COLLEGE DOCTORAL BRETAGNE — MATHS, TELECOMS, INFORMATIQUE, SIGNAL, SYSTEMES, ELECTRONIQUE

Université de Rennes

THÈSE DE DOCTORAT DE

L'UNIVERSITÉ DE RENNES

ÉCOLE DOCTORALE Nº 601
Mathématiques, Télécommunications, Informatique, Signal, Systèmes, Électronique
Spécialité : *Mathématiques*

Par
Alice BOUILLET

Modules de groupes finis plats en caractéristique $p > 0$

Thèse présentée et soutenue à Rennes, le 22 juin 2023
Unité de recherche : IRMAR (UMR CNRS 6625)

Rapporteurs avant soutenance :

Cyril DEMARCHE	Maître de conférences, Sorbonne Université
Paul SOBAJE	Assistant Professor, Georgia Southern University

Composition du Jury :

Présidente :	Ariane MEZARD	Professeure des universités, Sorbonne Université
Examinateurs :	Cyril DEMARCHE	Maître de conférences, Sorbonne Université
	Philippe GILLE	Directeur de recherches, Université Claude Bernard Lyon 1
	Cédric PEPIN	Maître de conférences, Sorbonne Université Paris Nord
	Paul SOBAJE	Assistant Professor, Georgia Southern University
Dir. de thèse :	Matthieu ROMAGNY	Professeur des universités, Université Rennes 1

REMERCIEMENTS

J'aimerais commencer ce manuscrit en exprimant ma gratitude à celles et ceux qui ont participé, de près ou de loin, à l'écriture de cette thèse.

C'est naturellement que je commence par remercier mon super directeur de thèse Matthieu. Je te suis vraiment reconnaissante d'avoir accepté d'encadrer ma thèse, et je me sens grandement chanceuse d'avoir eu un maître de thèse comme toi. Merci pour tes qualités de mathématicien et toutes tes qualités humaines. En particulier, merci d'avoir si bien mis en pratique ta phrase, connue et appréciée de tous, "la répétition est à la base de toute pédagogie". J'ai eu la chance de me sentir à l'aise pour poser mes questions grâce à ta patience à toute épreuve et ton immense disponibilité. Merci pour tes relectures rigoureuses, tes précieux conseils, tes nombreuses idées et ton écoute. Tout comme ton ancien étudiant, je souhaite à tout le monde d'avoir un directeur de thèse comme toi.

Je tiens à remercier Cyril Demarche et Paul Sobaje de s'être intéressés à mon travail et d'avoir accepté de rapporter ma thèse. En particulier, merci beaucoup Cyril pour tes remarques pertinentes. Je remercie grandement Philippe Gille, Ariane Mézard et Cédric Pépin d'avoir accepté de faire partie de mon jury.

Durant ces trois années de thèse, j'ai eu la chance de pouvoir exposer mes travaux à de nombreuses reprises, et également en discuter avec plusieurs personnes. Ainsi je tiens à remercier Salim Rostam, Ronan Terpereau, Andrea Fanelli, Philippe Gille, Florence Fauquant-Millet, Julia Schneider, Xavier Roulleau, Marc Chardin, Xavier Caruso et Dajano Tossici.

J'aimerais à présent remercier la personne qui m'a fait aimer les mathématiques dès le collège : merci infiniment Dorina de m'avoir transmis ta vocation. Merci pour tes superbes cours, ton énergie, ton enthousiasme débordant et ta pédagogie. Merci d'avoir continué à alimenter ma passion pendant le lycée. Je remercie ensuite chaleureusement Mihai pour avoir pris le relais. Merci de m'avoir suivie, conseillée et écoutée durant mes études supérieures. Je vous remercie tous les deux pour avoir toujours cru en moi et pour m'avoir poussée vers le haut.

J'aimerais ensuite remercier tous les professeurs qui ont marqué ma scolarité, et qui ont confirmé mon goût pour l'algèbre et la géométrie. Tout d'abord merci à Bernard Le Stum, Dominique Cerveau, Elisa Lorenzo Garcia et Tobias Schmidt pour leur cours et/ou leur aide durant ma thèse. Merci en particulier à David pour m'avoir encadrée durant mon tout premier stage. De manière plus générale, merci pour tes encouragements tout le long de ma scolarité, et merci pour ta bienveillance (malgré quelques dérapages à cause de la mauvaise influence de Goulwen). Merci Delphine pour ta superbe pédagogie, ta disponibilité et tes aides. Merci Benoit pour m'avoir initiée aux courbes elliptiques lors du TER. Malgré tout, merci à Goulwen et Eric pour leurs (très bons) cours et leur bienveillance à mon égard qui a duré environ 10 minutes.

Je tiens à remercier sincèrement toutes les personnes du laboratoire avec qui j'ai eu le plaisir d'échanger quotidiennement. Plus particulièrement merci à Marie-Aude, Hélène, Aude, Florian, Caroline, Adeline et Eric (même si son absence à ma soutenance est impardonnable) pour leur grande efficacité. Merci à Dominique et Maryse de continuer à m'accepter à la bibliothèque malgré mes retards systématiques. Merci à Marie-Françoise et Nicoletta pour la confiance que vous m'avez accordée. Merci à Miguel, Léo (même si son absence à ma soutenance est impardonnable) et Paul pour les discussions intéressantes et bien sûr les gossips. Merci beaucoup pour votre écoute et vos précieux conseils. Merci à Vincent pour les enseignements partagés, et pour m'avoir presque permise de faire un cours magistral. Merci aux footeuses et aux footeux qui m'ont tolérée le vendredi soir malgré mes piètres performances. Un très grand merci également à toutes les personnes (nouvelles et anciennes) de la commission parité pour votre accueil, votre compréhension et pour nos échanges fortement enrichissants.

Cette thèse ayant été riche en rencontres, j'aimerais remercier toutes celles et ceux que j'ai connus grâce aux conférences ou autres évènements mathématiques. Je pense bien sûr à mon chouchou Andrès, à Marion (merci pour tes nombreux conseils et tes encouragements : je me sens forte après nos discussions !), Lucie : hâte que tu m'apprennes de nouvelles chansons, Yann, Neige et Bianca. Merci à l'incroyable team angevine d'avoir égayé mes passages dans leur laboratoire : je pense en particulier à Antoine et au manuel-et-musclé Jean-Baptiste.

Merci aux doctorantes et doctorants des années supérieures pour leurs précieux conseils. Je pense notamment à Fabien, Mercedes, Mégane, Ronan, Antoine et Titouan.

