivial rotation of $\mathbb{H}^2$. Moreover, f comes from an element of $\pi_1(S)$ and has therefore infinite order: $\forall n \in \mathbb{N}$, $f^n(z_0) = z_0$. The set $\{f^n | n \in \mathbb{N}\}$ is infinite, which is not compatible with a properly discontinuous action: the action is free.

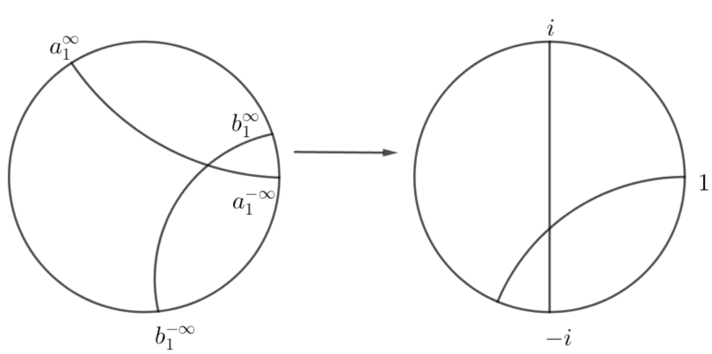
$\rho(\pi_1(S))$ is torsion free since $\pi_1(S)$ is and its action on $\mathbb{H}^2$ is free and properly discontinuous thus $\mathbb{H}^2 / \rho(\pi_1(S))$ is a hyperbolic surface X, with fundamental group $\rho(\pi_1(S))$ and universal cover $\mathbb{H}^2$. As $\mathbb{H}^2$ is contractible, X is a Eilenberg-MacLane space and according to *Theorem 1.1.7* there is a homotopy equivalence $\varphi : S \to X$ such that $\varphi_* = \rho$. Such a map is continuous and between compact surfaces with no boundary so it is proper and according to *Theorem 1.1.4*, φ is homotopic to a diffeomorphism Φ and finally we obtained (X, Φ) a hyperbolic marked structure on S.

This marked structure is defined up to the choice of ρ on its conjugacy class. If $\rho' = \nu\rho\nu^{-1}$ is conjugated to ρ then the following map is an isometry and the marked structures from ρ and ρ' are the same element of $\mathcal{T}(S)$.

$$\begin{array}{ccc} \mathbb{H}^2 / \rho(\pi_1(S)) & \to & \mathbb{H}^2 / \rho'(\pi_1(S)) \\ [z] & \mapsto & [\nu(z)]. \end{array}$$

□

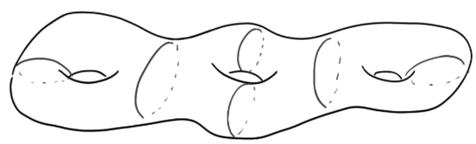
We are going to use this bijection to endow $\mathcal{T}(S)$ with a topology. If S is the genus $g \geq 2$ surface then $\pi_1(S) = \langle a_1, ..., a_g, b_1, ..., b_g | \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle$. Consider $DF(\pi_1(S), PSL_2(\mathbb{R}))$ as a subset of $\text{Hom}(\pi_1(S), PSL_2(\mathbb{R})) \equiv \{A_1, .., A_g, B_1, ..., B_g \in PSL_2(\mathbb{R}) | \prod_{i=1}^{g} [A_i, B_i] = Id\} \subset PSL_2(\mathbb{R})^{2g}$. If we have a section of $DF / PGL_2 \to DF$ then we have an identification of $\mathcal{T}(S)$ as a subset of $PSL_2(\mathbb{R})^{2g}$ where there is a well-known topology. If $A_1, ..., A_g, B_1, ..., B_g$ represent an element of $DF(\pi_1(S), PSL_2(\mathbb{R}))$ then they are hyperbolic elements of $PSL_2(\mathbb{R})$, all of them fix a geodesic $\mathfrak{a}_i$ or $\mathfrak{b}_i$ in $\mathbb{H}^2$ which have an attractive and a repulsive endpoint a_i^{∞} and $a_i^{-\infty}$ or b_i^{∞} and $b_i^{-\infty}$. In the conjugacy class of $A_1, ..., B_g$ there is a unique element such that $a_1^{\infty} = i$, $a_1^{-\infty} = -i$ and $b_1^{\infty} = 1$ since the action of $PGL_2(\mathbb{R})$ on $\overline{\mathbb{H}}^2$ is three transitive.



The map from DF / PGL_2 to DF which associates to every element the representative described above provides us section of the quotient and thus we have endow $\mathcal{T}(S)$ with a topology.

2.4 The Fenchel Nielsen coordinates

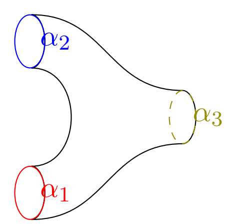
The Fenchel Nielsen coordinates on $\mathcal{T}(S)$ consist in associating to each marked hyperbolic structure X on S a unique system of coordinates based on a fixed pants decomposition of S. Since with a certain gluing of $2g-2$ pairs of pants we can construct the g-torus for every $g \geq 2$, *Theorem 1.1.1* makes clear that every hyperbolic closed surface admits at least a pants decomposition. We have some precision about this decomposition in [FM11] pages 248-249.



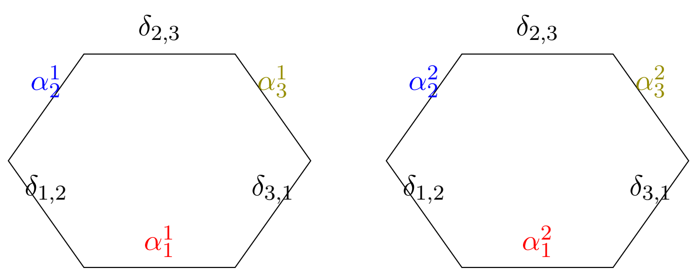
Theorem 2.4.1. *Let S be a closed surface of genus $g \geq 2$, S admits a pants decomposition made of $2g-2$ pairs of pants which are obtained cutting along $3g-3$ simple closed curves and all the pants decomposition of S are of this type.*

Let P be a (topological) pair of pants with boundary components α_1, α_2, α_3 and Y a hyperbolic surface with totally geodesic boundary equipped with an orientation preserving homeomorphism $\Phi : P \to Y$ (*ie.* a hyperbolic marked structure on P).

Thus $\Phi(\alpha_1)$,$\Phi(\alpha_2)$ and $\Phi(\alpha_3)$ denote the geodesics of the boundaries components of Y (now we will forget the Φ). For every $i \neq j \in \{1, 2, 3\}$ there is a unique Y-geodesic $\delta_{i,j}$ joining α_i and α_j orthogonally.



Cutting along the $\delta_{i,j}$ one obtains a decomposition of the pair of pants into two right angled hexagons with three non consecutive edges of the first one equal to three non consecutive edges of the second one.



Proposition 2.4.2. *If a marked hyperbolic hexagon is a hexagon with one vertex marked then every hyperbolic right angled marked hexagon is uniquely determined, up to orientation preserving isometry, by the length of three non consecutive edges, starting at the marked vertex and going in the forward direction.*

A proof of this proposition is exposed in [FM11] page 289.

In our decomposition we mark the intersection point between α_1 and $\delta_{3,1}$, therefore the hexagon is uniquely determined by the length of $\delta_{3,1}$,$\delta_{2,3}$ and $\delta_{1,2}$ and the two hexagons in the decomposition of Y are the same and are uniquely determined by the length of α_1,α_2 and α_3. These leads to the following theorem.

Theorem 2.4.3. $\begin{array}{ccc} \mathcal{T}(P) & \to & \mathbb{R}_+^3 \\ \mathfrak{X} & \mapsto & (\ell_{\mathfrak{X}}(\alpha_1), \ell_{\mathfrak{X}}(\alpha_2), \ell_{\mathfrak{X}}(\alpha_3)) \end{array}$ *is an homeomorphism.*

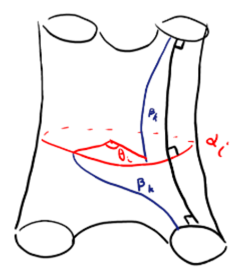
One can find more details about the study of the pair of pants in [FM11] section 10.5.

Now, go back to our closed oriented surface S and recall that it admits pants decompositions. Fix one such decomposition and call $\gamma_1,...,\gamma_{3g-3}$ the curves along which we cut. If (X, Φ) is a marked hyperbolic structure on S, the structure of each pant in the decomposition is uniquely determined by the lengths of the curves γ_i in X. It gives the first $3g-3$ coordinates of the Fenchel Nielsen system: the length parameters. The $3g-3$ last ones consist in specifying the way how we glue the pants: they are the twist parameters.

Let $\beta_1, ..., \beta_n$ be a set of curves in S such that in every pant of the decomposition the β_i consist in three disjoint arcs joining the boundary components two by two.

The idea of the twist parameters is to fix a "canonical way" to glue the pairs of pants and measure how the way we glue to obtain X is different from the canonical gluing.

We already know that in every pant of the decomposition their are three unique geodesics which join orthogonally and two by two the boundary components. As seen before they cut the boundary components into two equal arcs: say that the canonical way to glue the pants is to match the feet of the geodesics. By this way the different arcs of the β_i in the pants will not necessarily correspond: the twist parameter θ_i near α_i is the angle we have to turn the pants near α_i to glue the arcs of the β_j together.



