243. Power Series: convergence and properties. Examples and Applications.

* **Motivations**: generalize polynomials, show smoothness, solve ODE, generating function..

I] Definition and first properties.

1) Convergences: the radius of convergence.

def 1: a power series is a formal series $\sum_{n\in\mathbb{N}} a_n z^n$ where $(a_n)_{\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ and the variable $z\in\mathbb{C}$.

def-thm 2: the radius of convergence of a power series $\sum_{n\in\mathbb{N}} a_n z^n$ is defined equivalently by:

- $R = \sup\{r \geq 0 \,/\,$ the sequence $|a_n| r^n$ is bounded$\}$
- $R = \sup\{r\geq 0 \,/\, \sum_{n\in\mathbb{N}} a_n z^n$ converges absolutely for $|z|\leq r\}$
- $R = \sup\{r \geq 0 \,/\, \sum_{n\in\mathbb{N}} a_n z^n$ converges normally on $B(0,r)\}$
- $R = \inf\{r\geq 0 \,/\, \forall\, |z|>r,\ \sum_{n\in\mathbb{N}} a_n z^n$ diverges$\}$

ex 3: polynomials are power series with radius $R=+\infty$.
$\sum_{n\in\mathbb{N}} z^n$ has radius 1 ; $\sum_{n\in\mathbb{N}} n!\, z^n$ has radius 0.

prop 4: (d'Alembert, Cauchy-Hadamard)

- ratio test: if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lambda$, the radius of $\sum_{n\in\mathbb{N}} a_n z^n$ is $\frac{1}{\lambda}$.
- n^{th}-root test: if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lambda$, the radius of $\sum_{n\in\mathbb{N}} a_n z^n$ is $\frac{1}{\lambda}$.

ex 5: $\sum_{n\in\mathbb{N}} \frac{z^n}{n!}$ has radius $+\infty$. We define $\exp(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$

$\forall \alpha\in\mathbb{R}$, $\sum_{n\in\mathbb{N}} n^\alpha z^n$ has radius 1.

⚠ The sequence should not satisfy [$u_n = 0$ infinitely often]
→ we cannot compute the radius of $\sum_{n\in\mathbb{N}} z^{n^2}$ with these tests.

Rq 6: If the series $\sum_{n\in\mathbb{N}} a_n z^n$ satisfies the ratio-test, then so does it for the n-th root test. The converse is false.

ex 7: $u_n = \begin{cases} 1/3^n & \text{if } n \text{ is even} \\ 4/3^n & \text{if } n \text{ is odd}\end{cases}$ satisfies $\sqrt[n]{u_n} \xrightarrow[n\to\infty]{} 1$ but $\frac{u_{n+1}}{u_n} = \begin{cases} 4/3 & \text{(even)} \\ 1/12 & \text{(odd)}\end{cases}$

Rq 8: To compute usual power series, we use the integral Taylor-Lagrange formula (cf Annex: usual power series).

2) Algebraic operations on power series.

prop 9: Let $\sum_{n\in\mathbb{N}} a_n z^n$ and $\sum_{n\in\mathbb{N}} b_n z^n$ two power series with radius $R, R' > 0$ and sum f, g. Then:

- $\sum_{n\in\mathbb{N}} (a_n + b_n) z^n$ is a power series with radius $R'' \geq \min(R, R')$ and sum $f+g$.
- $\sum_{n\in\mathbb{N}} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n$ is a power series with radius $R'' \geq \min(R,R')$ and sum $f\times g$
- For any $\alpha\in\mathbb{C}^*$, $\sum_{n\in\mathbb{N}} (\alpha a_n) z^n$ is a power series with radius R and sum $\alpha \times f$.
- If $g(0)=0$, $f\circ g$ has a power series expansion with radius $R''>0$.
- If $f(0)\neq 0$, $\frac{1}{f}$ has a power series expansion with radius $R''>0$.

ctrex 10: $\sum_{n\in\mathbb{N}} z^n$ and $\sum_{n\in\mathbb{N}} (-1)^n z^n$ has radius 1, but their sum satisfies $R'' = +\infty$.

ctrex 11: Define $u_n = \begin{cases} 2 & \text{if } n=0 \\ 2^n & \text{if } n\geq 1\end{cases}$, $v_n = \begin{cases} -1 & \text{if } n=0 \\ 1 & \text{if } n\geq 1\end{cases}$
$\sum_{\mathbb{N}} u_n z^n$ has radius $\frac{1}{2}$; $\sum_{\mathbb{N}} v_n z^n$ has radius 1 but the product has infinite radius.