Je tiens à remercier toutes les doctorantes et tous les doctorants de l'IRMAR avec qui j'ai partagé cette expérience. De manière plus spécifique, merci à mes mâles cuites préférées Marie et Emeline pour nos discussions et réflexions très enrichissantes, et merci à Matilde d'être la meilleure co-bureau et (demi-)sœur de thèse possible ! Je vous souhaite plein de réussite dans le futur. Merci à Sergio d'égayer mes journées avec ses musiques, à Rémi, Milan, Virgile et Mattia pour la bonne ambiance de notre super bureau. Merci à MAAAAAAARC de rester ami avec moi malgré les occurrences beaucoup trop nombreuses de cette blague. Merci à François, Rémi, Thomas et Pierre du bureau 232 pour les parties de SuperTuxKart. Merci aussi pour les parties de Cémantix. Merci pour celles de Pedantix. Bien sûr, merci aussi pour leurs homologues anglaises Cemantle et Pedantle. Ah oui merci aussi pour les parties de Flaggle, Wordle ou Numberle. Bon effectivement les apparences sont trompeuses mais on a quand même travaillé aussi durant ces trois années ! Merci aussi pour les parties de badminton partagées, même si je dois bien l'écrire : François, tu es le plus fort au badminton. Je l'acte sur ces remerciements (crois moi bien que ça me brise le cœur).

Je tiens également à remercier les ami.e.s que j'ai eu la chance de rencontrer durant mes études. Je pense notamment à Manuela et Kévin pour nos sessions travail tardives, Erwan, Enora, Maïwenn, Lucien et Olivier. Tout particulièrement, merci à Valentine pour les fructueuses sessions révisions, notamment l'année de l'agrégation. Merci pour ton accueil, et pour les superbes moments partagés ensemble. Merci à Aurore de toujours croire en moi, je te souhaite plein de réussite dans la suite de ta belle carrière de mathématicienne. Merci à toute la team de "Parmi nous" avec qui j'ai partagé une partie de ma scolarité : je pense en particulier à Pierre, Mériadec, Enguechou (aussi connu sous le nom de "Enguechouette"), Benjamin pour nos tendres restaurants en tête-à-tête (j'y arriverai, un jour !) et mon cher Grégouze.

J'embrasse également tous mes ami.e.s qui ont réussi à me supporter durant cette thèse malgré mes petits moments de stress : merci à ma princesse Elisa pour tous ces fous rires. Je te souhaite tout le bonheur que tu mérites. Merci à Elphège, Marie, Lisa et Romane pour toutes nos rigolades et pour les soirées inoubliables. Merci à Elora d'être la meilleure marraine pour Rima possible, et l'élève qui m'a demandé le moins de travail ! Merci pour toutes nos discussions enrichissantes et pour ta tendresse. Merci à Eli et Michaël pour les bons restaurants partagés. Merci à ma rideuse professionnelle préférée Lise. En particulier merci pour ta force, et pour tout ce que tu m'apportes. Merci à Nicolas et Théo pour tous les entraînements et les tournois de badminton partagés. J'ai trop hâte d'avoir un niveau de professionnelle reconnue (j'estime ça à dans 1 ou 2 ans max) ! De plus merci beaucoup, Théo de m'avoir, appris à mettre, les virgules aux bons endroits. Merci à mes voisins Ségolène et Florentin pour toutes vos petites attentions, et surtout merci d'avoir attendu que l'on parte de Rennes pour déménager ! Merci Youna et Toinette pour les sessions jeux partagées. Je te suis très reconnaissante Youna de rester copine avec moi malgré mes indénombrables trahisons au Time bomb. François : merci d'être un si bon ami depuis la L2. Merci pour ton écoute précieuse, tes conseils et tes encouragements durant ma scolarité et tout particulièrement durant cette thèse. Merci de me faire autant rire (jusque dans mon sommeil apparemment). Merci de m'avoir appris l'existence des Cadillac ou encore de l'agrégation (exercice laissé au lecteur : deviner lequel m'a été le plus utile). Je ne suis pas peu fière de t'annoncer que j'ai déjà trouvé le titre de notre article commun : "La classification des fonctions régulues en caractéristique p, la troisième va vous étonner. Et la quatrième aussi, et même la millième, et même... bon allez j'arrête !". Merci à Lucie de faire partie de ma vie depuis si longtemps. Merci pour ta bienveillance, ton intelligence et ton amour. Je suis si fière de me construire chaque jour avec toi.

Merci à toute ma famille pour tout ce que vous m'apportez. Merci en particulier à Matata et Chloé pour toute votre affection, ainsi qu'à ma très belle sœur Anne-Sophie et à mon très beau frère Arnaud. Je tiens également à remercier Rima d'avoir toujours veillé à ce que je me lève très tôt pour être sûre d'arriver au bout de cette thèse. Je te remercie Madi de prendre le relais les week-ends où je suis chez toi. Maman et Papa : merci d'avoir toujours cru en moi, et de m'avoir laissée choisir ma voie. Merci de me soutenir dans mes envies et mes goûts. Merci maman pour ta précieuse aide et tes relectures. Merci frérot pour toutes nos discussions enrichissantes. Merci Marie d'être la meilleure sœur que l'on puisse avoir. Merci pour ta confiance et ton amour. Merci de m'avoir mise sous ton aile et de continuer à m'y laisser une place.

Merci Silvère pour tout ce que tu m'apportes. Merci pour ton amour, ton écoute, ton soutien, ta douceur et ta bienveillance. Je suis très heureuse d'avoir fait cette thèse à tes côtés et je t'embrasse de tout mon cœur.

Table of Contents

Introduction

1 Contexte historique et motivations

Dans cette thèse, nous nous intéressons aux schémas en groupes finis localement libres, sur des bases quelconques. Ainsi notons S un schéma de base. Rappelons qu'un schéma en groupes $f : G \to S$ est dit *fini* si f est affine, et si pour tout ouvert affine $\mathrm{Spec}(R) \subset S$, avec image inverse $\mathrm{Spec}(A) \subset G$, le morphisme d'anneaux associé $R \to A$ fait de A un R-module fini. Les schémas en groupes finis localement libres ou, de manière équivalente, finis plats et localement de présentation finie sont des objets très intéressants en eux-mêmes. Par exemple, ceux qui sont commutatifs forment un cadre naturel pour la dualité de Cartier, dualité involutive, qui a été introduite par Pierre Cartier dans [Cart62]. De plus, ils donnent lieu à des questions sans réponse à ce jour. Par exemple, comme les schémas en groupes finis sont une généralisation des groupes abstraits finis (en effet à tout groupe fini Γ on peut associer un schéma en groupes fini G sur n'importe quel anneau de base R,