Theorem 2.4.4. *If their is a fixed system of curves α_i, β_j as above in S then the following map is a homeomorphism:* $\begin{array}{cccccc} FN & : & \mathcal{T}(S) & \to & \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \\ & & \mathfrak{X} & \mapsto & (\ell_{\mathfrak{X}}(\alpha_1), ..., \ell_{\mathfrak{X}}(\alpha_{3g-3}), \theta_1(\mathfrak{X}), ..., \theta_{3g-3}(\mathfrak{X})). \end{array}$ *As a consequence $\mathcal{T}(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$.*

One can find more details about this decomposition in [FM11] section 10.6.

Remark. *There is as many "Fenchel Nielsen" coordinates as ways to define the "canonical gluing". This canonical gluing can be seen as a section for $\mathcal{L} : \mathfrak{X} \mapsto (\ell_{\mathfrak{X}}(\alpha_1), ..., \ell_{\mathfrak{X}}(\alpha_{3g-3}))$ for which the fibers in $\mathcal{T}(S)$ are the orbits for the action by twist along the α_i of $\mathbb{R}^{3g-3}$. One can refer to [BBFS13] for more details about this construction.*

2.5 The 9g-9 theorem

As an extension of *Theorem 2.2.1* we can define a function from $\mathcal{T}(S)$ to $\mathbb{R}_+^{\mathcal{S}(S)}$ with

$$\begin{array}{ccccc} \ell & : & \mathcal{T}(S) & \to & \mathbb{R}_+^{\mathcal{S}(S)} \\ & & \mathfrak{X} & \mapsto & \ell_{\mathfrak{X}}. \end{array}$$

Theorem 2.5.1 (9g-9 theorem)**.** *There is a collection of simple closed curves $\alpha_1, ..., \alpha_{9g-9}$ on S such that the following map is a proper embedding.*

$$\begin{array}{cccc} \ell & : & \mathcal{T}(S) & \to & \mathbb{R}_+^{9g-9} \\ & & \mathfrak{X} & \mapsto & (\ell_{\mathfrak{X}}(\alpha_i))_{i=1,...,9g-9} \end{array}$$

We will not prove this theorem but the reader can refer to [FM11] page 300. However we can use it to study divergent sequences in $\mathcal{T}(S)$.

Corollary 2.5.2. *If $(\mathfrak{X}_n)_{n\in\mathbb{N}} \in \mathcal{T}(S)^{\mathbb{N}}$ is unbounded then there exists α a simple closed curve in S such that the sequence $\ell_{\mathfrak{X}_n}(\alpha)$ is unbounded in $\mathbb{R}_+$.*

3 Gromov hyperbolic spaces

In this section we are going to use Gromov hyperbolicity to deduce properties of hyperbolic surfaces. In particular, this will allow us to study the action of homeomorphisms in the boundary of the universal cover of a surface and on its geodesics. Gromov hyperbolic spaces are studied in greater detail in [GdlH90].

3.1 Quasi-isometries

The main topic of this section are quasi-isometries: if X and X' are hyperbolic surfaces and $\tilde{\varphi} : \tilde{X} \to \tilde{X}'$ a lift on the universal covers of a homeomorphism $\varphi : X \to X'$ then $\tilde{\varphi}$ is a quasi-isometry.

Definition 3.1.1. *Let (X, d) and (X', d') be two metric spaces. They are said to be quasi-isometric if there exist two positive constants c, λ and two maps $f : X \to X'$, $g : X' \to X$ such that the following holds:*

- $\forall\, x, y \in X,\ d'(f(x), f(y)) \leq \lambda d(x, y) + c,$
- $\forall\, x', y' \in X',\ d(g(x'), g(y')) \leq \lambda d'(x', y') + c,$
- $\forall\, x \in X,\ d(g(f(x)), x) \leq c,$
- $\forall\, x' \in X',\ d(f(g(x')), x') \leq c.$

Now give a characterisation of this property in terms of maps called quasi-isometries.

Definition 3.1.2. *With (X, d) and (X', d') two metric spaces, a map $f : X \to X'$ is a (λ, c)-quasi-isometric embedding if*

$$\forall\, x, y \in X,\ \frac{1}{\lambda} d(x, y) - c \leq d'(f(x), f(y)) \leq \lambda d(x, y) + c.$$

Two (λ, c)-quasi-isometric embeddings $f : X \to X'$ and $g : X' \to X$ are (λ, c)-quasi-inverses if there is a positive constant δ such that:

- $\forall\, x \in X,\ d(g(f(x)), x) \leq \delta,$
- $\forall\, x' \in X',\ d'(f(g(x')), x') \leq \delta.$

Hence a map $f : (X, d) \to (X', d')$ is a (λ, c)-quasi-isometry if it admits a (λ, c)-quasi-inverse.

A computation proves that every f and g as in *Definition 3.1.1* are $(\lambda, \max(c, 3c/\lambda))$-quasi-inverses. Hence, two metric spaces are quasi-isometric if and only if there exists a pair of quasi-isometries between them.

Example. *The natural injection and the integer part are quasi-inverses between* $\mathbb{Z}$ *and* $\mathbb{R}$

The main example of quasi-isometric metric spaces may be constructed endowing finitely generated groups with metric structures.

Let G be a finitely generated group. A finite set of generators S is said to be symmetric if it does not contain 1_G and it is its own inverse (*ie.* $S^{-1} = S$). If S is such a generating set then define $\ell_S(g)$, for all $g \in G$, the S-length of g, as the length of the shortest reduced word in the alphabet S which is equal to g. For example $\ell_S(1_G) = 0$ and if $s \in S$ then $\ell_S(s) = 1$.

Definition 3.1.3. *If S is a symmetric generating set of G then ℓ_S endows G with a* <u>*left invariant metric associated to S*</u>*:*

$$d_S(h, g) = \ell_S(g^{-1}h)$$

For example, the fundamental group of a hyperbolic closed surface admits a metric space structure with the previous construction.

What is important is that, up to quasi-isometries, we can talk about the metric of a finitely generated group.

Proposition 3.1.4. *If S and S' are to symmetric generating sets of G then $id : (G, d_S) \to (G, d_{S'})$ is a quasi-isometry.*

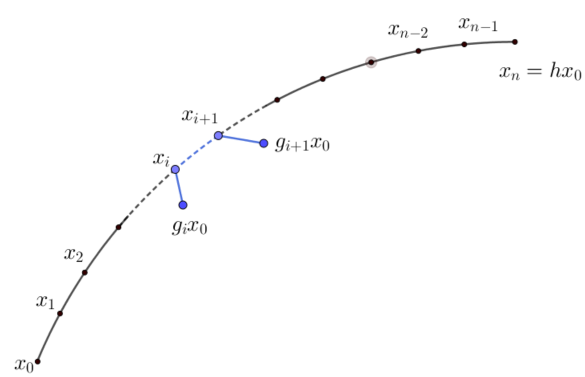
Proof. If $l = \max\limits_{s\in S}(\ell_{S'}(s))$ and $l' = \max\limits_{s\in S'}(\ell_S(s))$ then for every element g in the group, $\ell_S(g) \leq l.\ell_{S'}(g)$ and $l_{S'}(g) \leq l.\ell_S(g)$ and the identity map is a $(\max(l, l'), 0)$-quasi-isometry. □

The reason why this example is interesting is because of the Švarc-Milnor lemma ([BH99] page 140): it implies that a group acting on a space is, under certain conditions, quasi-isometric to the space it acts on.

Theorem 3.1.5 (Švarc-Milnor Lemma)**.** *Let (X, d) be a proper simply connected geodesic metric space and a group G whose action on X is cocompact, via isometries, and properly discontinuous. G is finitely generated and for every $x_0 \in X$ the map $g \in G \mapsto g \cdot x_0 \in X$ is a quasi-isometry.*

Proof. As the action of G is cocompact the diameter d of ${}^{X}\!/_{G}$ is finite.

For $x_0 \in X$ fixed, consider the following set: $S = \{g \in G | d(x_0, gx_0) \leq 3d\}$.
If h is an element of G then cut a geodesic between x_0 and hx_0 into segments of length at most d and note x_i the successive points of this decomposition with $x_n = hx_0$. By definition of d, there exists for all $i \in \{0, ..., n\}$ $g_i \in G$ such that $d(x_i, g_ix_0) \leq d$, with $g_0 = 1_G$ and $g_n = h$. So for every i we have $d(g_ix_0, g_{i+1}x_0) \leq d(g_ix_0, x_i) + d(x_i, x_{i+1}) + d(x_{i+1}, g_{i+1}x_0) \leq 3d$.



Since the action is via isometries, $g_i^{-1}g_{i+1} \in S$. Thus $g_{i+1} = g_is_i$ where $s_i \in S$ and $h = g_n = s_{n-1}...s_1$. Finally S is a generating set for G.

The action is properly discontinuous so S have to be finite and G is finitely generated.

As the action is by isometries S is symmetric and induces a notion of length ℓ_S on G, and with the preceding, for h a point in G we obtain $\ell_S(h) \leq \frac{d(x_0,hx_0)}{d} + 1$.