3) Tauberian theorem

Th 12 Let $f(z) = \sum_{n\geq 0} a_n z^n$, with a radius 1 such that $\sum_{n\geq 0} a_n$ converges. Let $\theta_0 \in [0, \pi/2[$, we set $\Delta_{\theta_0} = \{z\in\mathbb{C} : |z|<1\} \cap \{1-\rho e^{i\theta} ; \rho>0, \theta\in[-\theta_0; \theta_0]\}$.
Then $S(z) \xrightarrow[\substack{z\to 1 \\ z\in\Delta_{\theta_0}}]{} \sum_{n=0}^{\infty} a_n$.

Cor 13 $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \pi/4$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln(2)$

Th 14 Let $f(z) = \sum_{n\geq 0} a_n z^n$, with a radius 1, and we assume there exists $S\in\mathbb{C}$ s.t. $f(x) \xrightarrow[x\to 1^-]{} S$. If $a_n = o(1/n)$, then $\sum_{n\geq 0} a_n$ converges and $S = \sum_{n=0}^{\infty} a_n$.

II Smoothness of the sum and the holomorphic point of view

1) Continuity, derivative and antiderivative

Theoreme 15: The function $z \mapsto \sum_{n=0}^{+\infty} a_n z^n$ is continuous on the convergence disc $\{z \in \mathbb{C}, |z| < R\}$

Example 16: $z \mapsto \exp(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$ is continuous on $\mathbb{C}$ because $R = +\infty$

Thrm 17: The function $f :]-R;R[\to \mathbb{C}$, $x \mapsto \sum_{n=0}^{+\infty} a_n x^n$ is continuously differentiable

$\sum_{n=1}^{+\infty} n a_n x^{n-1}$ has the same radius than $\sum_{n=0}^{+\infty} a_n x^n$

and $f'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$

Example 18: $\frac{1}{(1-x)^2} = \sum_{n=0}^{+\infty} (n+1) x^n \quad \forall x \in]-1;1[$

Corollary 19: The same f is, in fact, infinitely differentiable and all the derivatives have the same radius.

Also $\forall p \in \mathbb{N}$ $a_p = \frac{f^{(p)}(0)}{p!}$ then $\forall z \in \mathbb{C}$ $f(z) = \sum_{p=0}^{+\infty} \frac{f^{(p)}(0)}{p!} z^p$

Thrm 20 The function $F :]-R;R[\to \mathbb{C}$, $x \mapsto \sum_{n=0}^{+\infty} \frac{a_n}{n+1} x^{n+1}$ is a primitive of $f :]-R;R[\to \mathbb{C}$, $x \mapsto \sum_{n=0}^{+\infty} a_n x^n$ and F has the same radius R than f.

Example 21 $-\ln(1-x) = \sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1} \quad \forall x \in]-1;1[$

2) Real analytic function.

Def 22 Let I be an open interval containing $a \in \mathbb{R}$ and $f : I \to \mathbb{R}$ C^∞. The power series around the point a given by

$$\sum_{k=0}^{+\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$ is called the Taylor series of f around a.

Rmk 23 The Taylor series of f around a converges to $f(a)$ but it may not converge to $f(x)$ for other point $x \in I$ and may be divergent.

Example 24 • $f(x) = |x|$ around 1 gives $1 + (x-1)$ not equal to $f(x)$ when $x < 0$

• $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{si } x \neq 0 \\ 0 & \text{si } x = 0 \end{cases}$ $f^{(k)}(0) = 0 \ \forall k \in \mathbb{N}$ but $f \not\equiv 0$

Def 25 If the Taylor series of f around each point of I converges to $f(x)$ for every $x \in I$ then f is said to be a real analytic function on I.

Example 26 • $f(x) = \exp(x)$ is a real analytic function on $\mathbb{R}$

• $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{si } x \neq 0 \\ 0 & \text{si } x = 0 \end{cases}$ is not.