$$G = \coprod_{\gamma \in \Gamma} \mathrm{Spec}(R)$$

en définissant une multiplication provenant de celle de Γ), il est alors naturel de se demander si le théorème de Lagrange y est toujours vérifié. Pierre Deligne a démontré qu'en effet un schéma en groupes fini localement libre de rang n était annulé par la puissance n-ième, dans le cas commutatif (il a utilisé la dualité de Cartier), et sur n'importe quelle base. Une preuve est faite dans [TO70]. Cependant nous ne savons toujours pas si le résultat est vrai pour un groupe non commutatif, sur une base non réduite (le cas de la base réduite est démontrée dans [SGA3], exposé VII, Proposition 8.5). Sur des corps de base, de nombreux résultats sur les groupes finis sont connus. Par exemple en caractéristique nulle, Pierre Cartier a montré qu'ils étaient tous étales. En caractéristique $p > 0$, nous avons des résultats partiels. Pour en citer quelques-uns, John Tate et Frans Oort ont classifié les groupes finis d'ordre p sous de faibles hypothèses dans [TO70]. De plus, des résultats sur les corps parfaits sont connus : par exemple tout groupe fini abélien commutatif défini sur un corps parfait se décompose de manière unique en produit

$$G = G_{(e,e)} \times G_{(e,c)} \times G_{(c,e)} \times G_{(c,c)}$$

où $G_{(e,e)}$ est de type étale avec dual de Cartier étale, $G_{(e,c)}$ est de type étale avec dual de Cartier connexe, $G_{(c,e)}$ est connexe avec dual de Cartier étale, et $G_{(c,c)}$ est connexe avec dual de Cartier connexe (voir [W79], section 6.8). La théorie de Galois permet de comprendre les deux premiers types, et après dualisation de Cartier, elle permet également de comprendre le troisième type. La théorie des modules de Dieudonné permet de classifier la partie "connexe-connexe". Nous pouvons également citer A.J. de Jong ([D93]) qui a caractérisé certains schémas en groupes finis localement libres commutatifs de caractéristique $p > 0$ sur des bases plus générales.

De plus, ces objets apparaissent naturellement en géométrie algébrique, et ce, de différentes manières. Par exemple soit A une variété abélienne de dimension g définie sur un corps k algébriquement clos. Pour tout entier naturel n, l'ensemble des points de n-torsion

$$A(k)[n] = \ker(n : A \to A)$$

est un groupe fini (par exemple si $\mathrm{car}(k) \nmid n$, on sait que $A(k)[n] = (\mathbb{Z}/n\mathbb{Z})^{2g}$). De plus tout groupe fini localement libre est lié à une variété abélienne de la façon suivante : pour tout schéma de base S et tout schéma en groupes commutatif $G \to S$ fini localement libre, alors localement pour la topologie de Zariski, il existe un schéma abélien projectif A et une S-immersion fermée $G \hookrightarrow A$ (voir [BBM82], Théorème 3.1.1).

Les schémas en groupes finis localement libres sont également très utiles pour étudier les schémas en groupes lisses connexes en général. D'intérêt particulier parmi ceux-ci sont les schémas en groupes lisses connexes sur un corps de base, ce qui inclut les groupes réductifs, les groupes unipotents, les variétés abéliennes, et les extensions de tels objets. Le théorème de Chevalley et le théorème de Rosenlicht donnent des résultats de structure en termes de ces classes particulières (voir [Br13], sous-parties 1.1 et 1.2). On rencontre des difficultés lorsqu'on étudie ces schémas en groupes, comme par exemple :

- manque d'une classification générale, sur un corps algébriquement clos
- échec de la représentabilité des groupes d'automorphismes, dû essentiellement aux groupes unipotents
- échec de la finitude de la cohomologie (cohomologie des G-modules cohérents)
- échec de l'algébricité du champ classifiant (liée à ce qui précède).

Pour s'attaquer à ces difficultés, en caractéristique p on dispose d'un atout : le morphisme de Frobenius, qui à son tour fournit la famille $\{G_r\}$ des noyaux de Frobenius, qui sont des schémas en groupes finis localement libres. Par exemple on associe à $\mathbb{G}_a$ la collection $\{\alpha_{p^n}, n \in \mathbb{N}^*\}$, et on associe à $\mathbb{G}_m$ la collection $\{\mu_{p^n}, n \in \mathbb{N}^*\}$. Il s'avère que la famille $\{G_r\}$ est reliée de manière très étroite au groupe G initial, et ce, par plusieurs manières comme par exemple :

- en géométrie : la collection $\{G_r\}$ permet de déterminer le revêtement universel de G, voir [Su78] ;
- en théorie des représentations : les restrictions de certains G-modules simples fournissent des G_r-modules simples, et tous les G_r-modules simples apparaissent de cette manière,

voir [Ja03], § II.3 ;

- en cohomologie : la cohomologie d'un G-module est la limite inverse des cohomologies des G_r-modules restreints associés, voir [FP87].

Voici ainsi nos motivations principales pour étudier ce type de groupes.

Ces objets ont beaucoup été étudiés sur des corps algébriquement clos, ou sur des corps plus généraux pour des problèmes arithmétiques. Cependant, l'étude de ces objets *en famille* est quelque chose de moins documenté dans la littérature. Voici pourquoi dans cette thèse, nous proposons d'étudier ces objets avec le point de vue des *espaces de modules*. Un espace de modules est un schéma (ou plus généralement un espace algébrique) qui classifie et mieux, qui paramétrise des classes d'équivalences de certains objets géométriques. Plus précisément, nous serons à la recherche de schémas ou d'espaces algébriques dont les points sont les objets qui nous intéressent. Le j-invariant qui classifie les courbes elliptiques ou encore la variété Jacobienne d'une courbe algébrique C qui paramétrise les diviseurs de degré 0 sur C en sont des exemples typiques. Ainsi dans la suite nous posons S un schéma de base de caractéristique $p > 0$, et nous allons étudier ces foncteurs :

$$\left\{\begin{array}{l}\text{Schémas en groupes}\\ \text{lisses connexes sur } S\\ \text{de dimension } n\end{array}\right\} \xrightarrow{\ker(\mathrm{Frob})} \left\{\begin{array}{l}\text{Schémas en groupes}\\ \text{finis loc. libres sur } S\\ \text{d'ordre } p^n\text{, de ht. 1}\end{array}\right\}$$

$$\updownarrow \text{Lie}$$

$$\left\{\begin{array}{l}\mathcal{O}_S\text{-}p\text{-algèbres de Lie}\\ \text{finies loc. libres}\\ \text{sur } S \text{ de rang } n\end{array}\right\}$$

ainsi que celui-ci :

$$\left\{\begin{array}{l}\mathcal{O}_S\text{-}p\text{-algèbres de Lie}\\ \text{finies loc. libres}\\ \text{sur } S \text{ de rang } n\end{array}\right\} \xrightarrow{\text{oubli}} \left\{\begin{array}{l}\mathcal{O}_S\text{-algèbres de Lie}\\ \text{finies loc. libres}\\ \text{sur } S \text{ de rang } n\end{array}\right\}.$$

On note GLC le champ ci-dessus qui associe à un schéma S la catégorie des schémas en groupes lisses connexes sur S, on note $\mathcal{L}ie_n$ le champ des algèbres de Lie finies localement libres de rang n, et p-$\mathcal{L}ie_n$ celui des p-algèbres de Lie de ce type. On note également $\mathcal{G}_n^r$ le champ des schémas en groupes finis d'ordre p^n, localement libres de hauteur r. Notons que GLC n'est pas algébrique alors que p-$\mathcal{L}ie_n$, $\mathcal{L}ie_n$ et $\mathcal{G}_n^r$ le sont. Voici quelques questions naturelles :

1) Quelle est la structure algébro-géométrique du champ $\mathcal{G}_n^r$?