We now want to prove that $ev_0 : g \in G \mapsto gx_0 \in X$ is a quasi-isometry, for that, consider $m : x \in X \mapsto g_x \in G$ where g_x is such that $d(x, g_x x_0) \leq d$. For $x, y \in X$ and $g, h \in G$ we have

$$\begin{array}{rcl} d_S(mx, my) & = & \ell_S(g_x^{-1} g_y) \\ & \leq & \dfrac{d(g_x x_0, g_y x_0)}{d} + 1 \\ & \leq & \frac{1}{d}(d(g_x x_0, x) + d(x, y) + d(g_y x_0, y)) + 1 \\ & \leq & \frac{1}{d} d(x, y) + 3 \end{array}$$

$$\begin{array}{rcl} d(x_0, gx_0) & = & d(x_0, s_n ... s_1 x_0) \text{ since } S \text{ generates } G \\ & \leq & d(x_0, s_n s_0) + d(s_n x_0, s_n s_{n-1} x_0) \\ & & +d(s_n s_{n-1}, s_n s_{n-1} s_{n-2} x_0) + ... \\ & & +d(s_n .. s_{n-k} x_0, s_n .. s_{n-k} s_{n-k-1} x_0) + ... \\ & & +d(s_n ... s_2 x_0, s_n ... s_1 x_0) \\ & = & d(x_0, s_n x_0) + ... + d(x_0, s_{n-k} x_0) + ... + d(x_0, s_1 x_0) \\ & \leq & 3d\ell_S(g) \end{array}$$

Finally $d_S(mx, my) \leq \max(\frac{1}{d}, 3d) d(x, y) + 3$ and $d(ev_0 h, ev_0 g) \leq \max(\frac{1}{d}, 3d) d_S(h, g) + 3$. Moreover $d(x, ev_0 g_x) = d(x, g_x x_0) \leq d$ and $d_S(g, g_{ev_0 g}) = d_S(g, g_{gx_0}) \leq \frac{d(gx_0, g_{gx_0} x_0)}{d} + 1 \leq 2$ by definition of g_x and ev_0. Thus, G and X are quasi-isometric and ev_0 is a quasi-isometry whatever x.

□

Therefore, if X is a closed hyperbolic surface with its fundamental group acting by deck transformations on the universal cover $\tilde{X}$ then $\pi_1(X)$ is quasi-isometric to $\tilde{X}$ and to $\mathbb{H}^2$. Following this remark, let us go back to hyperbolic surfaces.

Corollary 3.1.6. *Let X and X' be closed hyperbolic surfaces and $\varphi : X \to X'$ a homeomorphism, every lift $\tilde{\varphi} : \tilde{X} \to \tilde{X}'$ of φ is a quasi isometry.*

Proof. Fix some base-points in X and X' and a lift of this base-points in the universal covers, all of them noted $*$.

As φ is a homeomorphism the induced map $\varphi_* : \pi_1(X, *) \to \pi_1(X', *)$ is an isomorphism and with well chosen generating sets in the fundamental groups φ_* is an isometry.

According to the Švarc-Milnor Lemma *3.1.5*, $g \in \pi_1(X, *) \mapsto g* \in \tilde{X}$ and $g \in \pi_1(X', *) \mapsto g* \in \tilde{X}'$ are quasi-isometries. We even know that lifted morphisms are π_1-equivariant, in other words for $x \in \tilde{X}$ and $g \in \pi_1$ we have $\tilde{\varphi}(gx) = \varphi_*(g)\tilde{\varphi}(x)$. As a consequence the following diagram commutes and $\tilde{\varphi}$ is a quasi-isometry as a composition of quasi-isometries.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\varphi}} & \tilde{X}' \\ \uparrow & \circlearrowleft & \uparrow \\ \pi_1(X, *) & \xrightarrow[\varphi_*]{} & \pi_1(X', *) \end{array}$$

□

3.2 Gromov hyperbolic spaces

Quasi-isometries have a particular behavior in spaces called Gromov hyperbolic spaces. From now on suppose that all the metric spaces are geodesics, meaning that there is a shortest path between every two points: if $x, y \in X$ are at distance d then there is a isometry $g : [0, d] \to X$ with base-point x and endpoint y. Such a geodesic path will be noted $[x, y]$. A map $g : \mathbb{R} \to X$ is a minimizing geodesic if every sub-segment is a geodesic path.

We note that the geodesic path g between x and y is not necessarily unique: in $\mathbb{H}^2$ there is a unique geodesic path between two points while not in $\mathbb{S}^2$ (between the poles there are infinitely many).

Definition 3.2.1. *Let (X, d) be a geodesic metric space and δ a positive constant.*

The space X is said to be δ-hyperbolic if in every geodesic triangle $\Delta = [x, y] \cup [y, z] \cup [z, x]$ all edges verifies that all of its points are at distance at most δ of the union of the other two.

One says that X is (Gromov)-hyperbolic if it is δ-hyperbolic for some δ.

The above condition over triangles is called a Rips condition. There are equivalent conditions to define hyperbolic spaces. One introducing the Gromov product: $(y|z)_x = \frac{1}{2}(d(x, y) + d(x, z) - d(y, z))$. It is equal to 0 if $x \in [y, z]$ and $\min(d(x, y), d(x, z))$ if not. In that case, X is δ-hyperbolic if for every $x, y, z, w \in X$ $(x|z)_w \geq \min((x|y)_w, (y|z)_w) - \delta$. There exists other definitions introducing some quantities on triangles. To find more informations see [GdlH90] chapter 2.

Theorem 3.2.2. *The hyperbolic plane is Gromov-hyperbolic.*

Proof. A property of hyperbolic triangles is that their area is π minus the sum of their angles, as a consequence, there is no hyperbolic triangle of area grater than π.

Let δ be three times the radius of a π area disc in $\mathbb{H}^2$. If Δ is a triangle with a point of an edge at distance larger than δ of the other two then Δ contains a half disc of radius δ (centered at this point) which strictly contains a disc of radius $\delta/3$. Thus, the area of Δ is larger than the area of this disc: the area of Δ is strictly larger than π, we noticed earlier that it was impossible. We have proved that $\mathbb{H}^2$ is δ-hyperbolic. □

We will from now on call quasi-geodesic (resp. quasi-segment, resp. quasi-ray) a quasi-isometric embedding from $\mathbb{R}$ (resp. $[a, b]$, resp. $[0, \infty)$). We want to know how far from geodesic paths are quasi-segments. The following theorem is admitted here, one can find proofs in [GdlH90] pages 82 and 101, but we are going to apply it to prove that quasi-isometries preserve Gromov hyperbolicity.

For this we introduce the Hausdorff distance. If (X, d) is a metric space then we have an induced function on the set of nonempty closed subsets of X called the Hausdorff distance:

$$d_H(U, V) = \inf\{\varepsilon > 0 | \forall (u, v) \in U \times V,\ d(u, V) \leq \varepsilon,\ d(v, U) \leq \varepsilon\}.$$

It is not a distance as it might take infinite values but if verifies the triangle inequality.

Theorem 3.2.3. *Let $\delta \geq 0$, $\lambda \geq 1$ and $c > 0$ be three constants.*

There is a constant $H(\delta, \lambda, c)$ such that in every δ-hyperbolic space (X, d) and for every $x, y \in X$, if g is a (λ, c)-quasi-geodesic and g' a geodesic between x and y, then $d_H(Im(g), Im(g')) \leq H(\delta, \lambda, c)$.

There exists a constant $\tilde{H}(\delta, \lambda, c)$ such that in every proper δ-hyperbolic space (X, d) all the (λ, c)-quasi-geodesics are at distance at most $\tilde{H}(\lambda, \delta, c)$ of a minimizing geodesic.

The same result is true for quasi-rays.

We are now able to apply it to prove that quasi-isometries preserve Gromov hyperbolic spaces.

Corollary 3.2.4. *Let (X, d) and (X', d') be geodesic metric spaces with $f : X \to X'$ and $g : X' \to X$ some (λ, c)-quasi-isometries. Suppose that X is δ-hyperbolic then X' is Gromov hyperbolic.*

Proof. With the same notations as in the corollary consider $\Delta = [x, y] \cup [y, z] \cup [z, x]$ a geodesic triangle in X' and $\Delta' = [gx, gy] \cup [gy, gz] \cup [gz, gx]$ a geodesic triangle in X with vertices gx, gy and gz.

$$\begin{array}{rcll} d_H([x,z],[y,z]\cup[y,x]) & \leq & d_H([x,z],fg[x,z]) & \\ & + & d_H(fg[x,z],fg([y,z]\cup[x,y])) & \\ & + & d_H(fg([x,z]\cup[x,y]),[x,y]\cup[y,z]) & \\ & \leq & c & f \text{ and } g \text{ are } (\lambda,c)\text{-quasi-isometries} \\ & + & \lambda d_H(g[x,z],g([x,y]\cup[y,z]))+c & f \text{ is a } (\lambda,c)\text{-quasi-isometric embedding} \\ & + & c & f \text{ and } g \text{ are } (\lambda,c)\text{-quasi-isometries} \end{array}$$

$$\begin{array}{rcll} d_H(g[x,z],g([x,y]\cup[y,z])) & \leq & d_H(g[x,z],[gx,gz]) & \\ & + & d_H([gx,gz],[gx,gy]\cup[gy,gz]) & \\ & + & d_H([gx,gy]\cup[gy,gz],g([x,y]\cup[y,z])) & \\ & \leq & H(\delta,\lambda,c) & \text{By the previous theorem} \\ & + & \delta & X \text{ is a } \delta\text{-hyperbolic space} \\ & + & H(\delta,\lambda,c) & \text{By the previous theorem} \end{array}$$

We have therefore proved that $d_H([x,z],[y,z]\cup[y,x]) \leq 3c+\lambda(\delta+2H(\delta(\delta,\lambda,c))$ and with the same computations every side of δ is at distance at most $3c+\lambda(\delta+2H(\delta(\delta,\lambda,c))$ of the union of the other two. So, we have proved that X' is Gromov hyperbolic with $\delta' = 3c+\lambda(\delta+2H(\delta(\delta,\lambda,c))$. □

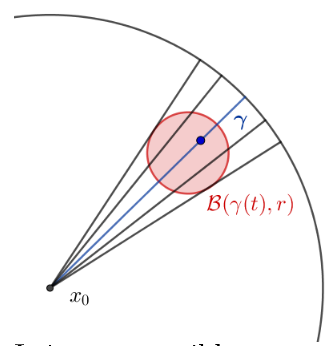
3.3 Notion of boundary

Since we want to extend lifted homeomorphisms to the boundary of $\mathbb{H}^2$ we need to study the boundary of a Gromov hyperbolic space.