3) Links with the holomorphic functions

Thrm 27: $f : D(0,R) \to \mathbb{C}$, $z \mapsto \sum_{n=0}^{+\infty} a_n z^n$ is holomorphic on $D(0,R)$

and analytic on $D(0,R)$

$a \in D(0,R)$, $|z-a| < R - |a| \Rightarrow z \in \Omega$ and $f(z) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(a)}{k!} (z-a)^k$

Thrm 28 Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ be a power series. If there exists $(z_p)_{p \in \mathbb{N}} \in (\mathbb{C}^*)^{\mathbb{N}}$ such that $z_p \xrightarrow[p \to +\infty]{} 0$ and $f(z_p) = 0$ for all p in $\mathbb{N}$.

Then $a_n = 0$ for all n in $\mathbb{N}$.

Corollary 29 Two power series that are equal on a neighbourhood of zero have the same coefficients.

Thrm 30 Let $f(z) = \sum a_n z^n$ be a power series of radius $R > 0$

Then $\forall r \in]0;R[$, $\forall n \in \mathbb{N}$ $2\pi r^n a_n = \int_0^{2\pi} f(re^{i\theta}) e^{-ni\theta} \, d\theta$

III) Applications.

1) Ordinary differential equations.

App31: The function $f : x \in]-1,1[\mapsto (\mathrm{Arcsin}\, x)^2$ has a power series expansion with positive radius. It satisfies the following equation:

$$(1-x^2)\, f'(x)^2 = 4 f(x)$$

Then for any $x \in]-1,1[$, $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ with:

$$a_{2n+1} = 0 \quad , \quad a_{2n} = \frac{2^{2n-2}\left((n-1)!\right)^2}{n\,(2n-1)!}$$

Similarly, for any $x \in]0,1[$, $\dfrac{\mathrm{Arcsin}\sqrt{x}}{\sqrt{x(1-x)}} = \sum_{n=0}^{+\infty} \dfrac{2^{2n}(n!)^2}{(2n+1)!} x^n$

1st DEV

App32: (Bessel function). Let $J_0 : x \in \mathbb{R} \mapsto \frac{1}{\pi}\int_0^{\pi} \cos(x \sin\theta)\, d\theta$. Then:

(i) $J_0(x) = \sum_{n=0}^{+\infty} \dfrac{(-1)^n}{4^n (n!)^2} x^{2n}$

(ii) J_0 is solution of the equation (B) $xy'' + y' + xy = 0$ on $\mathbb{R}$.

(iii) The solutions of (B) on $I_\alpha =]0,\alpha[$ where $\alpha > 1$ and $J_0 \neq 0$ on I satisfy: $\exists C, D \in \mathbb{R} \,/\, \forall x \in I_\alpha,\ y(x) = C J_0(x) \int_1^x \dfrac{dt}{t (J_0(t))^2} + D J_0(x)$

(iv) $\int_1^x \dfrac{dt}{t (J_0(t))^2} = \ln x + \dfrac{x^2}{4} + o(x^2)$

thm33: The closure of the sets of polynomials on $\overline{B_{\mathbb{C}}(0,1)}$ with respect to uniform convergence is the set of continuous functions f on $\overline{B_{\mathbb{C}}(0,1)}$ having a power series expansion on $B_{\mathbb{C}}(0,1)$.

2) Combinatorics.

App34: let $n \in \mathbb{N}$ and D_n be the number of fixed-point-free permutations of $[\![1,n]\!]$. Then: $D_n = \left\lfloor \dfrac{n!}{e} + \dfrac{1}{2} \right\rfloor$

App35: The number of valid parentheses expressions of length $n \in \mathbb{N}$ is $C_n = \dfrac{1}{n}\dbinom{2n-2}{n-1}$

App36: (Bell numbers) Let $n \in \mathbb{N}$ and B_n the number of partitions of $[\![1,n]\!]$. Then: $B_n = \dfrac{1}{e}\sum_{k=0}^{+\infty} \dfrac{k^n}{k!}$

2nd DEV

3) Power Series in Probability.

Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X(\Omega) \subset \mathbb{N}$.

Def37: The generating function of X is the power series: $G_X(z) = \sum_{n=0}^{\infty} \mathbb{P}(X=n) z^n = \mathbb{E}(z^X)$, with radius ≥ 1.

Prop38: G_X characterises the law of X: $\mathbb{P}_X$.