2) Quelle est la structure algébro-géométrique du morphisme de champs algébriques

$$p\text{-}\mathcal{L}ie_n \xrightarrow{\text{oubli}} \mathcal{L}ie_n\,?$$

3) Quelle est l'image de GLC (qui n'est pas algébrique) dans $p\text{-}\mathcal{L}ie_n$ et $\mathcal{L}ie_n$? Est-elle algébrique ?

4) Les liens rappelés ci-dessus entre un groupe algébrique G et la famille $\{G_r\}$ s'étendent-ils à des schémas en groupes sur des bases S plus générales ?

5) Quelles sont les fibres du morphisme GLC $\xrightarrow{\text{Lie}(\ker(\text{Frob}))}$ $p\text{-}\mathcal{L}ie_n$, et peut-on distinguer les points d'une fibre donnée en augmentant la hauteur c'est-à-dire en étudiant le morphisme $\ker(\text{Frob}^r) : \text{GLC} \to \mathcal{G}_n^r$ pour $r \geq 2$?

Certaines de ces questions sont trop difficiles pour espérer une réponse en toute généralité, et l'étude d'exemples (petite dimension, groupes de type particulier) est nécessaire. C'est ce que nous proposons de faire dans cette thèse.

2 Résumé du premier chapitre

Dans le but d'explorer les objectifs et questions évoqués ci-dessus, nous avons commencé par étudier l'espace $\mathcal{G}_n^r$ des groupes de hauteur r, finis localement libres de rang n. En caractéristique p, nous pouvons munir canoniquement l'algèbre de Lie d'un groupe algébrique d'une structure supplémentaire, appelée *p-application*. De manière générale, l'algèbre de Lie d'un groupe en caractéristique p ne caractérise pas le groupe (par exemple si G est un groupe étale, alors son algèbre de Lie est nulle), mais munie de sa p-application, l'algèbre de Lie capture le premier noyau de Frobenius de notre groupe. En effet, le foncteur Lie donne une équivalence de champs $\mathcal{G}_n^1 \simeq p\text{-}\mathcal{L}ie_n$ (voir [SGA3], exposé VIIA, section 7) et ainsi donne un cadre naturellement plaisant pour mieux comprendre ces objets. De plus, les groupes de hauteur 1 permettent un "dévissage" dans le sens suivant : soit G_r le r-ième noyau de Frobenius d'un groupe lisse. Alors G_r est fini localement libre, de hauteur r dans toutes les fibres. Nous pouvons regarder son premier noyau de Frobenius $(G_r)_1$, qui est un groupe fini localement libre de hauteur 1, et le quotient $G_r/(G_r)_1$ est un groupe fini localement libre de hauteur $r-1$, et continuer à dévisser ce quotient. Ainsi dans ce chapitre, nous nous concentrons sur les groupes de hauteur 1 et leur p-algèbre de Lie.

Nous avons décrit la structure du morphisme $p\text{-}\mathcal{L}ie_n \to \mathcal{L}ie_n$ ainsi que le champ algébrique $p\text{-}\mathcal{L}ie_n$ dans le cas particulier de la dimension 3 (les dimensions 1 et 2 ne posent pas de problème particulier). Nous rappelons ici quelques définitions et résultats utiles.

Définition A. *Soit R un anneau de caractéristique $p > 0$, et $\mathfrak{l}$ une R-algèbre de Lie. On appelle p-application sur $\mathfrak{l}$ une application $x \mapsto x^{[p]}$ de $\mathfrak{l}$ dans $\mathfrak{l}$ qui vérifie :*

(AL1) pour tout $x \in \mathfrak{l}$, $\text{ad}_{x^{[p]}} = (\text{ad}_x)^p$

(AL2) pour tout $\lambda \in R$ and $x \in \mathfrak{l}$, $(\lambda x)^{[p]} = \lambda^p x^{[p]}$

(AL3) pour tout $x, y \in \mathfrak{l}$, $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$

$$\text{où pour tout } i,\ s_i(x,y) := -\frac{1}{i}\sum_u \mathrm{ad}_{u(1)}\,\mathrm{ad}_{u(2)}\ldots\mathrm{ad}_{u(p-1)}(y),$$

et u *varie parmi les applications de* $\{1,\ldots,p-1\}$ *dans* $\{x,y\}$*, prenant* i *fois la valeur* x*.*

L'algèbre de Lie $\mathfrak{l}$ *munie de cette application est appelée une p-algèbre de Lie, ou algèbre de Lie restreinte. Si l'algèbre de Lie* $\mathfrak{l}$ *peut être munie d'une p-application, on dit qu'elle est restreignable.*

Remarquons que l'on peut étendre cette définition à des objets géométriques :

Définition B. *Soit* S *un schéma de base de caractéristique* $p>0$ *et* $L\to S$ *un fibré vectoriel en algèbres de Lie. On dit qu'un morphisme de schémas* $(\cdot)^{[p]} : L \to L$ *est une p-application si pour tout* S*-schéma* T*, c'est une p-application sur* $L(T)$*.*

On peut voir que la différence de deux p-applications prend ses valeurs dans le centre $Z(L)$ (voir [SF88], Chapter 2, Section 2.2, Proposition 2.1.). Ainsi le centre joue un rôle clé dans le monde des p-algèbres de Lie, en contrôlant le défaut d'unicité de la p-application. Pour tout $n \in \mathbb{N}^*$, on note L_n le foncteur des algèbres de Lie de dimension n munies d'une base : on montre facilement qu'il est représentable par un schéma de type fini, défini sur $\mathbb{Z}$. Notons alors $\mathbb{L}_n := \mathbb{A}_{n,\mathrm{L}_n}$ l'algèbre de Lie universelle au-dessus de L_n, de dimension n. Notre premier résultat (voir Theorem 2.1.4 et Theorem 2.1.8), qui répond à la question 2), est obtenu après restriction à une stratification aplatissante du centre, et peut être énoncé ainsi :

Théorème A. *Le morphisme d'oubli* $\pi:\ p\text{-}\mathcal{L}ie_n \to \mathcal{L}ie_n$ *est, après passage à une stratification aplatissante du centre de* $\mathbb{L}_n$*, un fibré affine.*

Ensuite, nous cherchons des conditions suffisantes pour qu'une algèbre de Lie soit restreignable. Voici quelques cas où l'existence d'une p-application est connue :

- Algèbres de Lie associatives, avec le morphisme de Frobenius.
- L'algèbre de Lie d'un groupe algébrique.
- Les algèbres de Lie avec une forme de Killing non-dégénérée (Zassenhaus).
- Un cas que l'on peut définir comme opposé au précédent est le cas de l'algèbre de Lie abélienne, où le morphisme nul est une p-application.