Definition 3.3.1. *Let (X,d) be a Gromov hyperbolic space. Two quasi-rays are said to be equivalent if they are at finite Hausdorff distance: the boundary ∂X of X is the set of equivalence classes of quasi-rays.*

Fact. *The quasi-rays f and g are equivalent if and only if $\sup_{t\geq 0} d(f(t),g(t)) < \infty$.*

In $\mathbb{H}^2$, the ordinary boundary is the set of equivalence classes of geodesic rays, with *Theorem 3.2.3* it is clear that the previous definition of boundary is the same as the well-known one.



We can endow the boundary with a topology thanks to a neighborhood basis for every point of ∂X. Fix a base point $x_0 \in X$, in every class of ∂X there is at least a geodesic ray γ based at x_0.
If t, r are positive real numbers and γ a geodesic ray based at x_0, then the (t,r)-neighborhood of $[\gamma]$ is the set $\mathcal{V}_{t,r}([\gamma])$ of all classes of geodesic rays based at x_0 which path through the ball of center $\gamma(t)$ and radius r.
We construct a topology with the basis of neighborhood given above.

It is now possible to extend quasi-isometries to the boundary.

Theorem 3.3.2. *Let (X,d) and (X',d') be two proper δ-hyperbolic spaces. Every (λ,c)-quasi-isometry $f:(X,d)\to(X',d')$ induces a well defined homeomorphism ∂f between the boundaries:*

$$\begin{array}{rcccl} \partial f & : & \partial X & \to & \partial X' \\ & & [\alpha] & \mapsto & [f\circ\alpha]. \end{array}$$

Proof. If $f:(X,d)\to(X',d')$ is a (λ,c)-quasi-isometry and $\alpha:\mathbb{R}_+\to X$ a quasi-ray then $f\circ\alpha$ is a quasi-ray in X' and if $[\gamma]$ denotes the equivalence class of the quasi-ray γ then we set $\partial f([\alpha]) := [f\circ\alpha]$.

It remains to prove that ∂f is well defined. If α and β are two equivalent quasi-rays in X, $\sup_{t\geq 0} d(\alpha(t),\beta(t)) = h < \infty$ and for $t\in\mathbb{R}_+$

$$\begin{array}{rcl} d(f(\alpha(t)), f(\beta(t))) & \leq & \lambda d(\alpha(t), \beta(t)) + c \\ & \leq & \lambda h + c \end{array}$$

Thus $\sup_{t \geq 0} d(f \circ \alpha(t), f \circ \beta(t)) < \infty$ and $f \circ \alpha$ and $f \circ \beta$ are equivalent quasi-rays in X' and ∂f is well defined.

Moreover, if g is a quasi-inverse for f then $\partial g = \partial f^{-1}$. With the same computations as above it is clear that ∂f and ∂g are continuous for the above topology. $\square$

As a consequence, for X and X' two closed hyperbolic surfaces if $\varphi : X \to X'$ is a homeomorphism, it induces π_1-equivariant quasi-isometries $\tilde{\varphi} : \tilde{X} \to \tilde{X}'$ which lead to maps $\partial\tilde{\varphi} : \partial\tilde{X} \to \partial\tilde{X}'$.

Property 3.3.3. *Let X and X' be two closed hyperbolic surfaces, if $\varphi : X \to X'$ and $\phi : X \to X'$ are homotopic then they induced the same applications between $\partial\tilde{X}$ and $\partial\tilde{X}'$ for some lifts.*

Proof. Let $H : [0,1] \times X \to X'$ be a homotopy with $H_0 = \varphi$ and $H_1 = \phi$. This homotopy lifts to a homotopy $\tilde{H} : [0,1] \times \tilde{X} \to \tilde{X}'$ with $\tilde{H}_0 = \tilde{\varphi}$ a lift of φ and $\tilde{H}_1 = \tilde{\phi}$ a lift of ϕ. For every $\tilde{x} \in \tilde{X}$, $\tilde{H}_{t\in[0,1]}(\tilde{x})$ is a a continuous path between $\tilde{\varphi}(\tilde{x})$ and $\tilde{\phi}(\tilde{x})$ and so $d(\tilde{\varphi}(\tilde{x}), \tilde{\phi}(\tilde{x})) \leq \ell_{\tilde{X}'}(\tilde{H}_{t\in[0,1]}(\tilde{x})) = \ell_{X'}(H_{t\in[0,1]}(x))$. Since X is compact $x \in X \mapsto \ell_{X'}(H_{t\in[0,1]}(x))$ is bounded:

$$\exists d > 0 : \forall \tilde{x} \in \tilde{X} \; d(\tilde{\varphi}(\tilde{x}), \tilde{\phi}(\tilde{x})) \leq d.$$

As a consequence, if α is a quasi ray in $\tilde{X}$ its images through $\tilde{\varphi}$ and $\tilde{\phi}$ are at bounded distance and $\partial\tilde{\varphi} = \partial\tilde{\phi}$. $\square$

4 Laminations and currents

This section will be dedicated to the study of sets of geodesics. First, geodesic laminations which are compact subsets of surfaces made of geodesics and later geodesic currents which are measures on $\mathcal{G}(\tilde{X})$.

If X is a closed hyperbolic surface there is a canonical bijection between the set of unoriented geodesics $\mathcal{G}(\tilde{X})$ in $\tilde{X}$ and $\partial\tilde{X} \times \partial\tilde{X} - \Delta /_{\equiv}$ where Δ is the diagonal and $(x, y) \equiv (y, x)$. Then $\partial\tilde{\varphi}$ naturally induces a map $\tilde{\varphi}_g : \mathcal{G}(X) \to \mathcal{G}(X')$.

Corollary 4.0.1. *If $\varphi : X \to X'$ is a homeomorphism between closed hyperbolic surfaces then every lift $\tilde{\varphi}$ of φ induces a π_1-equivariant homeomorphism $\tilde{\varphi}_g : \mathcal{G}(\tilde{X}) \to \mathcal{G}(X')$ where the spaces of geodesics are equipped with the uniform convergence on compact sets.*

We note that $\partial\tilde{\varphi}$ maps a geodesic g to the unique geodesic of $\tilde{X}'$ which is at bounded distance of the quasi-geodesic $\tilde{\varphi} \circ g$.

Fact. *It follows directly from Property 3.3.3 that if $\varphi : X \to X'$ and $\phi : X \to X'$ are homotopic then they induce the same applications between $\mathcal{G}(\tilde{X})$ and $\mathcal{G}(\tilde{X}')$.*

If X and X' are two copies of a closed surface S of genus at least 2 endowed with different hyperbolic metrics then $id : X \to X'$ induces a homeomorphism between $\mathcal{G}(\tilde{X})$ and $\mathcal{G}(\tilde{X}')$. Thus it is possible to speak about $\mathcal{G}(\tilde{S})$ without reference to the metric.

Remark. *Remark that if X and X' are the same element of $\mathcal{T}(S)$ then there is an isometry $i : X \to X'$ and $\tilde{X}$ and $\tilde{X}'$ are isometric too. Isometries map geodesics to geodesics so, if $X = X'$ in $\mathcal{T}(S)$, then the map $\tilde{i}_g$ maps a geodesic to its image trough $\tilde{i}$.*

4.1 Geodesic laminations

Now we work on a closed hyperbolic surface X, it means a closed surface S of genus at least 2 endowed with a hyperbolic metric ρ. The main reference of this section is [CB88].

Definition 4.1.1. *A geodesic lamination λ on X is a non-empty compact subset of X which is a disjoint union of simple complete unoriented geodesics. The geodesics are called the leaves of λ and we will note $\mathcal{L}(X)$[1] the set of geodesics laminations on X.*

Example. *For instance a union of disjoint simple closed geodesics is a geodesic lamination.*

The surface X is closed and so compact, as a consequence the Hausdorff distance on $\mathcal{K}(X)$, the set of closed subsets of X, is a distance and induced a topology. $\mathcal{K}(X)$ is compact for this topology as X is compact. As a consequence, we can endow $\mathcal{L}(X)$ with a topology whose properties we will see later.

Given a geodesic γ in X, a lift $\tilde{\gamma}$ of γ is a complete geodesic in $\tilde{X}$ such that its projection is γ, note $\tilde{\tilde{\gamma}} = \pi^{-1}(\gamma)$ the set of all the lifts of γ with $\pi : \tilde{X} \to X$ the projection: it is a $\pi_1(X)$-invariant subset of $\mathcal{G}(\tilde{X})$. The same construction with geodesic laminations give a correspondence between a geodesic lamination λ and a closed non empty and $\pi_1(X)$-invariant subset $\tilde{\tilde{\lambda}}$ of $\mathcal{G}(\tilde{X})$ made of disjoint geodesics.