Prop39: Let $X_1, \dots, X_n$ be independent random variables. Then $G_{X_1 + \dots + X_n} = \prod_{k=1}^{n} G_{X_k}$

Ex40.
- $X \sim \mathcal{B}(1,p)$: $G_X(z) = 1 - p + pz$.
- $X \sim \mathcal{B}(n,p)$: $G_X(z) = (1-p+pz)^n$.
- $X \sim \mathcal{P}(\lambda)$: $G_X(z) = e^{\lambda(z-1)}$

Prop41: Let (X,Y) be a random variable with law $\mathcal{P}(\lambda) \otimes \mathcal{P}(\mu)$. Then $X + Y \sim \mathcal{P}(\lambda + \mu)$.

Thm42: $\mathbb{E}(|X|) < \infty$ if and only if G_X has a left-hand derivative at 1^-. In this case: $\mathbb{E}(X) = G_X'(1^-)$.

Thm43: Let $(X_n)_{n\geq 0}$ be iid random variables and N be a random variable independent of $(X_n)_{n\geq 0}$. Let S_N be: $S_N := \sum_{k=0}^{N} X_k$. Then we have: $G_{S_N}(z) = G_N \circ G_{X_1}(z)$.

Prop44: Let $(X_n)_n$ be iid with $X_n \sim \mathcal{B}(1,p)$ and $N \sim \mathcal{P}(\lambda)$ independent of $(X_n)_{n\geq 0}$. Then $S_N = \sum_{k=1}^{N} X_k \sim \mathcal{P}(\lambda p)$.

Thm45: Let $\mu_k := \mathbb{E}(|X|^k)$. If $\limsup_{k\to\infty} \dfrac{(\mu_k)^{1/k}}{k} < \infty$, then, the law $\mathbb{P}_X$ is characterized by its moments: $(\mathbb{E}(X^k))_{k\geq 1}$

Ex46: $\mathcal{N}(0,1)$ is characterized by its moments.

Appendix: Usual power series

$\forall x \in \mathbb{R} \quad e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots = \sum_{k=0}^{+\infty} \frac{x^k}{k!}$

$\forall x \in \mathbb{R} \quad \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^p \frac{x^{2p+1}}{(2p+1)!} + \dots = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

$\forall x \in \mathbb{R} \quad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^p \frac{x^{2p}}{(2p)!} + \dots = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k}}{(2k)!}$

$\forall x \in \mathbb{R} \quad sh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2p+1}}{(2p+1)!} + \dots = \sum_{k=0}^{+\infty} \frac{x^{2k+1}}{(2k+1)!}$

$\forall x \in \mathbb{R} \quad ch(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2p}}{(2p)!} + \dots = \sum_{k=0}^{+\infty} \frac{x^{2k}}{(2k)!}$

$\forall x \in]-1;1[\quad (1+x)^\alpha = 1 + \frac{\alpha}{1!} x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots + \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!} x^n + \dots$

In particular:

$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots = \sum_{k=0}^{+\infty} x^k$

$\frac{1}{1+x} = 1 - x + x^2 + \dots + (-1)^n x^n + \dots = \sum_{k=0}^{+\infty} (-1)^k x^k$

$\sqrt{1+x} = 1 - \frac{1}{2} x + \frac{1 \times 3}{2 \times 4} x^2 + \dots + (-1)^n \frac{1 \times 3 \times \dots \times (2n-1)}{2 \times 4 \times \dots \times (2n)} x^n + \dots$

$\forall x \in]-1;1[\quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^{n-1} \frac{x^n}{n} + \dots = \sum_{k=1}^{+\infty} (-1)^{k-1} \frac{x^k}{k}$

$\forall x \in \mathbb{R} \quad \arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$

$\forall x \in]-1;1[\quad argth(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2n+1}}{2n+1} + \dots = \sum_{k=0}^{+\infty} \frac{x^{2k+1}}{2k+1}$

$\forall x \in]-1;1[\quad \arcsin(x) = x + \frac{1}{2 \times 3} x^3 + \dots + \frac{1 \times 3 \times \dots \times (2n-1)}{2 \times 4 \times \dots (2n)(2n+1)} x^{2n+1} + \dots$

$\forall x \in]-1;1[\quad argsh(x) = x - \frac{1}{2 \times 3} x^3 + \dots + (-1)^n \frac{1 \times 3 \times \dots \times (2n-1)}{2 \times 4 \times \dots \times (2n)(2n+1)} x^{2n+1} + \dots$