Le dernier cas correspond à celui où l'algèbre de Lie dérivée L' est nulle. On étend alors le résultat au cas où $\mathrm{rg}(L')=1$ (voir Theorem 2.2.1) :

Théorème B. *Soit* $L\to S$ *un fibré vectoriel en algèbres de Lie, tel que* L' *soit un sous-fibré vectoriel de rang* 1*. Alors il existe une p-application, Zariski-localement sur* L*.*

Maintenant ces résultats démontrés, nous étudions plus en détails l'espace de modules $\mathcal{L}ie_3$ des algèbres de Lie de dimension 3. Or, pour tout n, le groupe GL_n agit naturellement sur L_n par changement de base, et il existe une présentation de champs :

$$\mathcal{L}ie_n = [\mathrm{L}_n / \mathrm{GL}_n].$$

Nous pouvons ainsi nous ramener à étudier les orbites de l'action naturelle de GL_n sur cet espace, c'est-à-dire les différentes classes d'isomorphismes d'algèbres de Lie. Historiquement, sur le corps des réels ou des complexes, les algèbres de Lie de dimension 3 ont été classifiées dès 1898 dans le papier de Bianchi [Bi98]. Nous pouvons ensuite citer plusieurs travaux, dans lesquels la structure de variété algébrique de l'ensemble L_n d'algèbres de Lie de dimension n a été étudiée, voir par exemple Vergne [V66], Carles [Carl79], Carles et Diakité [CD84], Kirillov et Neretin [KN84], et d'autres. Dans ces travaux, ce sont principalement les petites dimensions qui ont été étudiées. Pour obtenir une classification des algèbres de Lie sur des corps algébriquement clos quelconques, nous avons adapté la preuve de la classification des classes d'isomorphismes sur $\mathbb{C}$ que l'on peut trouver dans le livre de Fulton et Harris [FH91] Part 2 Section 10. En particulier, nous avons besoin de classifier les algèbres de Lie sur un corps de caractéristique positive, pour pouvoir également donner une classification des algèbres de Lie restreignables. Cela donne ce résultat, détaillé dans la sous-section 3.1 :

Théorème C. *Soit k un corps algébriquement clos de caractéristique $p > 0$. On note $\sim$ la relation d'équivalence sur k, donnée par $x \sim x'$ si et seulement si $x' = x$ ou $x' = x^{-1}$. Alors, toute algèbre de Lie de dimension 3 sur k est isomorphe à exactement une algèbre de Lie de la famille suivante : $\{\mathfrak{ab}, \mathfrak{h}_3, \mathfrak{r}, \mathfrak{s}, (\mathfrak{l}_t)_{t \in k/\sim}\}$.*

A partir de cette classification des algèbres de Lie, nous donnons la classification des p- algèbres de Lie en cherchant lesquelles d'entre elles sont restreignables. Ensuite, nous avons donné des résultats plus précis sur la structure schématique de l'espace de modules L_3 des algèbres de Lie de dimension 3, que nous avons défini sur $\mathbb{Z}$. Ce schéma avait été étudié auparavant mais sur le corps des complexes. Par exemple Laurent Manivel dans [Ma16] a donné une description des composantes irréductibles de L_3, en donnant une preuve géométrique : il a exploité la correspondance entre la structure d'algèbre de Lie sur un anneau R avec un élément de $w \in R^3 \otimes R^3$. Dans ce cas, il y a alors une décomposition

$$\begin{aligned} R^3 \otimes R^3 &\xrightarrow{\sim} S^2 R^3 \oplus \wedge^2 R^3 \\ w &\mapsto (\frac{1}{2}(w + \overline{w}), \frac{1}{2}(w - \overline{w})) \end{aligned}$$

mais ici nous ne pouvons pas l'utiliser car nous travaillons sur $\mathbb{Z}$. De plus, pour avoir des résultats plus fins sur les propriétés du schéma L_3 et de ses composantes irréductibles, nous avons utilisé la théorie de la liaison, comme l'ont développée Christian Peskine et Lucien Szpiro dans [PS74].

Lorsqu'un idéal J d'un anneau est *lié* à un autre idéal L (au sens où J est l'annulateur de L), alors on peut obtenir des informations fines de régularité sur J si on en connaît sur L. Dans notre cas, il s'avère que notre variété L_3 est en réalité un cas typique de schéma réductible, dont les composantes sont liées. Cela vient du fait que l'idéal de définition de L_3 est engendré par une suite régulière. Grâce à cette jolie théorie, nous sommes arrivés au résultat suivant :

Théorème D. 1) *Le schéma* L_3 *est plat sur* $\mathbb{Z}$.

2) *Le schéma* L_3 *a deux composantes irréductibles* $\mathrm{L}_3^{(1)}$ *et* $\mathrm{L}_3^{(2)}$, *toutes les deux plates sur* $\mathbb{Z}$, *et avec des fibres géométriques intègres Cohen-Macaulay, de dimension* 6.

Dans le Théorème A, nous étudions l'espace de modules des p-algèbres de Lie sur une stratification aplatissante du centre, car c'est dans ce contexte qu'apparaissent des résultats satisfaisants. Afin de toujours faire le lien avec les schémas en groupes de hauteur 1, nous avons étendu l'équivalence de catégories entre les groupes de hauteur 1 et leur algèbre de Lie, en montrant que les centres de ces deux objets se correspondaient. Cela nous a permis de déduire des résultats sur la topologie de l'espace $\mathcal{G}_3^1$ grâce aux résultats obtenus sur $\mathcal{L}ie_3$:

Théorème E. *Pour tout* $r \in \{0,1,2,3\}$, *on note* $\mathcal{G}_{3,r}$ *l'ensemble des schémas en groupes de hauteur* 1, *localement libres de rang* p^3, *dont le centre est localement libre de rang* r. *Alors* $\mathcal{G}_{3,r}$ *possède deux composantes irréductibles, que l'on note* $\mathcal{G}_{3,r}^{(1)}$ *et* $\mathcal{G}_{3,r}^{(2)}$, *et on a :*

- *Si* $p \neq 2$, $\mathcal{G}_{3,0}$ *est singulier, mais devient lisse après intersection avec* $\mathcal{G}_{3,0}^{(2)}$, *et si* $p = 2$, $\mathcal{G}_{3,0}$ *est lisse.*
- $\mathcal{G}_{3,1}$ *est singulier, mais devient lisse après intersection avec* $\mathcal{G}_{3,1}^{(1)}$.
- $\mathcal{G}_{3,2}$ *est vide.*
- $\mathcal{G}_{3,3}$ *est lisse.*