Proposition 4.1.2. *Let X and X' be two closed hyperbolic surfaces, every homeomorphism $\varphi \in \mathrm{Homeo}(X, X')$ induces a homeomorphism $\varphi_\Lambda : \mathcal{L}(X) \to \mathcal{L}(X')$.*

Proof. We even know by *Theorem 4.0.1* that every lift $\tilde{\varphi}$ of φ induces a π_1-equivariant homeomorphism $\tilde{\varphi}_g$ between $\mathcal{G}(\tilde{X})$ and $\mathcal{G}(\tilde{X}')$, this application will map closed non empty and $\pi_1(X)$-invariant subset of $\mathcal{G}(\tilde{X})$ made of disjoint geodesics to closed non empty and $\pi_1(X')$-invariant subset of $\mathcal{G}(\tilde{X}')$ made of disjoint geodesics. If we prove that this map does not depend on the lift then it will induce a well defined homeomorphism $\varphi_\Lambda : \mathcal{L}(X) \to \mathcal{L}(X')$.

Let $\tilde{\varphi}$ and $\tilde{\varphi}'$ be two lifts of φ, they differ from a deck transformation $\alpha \in \pi_1(X)$: $\varphi' = \varphi \circ \alpha$ and $\tilde{\varphi}'_g = \tilde{\varphi}_g \circ \alpha$. If $\tilde{\tilde{\lambda}}$ is the subset of $\mathcal{G}(\tilde{X})$ corresponding to a geodesic lamination λ then $\tilde{\varphi}'_g(\tilde{\tilde{\lambda}}) = \tilde{\varphi}_g(\alpha(\tilde{\tilde{\lambda}}))$, but $\tilde{\tilde{\lambda}}$ is $\pi_1(X)$-invariant so $\alpha(\tilde{\tilde{\lambda}}) = \tilde{\tilde{\lambda}}$ and thus $\tilde{\varphi}'_g(\tilde{\tilde{\lambda}}) = \tilde{\varphi}_g(\tilde{\tilde{\lambda}})$ and $\varphi_\Lambda(\lambda) = \pi \circ \tilde{\varphi}_g(\tilde{\tilde{\lambda}})$ does not depend on the lift and is a homeomorphism because $\tilde{\varphi}_g$ is a homeomorphism. □

Reasoning as with the geodesics, if X and X' are two copies of S endowed with different metrics, where S is a closed surface of genus $g \geq 2$, then we can pass from $\mathcal{L}(X)$ to $\mathcal{L}(X')$ homeomorphically: even if there is no notion of geodesic on S, we can talk about $\mathcal{L}(S)$ the set of geodesic laminations on S.

By definition, a geodesic lamination is a compact subset of X which admits a decomposition as union of geodesics. This decomposition is unique but to see this we first need the following observation.

Consider two geodesics which are close near a given point. Intuitively, if their directions are too different then they will meet. This observation leads to the following lemma.

Lemma 4.1.3. *If $\lambda = \bigcup_{x \in L} \gamma_x$ is a decomposition of a geodesic lamination into geodesics then the direction of γ_x at x varies continuously with X.*

As a consequence, a geodesic lamination $\lambda \in \mathcal{L}(X)$ is a strict subset of X. Indeed, as the direction of the geodesics varies continuously, a decomposition of λ into disjoint simple closed geodesics induced a non vanishing vector field on λ, but a vector field on a closed orientable (connected) hyperbolic surface admits at least one singularity so λ is proper.

[1] $\mathcal{L}$ will be used as a notation for the Liouville measure too, the context will prevent any confusion.

Lemma 4.1.4. *A geodesic lamination in a closed oriented hyperbolic surface has no interior and has a unique decomposition as a union of disjoint simple geodesics.*

Proof. Fix a decomposition of $\lambda \in \mathcal{L}(X)$ into geodesics and suppose that λ contains an open ball B. If $\tilde{\lambda}$ is a lift of λ and $\tilde{B}$ a lift of B then we can consider α an arc into $\tilde{B}$ such that at every point x of α, the lifted geodesic $\tilde{\gamma_x}$ is transversal to α. This is possible thanks to *lemma 4.1.3*. See $\tilde{X}$ as $\mathbb{H}^2$ and define $\begin{array}{cccc} \phi & : & [0,1]\times\mathbb{R} & \to & \tilde{\lambda} \subset \mathbb{H}^2 \\ & & (s,t) & \mapsto & \tilde{\gamma}_{\alpha(s)}(t) \end{array}$. The image of ϕ is a region of $\mathbb{H}^2$ bounded by $\gamma_{\alpha(0)}$ and $\gamma_{\alpha(1)}$ which contains balls of arbitrarily large diameter. Take a ball of diameter larger than the one of X, we have a lift of X covered by $\tilde{\lambda}$ and by projection X is covered by λ: we have seen above that it is impossible. Hence we have proved that λ has no interior.

The first part of the proof is based on the existence of a transverse arc so, with the same reasoning we prove that there is a unique decomposition into geodesics. □

The only example of geodesic lamination we gave is the union of simple closed curves. The following property gives a way to construct some others.

Proposition 4.1.5. *The closure of a non-empty disjoint union of simple geodesics in X is a lamination.*

Proof. Let L be a disjoint union of simple geodesics and $\overline{L}$ its closure, we have to prove that each point of $\overline{L}\backslash L$ belongs to a geodesics γ included in $\overline{L}$ and that all the geodesics in $\overline{L}$ are disjoints.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in L which converge to a point x of X and note γ_n the geodesic through x_n in L for every n.

For every n the direction of γ_n at x_n is a point of $\mathbb{RP}^1$ which is a compact set then the sequence of the directions admits a sub-sequence (we may forget that it is a sub-sequence for the rest) which converges. Let γ be the unique geodesic through x with this limit direction.

If y is a point at distance d from x in γ and y_n at same distance from x_n in γ_n, as geodesics are solutions of an ordinary differential equation whose flow is continuous in every variable, the y_n converges to y and then γ belongs to $\overline{L}$.

In light of the foregoing, every geodesic of $\overline{L}$ can locally be approached by geodesics of L. Then, if two geodesics γ, γ' of $\overline{L}$ intersect transversely in x (if the intersection is not transversal then they are equal), they can be approach as close as possible by two geodesics β, β' of L and such β and β' will intersect transversely but that is excluded. A similar argument can be made to prove that the geodesics of $\overline{L}$ are simple. We have proved that $\overline{L}$ is a geodesic lamination. □

The last point on our brief study of geodesics laminations is to look at the topology. The natural topology on $\mathcal{L}(X)$ is the Hausdorff topology induced by the Hausdorff metric on $\mathcal{K}(X)$. But another way to deal with the topology on $\mathcal{L}(X)$ is to consider it as a closed subset of the projective tangent bundle $PT(X)$ where $PT(X) = \{(x,d) | x \in X,\, d \in \mathbb{RP}^1\}$. This set appears as a bundle on X with the following commutative diagram:

$$\begin{array}{ccc} PT(U) & \xrightarrow{\sim} & U \times \mathbb{RP}^1 \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & U \subset X. & \end{array}$$

So $PT(X)$ is a compact 3-manifold and is metrizable such that the map p is continuous, as a consequence $\mathcal{K}(PT(X))$ is compact for the topology of the Hausdorff distance. Now, we can embed $\mathcal{L}(X)$ into $\mathcal{K}(PT(X))$.

Every geodesic γ of X admits a lift $\hat{\gamma} = \{(x,d) | x \in \gamma,\, d \text{ direction of } \gamma \text{ at } x\}$ in $PT(X)$ such that every point of $PT(X)$ lies in a unique lifted geodesic. It is now possible to lift a geodesic lamination with $\hat{\lambda} = \{\hat{\gamma} | \gamma \subset \lambda\}$, thereby we obtain a closed subset of $PT(X)$ and the set $\mathcal{L}(X)$ lifts with $\hat{\mathcal{L}}(X) = \{\hat{\lambda} | \lambda \in \mathcal{L}(X)\} \subset \mathcal{K}(PT(X))$.

Theorem 4.1.6. *The Hausdorff distances on $\mathcal{K}(X)$ and $\mathcal{K}(PT(X))$ induce the same topology on $\mathcal{L}(X)$ and $\mathcal{L}(X)$ is compact for this topology.*

Proof. The continuous function $p : PT(X) \to X$ induces a continuous function $p_\lambda : \hat{\lambda} \to \lambda$, for every geodesic lamination λ and we can consider its inverse $\begin{array}{cccc} p_\lambda^{-1} & : & \lambda & \to & \hat{\lambda} \\ & & x & \mapsto & (x,\text{direction of } \gamma_x \text{ at } x) \end{array}$ which is continuous by *Lemma 4.1.3*. So p_λ is a homeomorphism and $\hat{\lambda}$ is compact as λ is compact and then closed too. We can consider $\hat{\mathcal{L}}(X)$ as a subset of $\mathcal{K}(PT(X))$.

Hence, $\begin{array}{ccccc} p_* & : & \hat{\mathcal{L}}(X) & \to & \mathcal{L}(X) \\ & & \hat{\lambda} & \mapsto & p_\lambda(\hat{\lambda}) \end{array}$ is bijective and continuous.

If we prove that $\hat{\mathcal{L}}$ is compact then p_* will be a homeomorphism and $\mathcal{L}$ will be compact: that is what we want to prove.

Consider a sequence of lifted geodesic laminations $(\hat{\lambda}_n)_{n\in\mathbb{N}}$ and $A \in \mathcal{K}(PT(X))$ the limit of this sequence. We want to show that A is a lifted geodesic lamination: $A = \hat{\lambda}$.