3 Résumé du deuxième chapitre

Dans la deuxième partie de la thèse, nous nous intéressons de nouveau au champ p-$\mathcal{L}ie_n$, et à celui $\mathcal{G}_n^r$ des schémas en groupes finis localement libres infinitésimaux de hauteur r. De tels champs sont non séparés. Ceci peut être mesuré de diverses manières : soit par l'étude des fibres de la diagonale, soit par l'étude du défaut d'unicité dans le critère valuatif de séparation. Nous étudions plutôt le deuxième point de vue, qui mène à l'étude des modèles des schémas en groupes, que nous appellerons plutôt prolongements pour éviter de petites ambiguïtés liées à la proximité entre les mots *module* et *modèle*. Nous nous plaçons sur un anneau de valuation discrète d'égale caractéristique. Soit alors k un corps et notons $R := k[\![t]\!]$ l'anneau des séries formelles et $K := k(\!(t)\!)$ le corps des séries de Laurent. Etant donnés un K-schéma en groupes fini G_K et une k-algèbre A, on appelle *prolongement de* G_K *à* A un schéma en groupes fini localement libre sur $A[\![t]\!]$, dont la fibre générique, définie comme $G \otimes A(\!(t)\!)$ est munie d'un isomorphisme avec $G_K \otimes A(\!(t)\!)$. Nous notons $\mathcal{P}_{G_K}$ ou simplement $\mathcal{P}$ lorsque le contexte est clair, le faisceau des prolongements de G_K. En utilisant la représentabilité des grassmanniennes affines, on obtient le résultat suivant (voir Theorem 1.2.4) :

Théorème F. *Le foncteur $\mathcal{P}$ est représentable par un ind-schéma, ind-projectif sur k.*

Ce faisceau est alors un objet géométrique, avec lequel nous pouvons travailler. Nous aimerions alors en savoir plus sur sa structure : est-il un schéma formel (i.e. $\mathcal{P}_{\text{red}}$ est-il un schéma) ? Si c'est le cas, son schéma réduit $\mathcal{P}_{\text{red}}$ est-il de type fini ? Quelle est sa dimension ? Quelles sont ses composantes connexes, irréductibles ?

Pour cela, nous allons étudier sa structure algébrique interne, en introduisant plusieurs outils. Tout d'abord nous introduisons une *adhérence schématique plate*, qui est l'analogue de l'adhérence schématique classique, mais dans notre espace de module. On montre dans le lemme 2.1.4 que le foncteur des adhérences schématiques plates est représentable :

Lemme A. *Soit A une k-algèbre et $G \in \mathcal{P}(A)$. Soit $H_K \longrightarrow G \otimes A((t))$ un sous-schéma fermé de $G_K \otimes A((t))$. Le foncteur suivant :*

$$\overline{H_K}^G : \{\mathrm{Aff}_A\} \to \mathrm{Set}$$
$$A' \mapsto \{\textit{adhérences schématiques de } H_K \textit{ dans } H \otimes A'[\![t]\!]\}.$$

est représentable par un ind-schéma. De plus le morphisme de ind-schémas

$$\overline{H_K}^G \to \mathrm{Spec}(A)$$

est un monomorphisme bijectif.

De plus, nous remarquons que le groupe Γ égal à la restriction de Weil du groupe d'automorphismes de G_K agit naturellement sur l'espace des prolongements $\mathcal{P}$, en changeant l'isomorphisme en fibre générique. Nous calculons explicitement les orbites pour cette action dans la sous-section 2.4 :

Théorème G. *Pour tout $G \in \mathcal{P}(A)$, l'orbite de G sous l'action de Γ est le faisceau fpqc associé au préfaisceau*

$$\Omega(G) = \{A\text{-}\mathrm{Alg}\} \to \mathrm{Set}$$
$$B \mapsto \mathrm{Aut}_{k[\![t]\!]}(G)(B((t)))/\,\mathrm{Aut}_{k[\![t]\!]}(G)(B[\![t]\!]).$$

Ensuite, nous introduisons une relation d'ordre naturelle sur l'espace des prolongements : soient G' et G deux prolongements de G_K. On note $G' \geq G$ s'il existe un morphisme de prolongements $G' \to G$. L'adhérence schématique plate nous permet de montrer l'existence de borne inférieure et borne supérieure pour cette relation d'ordre lorsque G_K est infinitésimal.

En particulier, grâce au fait que la platitude est bien maîtrisée sur les anneaux réguliers de dimension ≤ 2, nous pouvons montrer que les ensembles $\mathcal{P}(k)$ et $\mathcal{P}(A)$ où A est un anneau de valuation discrète sont connexes pour cette relation d'ordre, dans le sens où toute paire de prolongements possède une borne supérieure (voir Corollaire 2.2.6) :

Théorème H. *Soit A un anneau régulier de dimension au plus 1. Soient $G, G' \in \mathcal{P}_{G_K}(A)$ deux prolongements d'un groupe G_K. Alors, il existe une borne supérieure pour G et G' dans $\mathcal{P}(A)$.*

Pour étudier le cas des bornes inférieures, nous avons besoin d'utiliser la dualité de Cartier, qui ne fonctionne pas bien dans notre cas car nous ne supposons pas nos groupes commutatifs. Voici pourquoi nous définissons également l'espace de modules des prolongements d'algèbres de Hopf (pas forcément co-commutatives, ni même commutatives). En effet dans cet espace, la dualité de Cartier fonctionne sans condition. Grâce à cet outil, nous montrons que les éléments de $\mathcal{P}(A)$ possèdent également une borne inférieure pour la relation d'ordre, pour tout anneau A régulier de dimension au plus 1. Nous généralisons ainsi le résultat obtenu par Michel Raynaud dans [Ra74], section 2 qui obtient des résultats similaires mais dans le cas des groupes commutatifs.

De plus sur les points à valeurs dans un corps, cette relation d'ordre est engendrée par les *dilatations classiques*, comme c'est démontré dans [WW80], théorème 1.4. Ainsi pour attraper toutes les prolongements d'un groupe sur un corps de base, convenons d'appeler *famille minimale de prolongements de G_K* une famille $\{M_i\}$ de $k[\![t]\!]$-schémas en groupes, prolongements de G_K, telle que tout prolongement de G_K possède un morphisme de prolongements vers l'un des M_i. Par exemple, les groupes diagonalisables

$$D_K := \prod_{i=1}^{n} \mu_{p^{a_i},K}$$

et les groupes

$$A_K := \prod_{i=1}^{n} \alpha_{p^{a_i},K}$$

possèdent des familles minimales composées respectivement d'un élément D_R et d'une famille infinie de prolongements dont le groupe sous-jacent est A_R (ceci se voit facilement ; pour $\mathbb{G}_m$ et $\mathbb{G}_a$, voir [WW80], Lemma 2.1. et Corollary 2.4.). Supposons alors connue une famille minimale de prolongements pour notre groupe G_K. Alors la propriété de factorisation par les dilatations fournit un procédé de construction de tous les modèles de G_K. Ainsi, pour décrire un à un tous les prolongements d'un groupe G_K à $k[\![t]\!]$, nous pouvons nous restreindre à étudier et lister ses dilatations successives (qui peuvent se calculer explicitement), et nous pouvons nous arrêter dès lors que l'on rencontre un prolongement G_1 dont le groupe sous-jacent est le même qu'un prolongement G_2 déjà rencontré auparavant. En effet, cela implique que G_1 et G_2 sont dans la même orbite pour l'action du groupe d'automorphismes de G_K, et ainsi on peut montrer que leurs dilatations, en tant que prolongements, sont également dans la même orbite sous l'action du groupe d'automorphismes.

Dans notre cas, comme l'espace de modules des prolongements d'un groupe G_K est représentable, on peut avoir une définition de dilatation comme objet de cet espace, et obtenir ainsi des dilatations sur des bases plus générales. Ainsi pour tout prolongement $G \in \mathcal{P}(A)$, pour tout groupe fermé H de la fibre spéciale de G, notons $\mathcal{P}^H_{\geq G}$ le sous-espace des prolongements munis d'un morphisme vers G qui envoie la fibre spéciale dans H. Ce foncteur est également représentable par un ind-schéma, ind-projectif, car l'inclusion naturelle suivante est représentable par une immersion fermée :

$$\mathcal{P}^H_{\geq G} \quad \hookrightarrow \mathcal{P} \xrightarrow[\text{ind-projectif}]{\text{ind- schéma}} \operatorname{Spec}(k) \ .$$

Si l'ensemble $\mathcal{P}^H_{\geq G}$ est non vide et s'il possède un minimum pour la relation d'ordre, nous le notons $D_G(H)$ et l'appelons *dilatation de H dans G*. De manière générale, il n'est pas simple de savoir si une dilatation existe. Cependant, grâce au fait que la platitude est bien maîtrisée sur les anneaux réguliers de dimension ≤ 2, nous arrivons à démontrer l'existence des dilatations dans $\mathcal{P}(A)$ avec A régulier de dimension au plus 1, et lorsque le groupe G_K est infinitésimal (voir Proposition 2.3.15) :

Proposition A. *Supposons G_K infinitésimal. Soit A un anneau régulier de dimension au plus* 1*. Alors pour tout sous-groupe H de la fibre spéciale de G, la dilatation $D_G(H)$ existe dans $\mathcal{P}(A)$.*

Dans ce cadre plaisant, nous obtenons alors un morphisme rationnel de l'espace des sous-groupes de la fibre spéciale de G vers l'espace des prolongements. Plus précisément (voir Corollaire 2.3.16) :

Théorème I. *Supposons G_K infinitesimal. Soit A une k-algèbre et soit $G \in \mathcal{P}(A)$. On note $\mathcal{S}_d$ l'espace de modules des sous-groupes fermés de $G_{|t=0}$, d'ordre p^d. Alors il existe un morphisme rationnel entre le normalisé $\widetilde{\mathcal{S}_d^{\mathrm{red}}}$ et l'espace $\mathcal{P}$, défini sur un ouvert non vide contenant tous les points de codimension* 1 *:*

$$\widetilde{\mathcal{S}_d^{\mathrm{red}}} \dashrightarrow \mathcal{P}.$$

De plus dans l'exemple 2.3.17, nous donnons un cas où ce morphisme n'est pas injectif.

Nous passons ensuite à l'étude d'espaces de prolongements particuliers. Les cas des prolongements de μ_p et de μ_{p^2} sur un anneau de valuation discrète de caractéristique résiduelle positive ont été traités en détail dans [T10], et ceux de μ_{p^n} sur un anneau de valuation discrète de caractéristique mixte ont été considérés dans [MRT13]. Pour le cas du groupe $(\alpha_{p,K})^n$, nous utilisons une nouvelle fois l'équivalence de catégories entre les groupes de hauteur 1 et celle des p-algèbres de Lie. En effet, on peut montrer que le foncteur Lie donne une équivalence entre la catégorie des prolongements d'un groupe de hauteur 1 et celle des prolongements de son algèbre de Lie, vue comme fibré vectoriel en p-algèbres de Lie. De plus, puisque nous ne considérons que des objets plats, la fibre générique d'un prolongement G de G_K (qui est donc isomorphe à

G_K) est schématiquement dense dans G. Grâce à cette propriété, nous savons qu'un prolongement d'une algèbre de Lie abélienne est également une algèbre de Lie abélienne. De la même façon tout prolongement d'une p-algèbre de Lie dont la p-application est l'application nulle est également une algèbre de Lie restreinte, avec comme p-application l'application nulle. Grâce à cette observation, en notant $G_K := (\alpha_{p,K})^n$, on obtient un isomorphisme :

$$\begin{aligned} \mathcal{P}_{G_K} &\xrightarrow{\sim} \mathcal{L}att_n \\ G &\longmapsto \mathcal{O}_{\mathrm{Lie}(G)} \end{aligned}$$