Take $\lambda = p(A)$, as A is a closed subset of a compact, A is compact and by continuity of p the set λ is compact and non-empty for the same reason.

Let (a, d_a) be a point of A, there is a sequence of points $(x_n, d_n) \in \hat{\lambda}_n$ which converges to (a, d_a) then the x_n converge to a and the directions d_n to d_a, and for the same reason as in 4.1.5 the closed set λ is a geodesic lamination and $A = p_*^{-1}(\lambda) = \hat{\lambda}$ is a lifted geodesic lamination.

We have then proved that $\hat{\mathcal{L}}$ is a closed subset of $\mathcal{K}(PT(X))$ which is compact, thus $\hat{\mathcal{L}}$ is compact and the proof is complete. □

We have seen that geodesic laminations are defined on a closed surface S, fix such a surface S of genus at least two, we want to add some measures to the geodesic laminations to obtain the set of measured laminations $\mathcal{ML}(S)$ on S.

Definition 4.1.7. *If λ is a geodesic lamination on S, a <u>transverse measure</u> on λ is a collection of Radon measures λ_I on I for every arc I transverse to L such that:*

- *if $J \subset I$ is a subarc of I then $\lambda_J = (\lambda_I)_{|J}$,*
- *if I and J are homotopic via a homotopy H fixing the leaves of L then $\lambda_J = (H_1 \circ H_0^{-1})_* \lambda_I$.*

A <u>measured lamination</u> is a a geodesic lamination endowed with a transverse measure. The pair will be noted λ too.

For example, consider $L = \alpha_1 \cup ... \cup \alpha_n$ a finite union of closed simple geodesics and assign to each geodesic a positive weight t_i, if λ consists in counting the intersections taking in account the weights then (L, λ) is a measured lamination: for A a Borel subset of I a transversal arc, $\lambda_I(A) = \sum_{i=1}^{n} t_n |E \cap L|$. If L is made of a single geodesic α with weight 1 then the corresponding geodesic lamination is noted μ_α.

We can endow $\mathcal{ML}(S)$, the set of measured laminations on S, with a topology: λ_n converge to λ if for every arc I which is not contained in any simple complete geodesic and every $f : I \to \mathbb{R}$ we have

$$\int_I f d\lambda_{nI} \underset{n\to+\infty}{\longrightarrow} \int_I f d\lambda_I.$$

Remark that measured laminations can be multiplied by positive numbers so we can defined the set of <u>projective measured laminations</u> $\mathbb{P}\mathcal{ML}(S) = {}^{\mathcal{ML}(S) \setminus \{0\}}\!/_{\mathbb{R}^+}$, it is this set that we will use for the compactification of $\mathcal{T}(S)$.

One can prove that $\mathbb{P}\mathcal{ML}(S)$ is a finite dimensional sphere. The reader can find a proof of the following statement in [AL10].

Theorem 4.1.8. *If S is closed oriented surface of genus at least 2 then $\mathbb{P}\mathcal{ML}(S)$ is homeomorphic to $\mathbb{S}^{6g-7}$.*

4.2 Geodesic currents

The last element we need for the compactification is the notion of currents. We will compactify $\mathcal{T}(S)$ by embedding $\mathcal{T}(S)$ and $\mathbb{P}\mathcal{ML}(S)$ in the set $\mathcal{C}(S)$ of geodesic currents.

Definition 4.2.1. *Let X be a hyperbolic surface, a geodesic current on X is a $\pi_1(X)$- invariant radon measure on $\mathcal{G}(\tilde{X})$.*

If S is a closed surface of genus at least 2 we can talk about geodesic currents without specifying the metric on S. As seen before, if we have two distinct metrics on S then there is a π_1-equivariant bijective correspondence between the associated sets of geodesics in the universal cover. As we can push-forward the measures it is not necessary to precise the metric over S.

The space of geodesic currents is endowed with the weak* topology: a sequence $(\mu_n)_{n\in N}$ of currents converges to a current μ if for every continuous $\mathbb{R}$-valued and compactly supported functions $f \in C_c(\mathcal{G}(\tilde{S}), \mathbb{R})$ the following holds:

$$\int_{\mathcal{G}(\tilde{S})} f d\mu_n \underset{n\to+\infty}{\longrightarrow} \int_{\mathcal{G}(\tilde{S})} f d\mu.$$

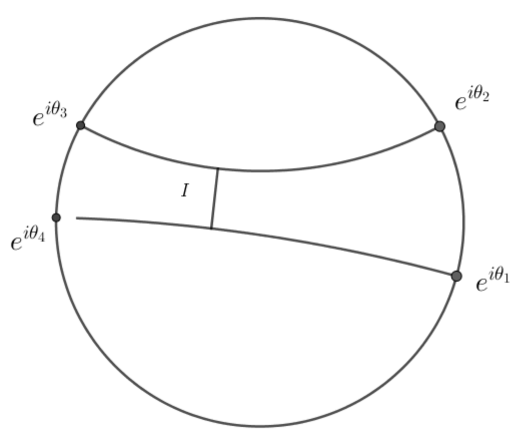
Example. *We can embed $\mathcal{S}(S)$ the set of free isotopy classes of simple closed curves into $\mathcal{C}(S)$ as follows. Let α be the geodesic representing an element of $\mathcal{S}(S)$ and $\tilde{\tilde{\alpha}}$ its preimage in $\tilde{S}$, $\tilde{\tilde{\alpha}}$ is a discrete $\pi_1(S)$-invariant subset of $\mathcal{G}(\tilde{S})$, as this set is $\pi_1(S)$-invariant the associated Dirac measure $\hat{\alpha}$ is a geodesic current. Using the same construction, every geodesic γ in S induces a current $\hat{\gamma}$.*

Remark. *We noticed that $\mathcal{G}(\tilde{S})$ identifies with ${}^{\partial\tilde{S}\times\partial\tilde{S}-\Delta}/_{\equiv}$ and thus to ${}^{\mathbb{S}^1\times\mathbb{S}^1-\Delta}/_{\equiv}$. An open basis for the topology on $\mathcal{G}(\tilde{S})$ with this identification is provided by the pairs $\{(e^{i\theta_1}, e^{i\theta_2}), (e^{i\theta_3}, e^{i\theta_4})\}$ such that the θ_i satisfy $\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ and $\theta_4 - \theta_1 \leq 2\pi$.*

Hence, to define a geodesic current it suffices to give the measure of such sets.

As an example, construct an injective map from $\mathcal{ML}(S)$ to $\mathcal{C}(S)$.

Suppose given a measured lamination λ of S. If $\{(e^{i\theta_1}, e^{i\theta_2}), (e^{i\theta_3}, e^{i\theta_4})\}$ is as in the remark, then there is a unique geodesic segment I which joins orthogonally the geodesics $(e^{i\theta_1}, e^{i\theta_4})$ and $(e^{i\theta_2}, e^{i\theta_3})$.



If this segment is transverse to $\tilde{\tilde{\lambda}}$, the preimage of λ in $\tilde{S}$, then this arc descends to an arc in S transversal to λ. We define the measure of the open set $\{(e^{i\theta_1}, e^{i\theta_2}), (e^{i\theta_3}, e^{i\theta_4})\}$ as the λ-measure of the transversal arc in S. If the arc is not transverse then the measure is 0. The corresponding current is noted $\hat{\lambda}$.

Proposition 4.2.2. *The previous construction is an injection of $\mathcal{ML}(S)$ into $\mathcal{C}(S)$.*

Proof. The construction of a measure associated to a measured lamination is described above. One can remark that this measure is supported on $\tilde{\tilde{\lambda}}$ as the λ-measures are supported on λ, proving that the application is injective. □

We can use the same process to define geodesic currents from measures on the unit tangent bundle $T^1(S)$ which are invariant under geodesic flip and geodesic flow.

Consider μ a flip and flow invariant measure on $T^1(S)$ and $\{(e^{i\theta_1}, e^{i\theta_2}), (e^{i\theta_3}, e^{i\theta_4})\}$ and I as above, the current $\hat{\mu}$ on $\mathcal{G}(\tilde{S})$ associated to μ satisfies

$$\hat{\mu}(\{(e^{i\theta_1}, e^{i\theta_2}), (e^{i\theta_3}, e^{i\theta_4})\}) = 2\tilde{\mu}(\bigcup_{t\in[0,1]} F_t(I))$$

where $\tilde{\mu}$ is a lift of μ on $T^1(\tilde{S})$.

This construction appears as a bijective correspondence illustrated in [ES] chapter 3.

Theorem 4.2.3. *If Σ is a hyperbolic surface then there is a bijective correspondence between $\mathcal{C}(\Sigma)$ and the set of geodesic flip and geodesic flow invariant measures on $T^1(\Sigma)$.*

We are going to use this identification to construct geodesic currents from elements of the Teichmüller space $\mathcal{T}(S)$.

Corollary 4.2.4. *Let S be a closed surface of genus at least 2, each element of $\mathcal{T}(S)$ identifies with an element of $\mathcal{C}(S)$.*

We will see later that the following construction is an embedding.

Proof. Liouville measure on $T^1(\mathbb{H}^2)$: There is a bi-invariant measure on $PSL_2(\mathbb{R})$ called the Haar measure. We can use it to construct a flip and flow invariant measure on $T^1(\mathbb{H}^2)$ called the Liouville measure.

The group $SL_2(\mathbb{R})$ acts on $T^1(\mathbb{H}^2)$ by homography with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, \vec{v}) = (\frac{az+b}{cz+d}, \frac{\vec{v}}{(cz+d)^2}).$$

This action is free and transitive and the stabilizer of $(i, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$ is $\pm I_2$, thus $PSL_2(\mathbb{R})$ identifies to $T^1(\mathbb{H}^2)$ via the orbit of $V_0 = (i, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$. Thereby, the Liouville measure $\mathcal{L}$ is defined on $T^1(\mathbb{H}^2)$ from the Haar measure. It remains to prove that it is flip and flow invariant.

If F is the geodesic flow then for every $t \in \mathbb{R}$ and g in $PSL_2(\mathbb{R})$ we have $\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \cdot V_0 = F_t(V_0)$ and $F_t(g \cdot V_0) = g \cdot F_t(V_0)$. Thus the action of the geodesic flow over $T^1(\mathbb{H}^2)$ correspond to the multiplication on the right by matrix of $SL_2(\mathbb{R})$. The Haar measure is invariant by right multiplication and thus $\mathcal{L}$ is geodesic flow invariant.

Now $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\cdot V_0 = (i, \begin{pmatrix}0\\ -1\end{pmatrix})$ and if $g\cdot V_0 = (z,\vec{v})$ then $g\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\cdot V_0 = (z,-\vec{v})$. As $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\in SL_2(\mathbb{R})$ the Liouville measure on $T^1(\mathbb{H}^2)$ is invariant under geodesic flip.

Note $\mathcal{L}$ the Liouville measure on $T^1(\mathbb{H}^2)$, it is a flip and flow invariant measure.

Liouville currents: Let ρ be a hyperbolic metric on S and note X the hyperbolic surface (S,ρ), it induces a isometric diffeomorphism $\varphi_\rho : \tilde{X}\to\mathbb{H}^2$ which itself define a homeomorphism $\varphi_{\rho,g} : \mathcal{G}(\tilde{X})\to\mathcal{G}(\mathbb{H}^2)$.

According to *Theorem 4.2.3* the Liouville measure $\mathcal{L}$ on $T^1(\mathbb{H}^2)$ descends to a current $\hat{\mathcal{L}}$ on $\mathbb{H}^2$: a Radon measure on $\mathcal{G}(\mathbb{H}^2)$ since $\mathbb{H}^2$ is its own universal cover. The Liouville measure is invariant under the action of $PSL_2(\mathbb{R})$ then the associated current is invariant too. We can pull-back the current $\hat{\mathcal{L}}$ via $\varphi_{\rho,g}$, we obtain a $\pi_1(S)$-invariant Radon measure on $\mathcal{G}(\tilde{S})$ since $\pi_1(S)$ identifies with a subgroup of $PSL_2(\mathbb{R})$. Moreover, if φ_ρ and φ'_ρ are to different isometries for ρ then they differ from a deck transformation which identifies with an element of $PSL_2(\mathbb{R})$ then the corresponding pull-back metrics are the same. We have then construct a current $\hat{\mathcal{L}}_\rho$ over S.

From $\hat{\mathcal{L}}_\rho$ to $\hat{\mathcal{L}}_{\mathfrak{T}}$: if ρ and ρ' are the same element $\mathfrak{T}$ in $\mathcal{T}(S)$ then there is an isometry isotopic to identity $i : (S,\rho) = X \to (S,\rho') = X'$. Note H the isotopy with $H_0 = id$ and $H_1 = i$. We can lift H to $\tilde{H} : [0,1]\times\tilde{S}\to\tilde{S}$. If $\varphi_{\rho'} : \tilde{X}'\to\mathbb{H}^2$ is an isometric diffeomorphism then $\varphi_{\rho'}\circ\tilde{H}_1 : \tilde{X}\to\mathbb{H}^2$ is an isometric diffeomorphism and we can take $\varphi_\rho = \varphi_{\rho'}\circ\tilde{H}_1$. Since $\tilde{H}_1$ is isotopic to identity then $\tilde{H}_{1g} = id$ and ρ and ρ' induce the same map on lifted geodesics: we can talk about $\hat{\mathcal{L}}_{\mathfrak{T}}$ instead of $\hat{\mathcal{L}}_\rho$. □

The compacity of $\mathcal{PC}(S)$ also derives from *Theorem 4.2.3*. With this theorem we can associate to every element of $\mathbb{P}\mathcal{C}(S)$ a unique probability measure in $T^1(S)$. The set of flip and flow invariant probability measures is a closed subset of $Prob(T^1(S))$, the set of all probability measures on $T^1(S)$ so, it suffices prove that $Prob(T^1(S))$ is compact. It is possible to prove that the set of probability measures over a compact set is weak* compact, since S is compact $T^1(S)$ is compact and it follows that $\mathbb{P}\mathcal{C}(S)$ is.

Corollary 4.2.5. $\mathbb{P}\mathcal{C}(S)$ *is compact for the quotient topology.*

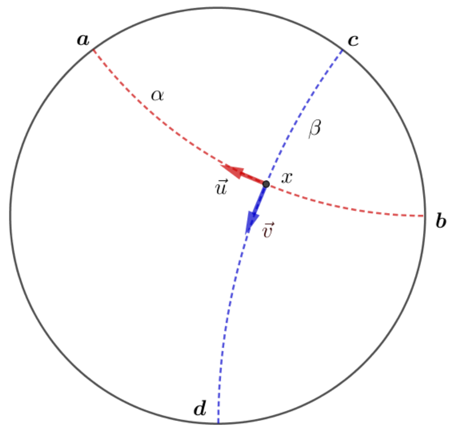
5 Compactification of Teichmüller spaces

In this section we explain how $\mathcal{T}(S)\cup\mathbb{P}\mathcal{ML}(S)$ can be seen as a compactification of $\mathcal{T}(S)$. Everything here will be based on the notion of intersection number between currents. It allows to prove that there is an embedding in 4.2.4 which extends to $\mathbb{P}\mathcal{C}(S)$. It also gives a characterisation of the currents which come from geodesic laminations. It is by bringing together these properties that we will be able to prove that divergent sequences in $\mathcal{T}(S)$ converge to projective measured laminations in $\mathbb{P}\mathcal{C}(S)$.

5.1 Intersection number

In order to construct the intersection number over $\mathcal{C}(S)$ we consider the subset $DG(\tilde{S})$ of $\mathcal{G}(\tilde{S})\times\mathcal{G}(\tilde{S})$ made of the pairs of geodesics which intersect transversely. We have some interpretations of this space.

$$\begin{aligned} DG(S) &= \{\alpha,\beta\in\mathcal{G}(\tilde{S})|\alpha \text{ and } \beta \text{ intersect transversely}\}\\ &= \{\{a,b\},\{c,d\}\in(\mathbb{S}^1)^2|a \text{ and } b \text{ are not in the same components of } \mathbb{S}^1\backslash\{c,d\}\}\\ &= \{(x,[u],[v])|(x,[u]),(x,[v])\in\mathbb{P}T(\tilde{S}),\ [u]\neq[v]\}\end{aligned}$$



The action of $\pi_1(S)$ on $\tilde{S}$ extends to a free and properly discontinuous action on $DG(\tilde{(S)})$ and ${DG(\tilde{S})}/{\pi_1(S)} = DG(S)$. For every pair of currents (μ, ν) in $\mathcal{C}(S)$ the measure $\mu \otimes \nu$ is a $\pi_1(S)$-invariant measure on $DG(\tilde{S})$ and induces a measure, also noted $\mu \otimes \nu$, on $DG(S)$.

Definition 5.1.1. *Let S be a closed surface of genus at least 2, if μ and ν are two currents then their intersection number is*

$$i(\mu, \nu) = \int_{DG(S)} d\mu \otimes d\nu$$

Theorem 5.1.2. *The intersection number is continuous, bilinear and symmetric from $\mathcal{C}(S) \times \mathcal{C}(S)$ to $\mathbb{R}_+$ and corresponds to the classical intersection number between geodesics.*

Proof. i is obviously symmetric and bilinear, we have to prove that it is continuous and finite and that it corresponds with the classical intersection number for geodesics.

i has values in $\mathbb{R}^+$: The only thing we have to prove is that i has finite values. Take μ and ν to currents, note $D\tilde{G(S)}$ a lift of $DG(S)$ in $DG(\tilde{S})$ and Δ a lift of S in $\tilde{S}$.

$$\begin{aligned} i(\mu, \nu) &= \mu \otimes \nu(DG(S)) \\ &= \mu \otimes \nu(D\tilde{G(S)}) \\ &\leq \mu(\{\text{geodesics in } \tilde{S} \text{ through } \Delta\}) \cdot \nu(\{\text{geodesics in } \tilde{S} \text{ through } \Delta\}) \end{aligned}$$

However, S is compact then Δ is compact and μ, ν are Radon measures, so $i(\mu, \nu) < \infty$.

Intersection number between geodesics: Take α and β two geodesics on S, they induce the currents $\hat{\alpha}$ and $\hat{\beta}$ with $i(\hat{\alpha}, \hat{\beta}) = \hat{\alpha} \otimes \hat{\beta}(DG(S)) = \delta_{\alpha \cap \beta}(DG(S)) = |\alpha \cap \beta|$. We find the classical definition of the intersection number between geodesics.

Continuity of i: Intuitively, the continuity of i may come from the continuity of the operator $\otimes$ and the fact that $DG(S)$ embeds in $\mathcal{G}(S) \times \mathcal{G}(S)$. The map $(\mu, \nu) \mapsto \mu \otimes \nu_{|DG(S)}$ is continuous and i is also continuous. One can find a detailed proof in [Bon86] section 4.2. □

Continue with the property of the intersection number when it applies to the Liouville currents. We start with a link between intersection number and length functions which allows us to say that the identification of $\mathcal{T}(S)$ as a subset of $\mathcal{C}(S)$ via Liouville currents is an injection.

Proposition 5.1.3. *Let S be a closed surface of genus at least 2, if $\mathfrak{X} \in \mathcal{T}(S)$ and α a simple closed curve in S then $i(\hat{\mathcal{L}}_{\mathfrak{X}}, \hat{\alpha}) = \ell_{\mathfrak{X}}(\alpha)$.*

Proof. Take α and $\mathfrak{T}$ as in the proposition and I a segment of $\tilde{\gamma}$ which cover γ once. We note $\phi : \tilde{S} \to \mathbb{H}^2$ an isomorphism done by $\mathfrak{T}$.

$$\begin{array}{rcl} i(\hat{\mathcal{L}}_{\mathfrak{T}}, \hat{\alpha}) & = & \hat{\mathcal{L}}_{\mathfrak{T}} \otimes \hat{\alpha}(DG(S)) \\ & = & \hat{\mathcal{L}}_{\mathfrak{T}}(\{\text{Geodesics ini } \tilde{S} \text{ meeting } I\}) \\ & = & \phi_* \hat{\mathcal{L}}_{\mathfrak{T}}(\phi(\{\text{Geodesics ini } \tilde{S} \text{ meeting } I\})) \\ & = & \hat{\mathcal{L}}(\{\text{Geodesics ini } \mathbb{H}^2 \text{ meeting } \phi(I)\}) \\ & = & C.\ell_{\mathbb{H}^2}(\phi(I)) \\ & = & C.\ell_{\mathfrak{T}}(\alpha) \end{array}$$

The constant C does not depends on the element $\mathfrak{T}$ of $\mathcal{T}(S)$ and disappears if we rescale the Liouville measure. □

Proposition 5.1.4. *With the same notations as above* $i(\hat{\mathcal{L}}_{\mathfrak{T}}, \hat{\mathcal{L}}_{\mathfrak{T}}) = \pi^2|\chi(S)|$.

Proof. We have a canonical projection $p : DG(\mathbb{H}^2) \to \mathbb{H}^2$ and we can push-forward $\hat{\mathcal{L}} \otimes \hat{\mathcal{L}}$ to a measure in $\mathbb{H}^2$. Since the Liouville measure is $PSL_2(\mathbb{R})$-invariant the measure we obtained by push-forward is a measure on $\mathbb{H}^2$ invariant by $PSL_2(\mathbb{R})$: it is a multiple of the volume form. However the area of a hyperbolic surface is a multiple of $|\chi(S)|$ then $i(\hat{\mathcal{L}}_{\mathfrak{T}}, \hat{\mathcal{L}}_{\mathfrak{T}})$, which consists in integrating along $\hat{\mathcal{L}}_{\mathfrak{T}} \times \hat{\mathcal{L}}_{\mathfrak{T}}$, is a fixed multiple of the area. One can prove that with the same rescaling of the Liouville measure as in *Proposition 5.1.3* the constant is π^2. □

About the rescaling of the Liouville measure, one can find more details in [Bon88].

Corollary 5.1.5. *If S is a closed surface with $g \geq 2$ then $\mathcal{T}(S) \to \mathcal{C}(S) \to \mathbb{P}\mathcal{C}(S)$ are embeddings.*

Proof. Applying the $9g-9$-theorem there is some simple closed curves $\gamma_1, ..., \gamma_{9g-9}$ in S such that two elements of the Teichmüller space are the same if and only if these $9g-9$ curves have the same length in each of them. Therefore, it follows from *Proposition 5.1.3* that $\mathcal{T}(S) \to \mathcal{C}(S)$ is injective, we don't prove the continuity here.

Now, if there is $t \in \mathbb{R}_+$ and two elements $\mathfrak{T}$ and $\mathfrak{T}'$ in $\mathcal{T}(S)$ such that $\hat{\mathcal{L}}_{\mathfrak{T}} = t\hat{\mathcal{L}}_{\mathfrak{T}'}$ then

$$\pi^2|\chi(S)| = i(\hat{\mathcal{L}}_{\mathfrak{T}}, \hat{\mathcal{L}}_{\mathfrak{T}}) = t^2 i(\hat{\mathcal{L}}_{\mathfrak{T}'}, \hat{\mathcal{L}}_{\mathfrak{T}'}) = t^2\pi^2|\chi(S)|.$$

As a consequence $t = 1$ and $\mathcal{T}(S) \to \mathbb{P}\mathcal{C}(S)$ is injective. □

To concludes this section we give a characterisations of the elements of $\mathcal{C}(S)$ which are measured laminations thanks to the intersection number.

Proposition 5.1.6. *Let μ be a current, there is $\lambda \in \mathcal{ML}(S)$ such that $\mu = \hat{\lambda}$ if and only if $i(\mu, \mu) = 0$.*

Proof. Start with μ a current such that $i(\mu, \mu) = 0$. Its support $Supp(\mu)$ is a closed $\pi_1(S)$-invariant subset of $\mathcal{G}(\tilde{S})$ since a current is a $\pi_1(S)$-invariant measure. If there is in $Supp(\mu)$ two geodesics α, β which intersect transversely then they descend to a pair $(\pi(\alpha), \pi(\beta))$ in $DG(S)$ such that $\mu^2(\pi(\alpha), \pi(\beta))) \neq 0$, its follows that $\mu^2(DG(S)) \neq 0$. As a consequence, the support of μ is the preimage in $\tilde{S}$ of a lamination λ. Moreover, μ naturally defined a transverse measure on λ inspired from the construction on $\hat{\lambda}$, such that $\hat{\lambda} = \mu$.

Reciprocally, if λ is a measured lamination the support of $\hat{\lambda} \otimes \hat{\lambda}$ in $DG(S)$ is among the elements $(\alpha, \beta) \in DG(S)$ where $\alpha \subset \lambda$, $\beta \subset \lambda$ with α and β intersecting transversely: such an element does not exist by definition of a lamination. As a consequence $i(\hat{\lambda}, \hat{\lambda}) = 0$. □

This property immediately proves that $\mathbb{P}\mathcal{ML}(S)$ is a closed subset of $\mathbb{P}\mathcal{C}(S)$, and we also deduce the following corollary.

Corollary 5.1.7. *$\mathcal{T}(S)$ and $\mathbb{P}\mathcal{ML}(S)$ are disjoint subsets of $\mathbb{P}\mathcal{C}(S)$.*

5.2 Effective compactification

Theorem 5.2.1 (Bonahon [Bon88]). *A compactification of $\mathcal{T}(S)$ as a subset of the compact set $\mathbb{P}\mathcal{C}(S)$ is $\mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$.*

Proof. Consider $(\mathfrak{T}_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{T}(S)$ which converges in $\mathbb{P}\mathcal{C}(S)$ to a point $[\mu] \in \mathbb{P}\mathcal{C}(S)\backslash\mathcal{T}(S)$. Going back to $\mathcal{C}(S)$, their exists a sequence of positive real numbers t_n such that the sequence $t_n.\hat{\mathcal{L}}_{\mathfrak{T}_n}$ converges to μ.

Applying *Theorem 2.5.1*, there is a simple closed curve α in S such that $\lim\limits_{n\to\infty} \ell_{\mathfrak{T}_n}(\alpha) = \infty$. But i is continuous so

$$t_n \ell_{\mathfrak{T}_n}(\alpha) = t_n i(\hat{\alpha}, \hat{\mathcal{L}}_{\mathfrak{T}_n}) = i(\hat{\alpha}, t_n \hat{\mathcal{L}}_{\mathfrak{T}_n}) \to i(\hat{\alpha}, \mu) < \infty$$

and $\lim\limits_{n\to\infty} t_n = 0$.

As a consequence,

$$i(\hat{\alpha}, \hat{\alpha}) = \lim_{n\to\infty} i(t_n \hat{\mathcal{L}}_{\mathfrak{T}_n}, t_n \hat{\mathcal{L}}_{\mathfrak{T}_n}) = \lim_{n\to\infty} t_n^2 i(\hat{\mathcal{L}}_{\mathfrak{T}_n}, \hat{\mathcal{L}}_{\mathfrak{T}_n}) = \pi^2 |\chi(S)| \lim_{n\to\infty} t_n^2 = 0.$$

Applying *Proposition 4.1.5*, μ is a measured lamination.

Moreover $\mathbb{P}\mathcal{ML}(S)$ is closed so $\mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$ is closed in a compact: it is compact. □

6 Notations

$\chi(S)$: Euler characteristic of S

$\mathcal{C}(S)$: Geodesic currents on S

$g(S)$: Genus of S

$\mathcal{G}(X)$: Sets of complete unoriented geodesics of X

G_0: Connected component of 1_G

$\mathcal{K}(S)$: Set of closed subsets of S

$\mathcal{L}(S)$: Set of geodesic laminations on S

$\mathcal{M}(S)$: Moduli space of S

$\mathrm{Map}^+(S)$: Mapping class group of S

$\mathcal{ML}(S)$: Set of measured laminations on S

$\pi_1(S)$: Fundamental group of S

$\pi_0(S)$: Connected components of S

$\mathbb{P}T(S)$: Projective tangent bundle of S

$\mathcal{S}(S)$: Free isotopy classes of simple closed curves on S

$\mathcal{T}(S)$: Teichmüller space of S

$T^1(S)$: Unit tangent bundle of S

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