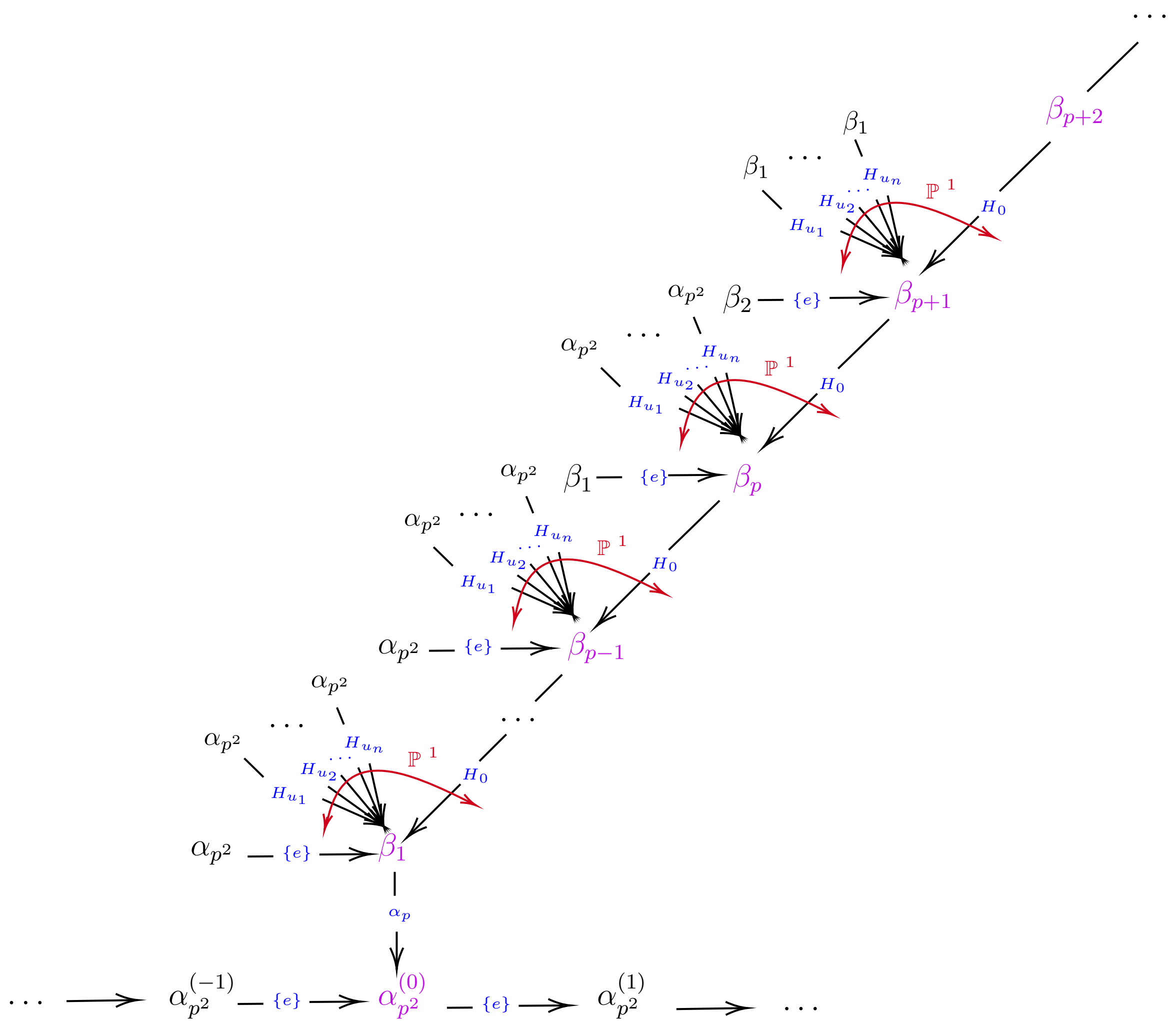
entre les prolongements de G_K et l'espace des *réseaux*, qui est un ind-schéma non réduit et ind-projectif sur $\mathrm{Spec}(k)$.

Nous étudions ensuite les prolongements du groupe $\alpha_{p^2,K}$. Tout d'abord, comme nous pouvons calculer explicitement le groupe d'automorphismes Γ de $\alpha_{p^2,K}$, et ainsi calculer l'orbite du prolongement standard (donné par α_{p^2} comme groupe sous-jacent, et par l'identité en fibre générique). Nous montrons alors que cette orbite est de dimension infinie. Nous obtenons donc ce résultat, qui répond à la question de la dimension de $\mathcal{P}$:

Théorème J. *L'espace $\mathcal{P}_{\alpha_{p^2,K}}$ est de dimension infinie sur k.*

Enfin, grâce à l'existence de la famille minimale discrète de $\alpha_{p^2,K}$ et au procédé de dilatation, nous pouvons calculer tous les prolongements de $\alpha_{p^2,K}$ à $k[\![t]\!]$. En effet, il suffit de faire toutes les dilatations possibles successivement. Grâce à l'action du groupe d'automorphismes Γ, nous pouvons nous arrêter lorsque nous rencontrons un prolongement dont le groupe sous-jacent est déjà apparu auparavant. Ainsi nous calculons alors l'ensemble $\mathcal{P}_{\alpha_{p^2}}(k)$ explicitement dans la sous-section 3.2, et nous pouvons résumer ces prolongements à l'aide d'un *arbre couvrant*, ce qui permet de de résumer de manière visuelle tous les prolongements.

Voici alors l'arbre couvrant des prolongements de $\alpha_{p^2,K}$. Comme indiqué ci-dessus, nous nous sommes arrêtés dès lors que nous avons rencontré un prolongement dont le groupe sous-jacent est déjà apparu dans l'arbre. Ainsi nous avons coloré en violet les prolongements nouveaux, et nous avons indiqué sur les flèches quels sont les groupes que nous avons dilatés. Toutes les notations et les calculs sont présentés dans la sous-section 3.2.



Enfin, comme le foncteur Lie donne une équivalence entre les prolongements des groupes et ceux des p-algèbres de Lie, nous voulions connaître toutes les algèbres de Lie et p-algèbres de Lie définies sur $k((t))$. En utilisant des outils cohomologiques, nous avons généralisé notre classification faite au Théorème 3.1 en classifiant tout d'abord les algèbres de Lie sur $k((t))$, avec $\mathrm{car}(k) \neq 2$ (lorsque $\mathrm{car}(k) = 2$, l'algèbre de Lis $\mathfrak{s}$ définie sur $\overline{k((t))}$ donne lieu à beaucoup de formes différentes que nous n'avons pas étudiées). Voici alors le résultat, qui montre que si on écarte le cas de la caractéristique 2, il n'y a qu'une nouvelle classe de d'algèbre de Lie qui apparaît :

Théorème K. *Soit k un corps algébriquement clos de caractéristique $p \neq 2$. Alors il existe une unique forme non triviale d'algèbre de Lie de dimension 3 sur $k((t))$.*

La question naturelle à se poser maintenant est de classifier les formes des p-algèbres de Lie. Grâce à la correspondance entre l'unicité de la p-application et la dimension du centre, nous savons que les algèbres de Lie $\mathfrak{s}$ et $\mathfrak{l}_\alpha$, avec $\alpha \in \mathbb{F}_p^*$ définies sur $\overline{k(\!(t)\!)}$ ne donnent pas lieu à des formes de p-algèbres de Lie sur $k(\!(t)\!)$, et pour les autres p-algèbres de Lie à centre non nul, des calculs sont faisables (mais deviennent fastidieux) pour étudier leurs potentielles formes de p-algèbres de Lie.

4 Ouvertures

Cette thèse amène à plusieurs questions ouvertes. Nous en listons ici quelques-unes, et donnons quelques pistes de réflexion.

Concernant le chapitre 1, de nombreuses questions sont naturelles. Par exemple en ce qui concerne le lieu restreignable d'une algèbre de Lie donnée :

- Comment le lieu restreignable d'une algèbre de Lie varie en fonction de la caractéristique p ?
- Peut-on trouver un schéma défini sur $\mathbb{Z}$ tel qu'il devienne le lieu restreignable de l'algèbre de Lie universelle sur $\mathbb{F}_p$ pour tout premier p ?

Concernant l'espace de modules des p-algèbres de Lie de dimension n :

- Peut-on connaître le nombre de ses composantes connexes ?
- Peut-on connaître le nombre de ses composantes irréductibles ?
- Peut-on connaître sa dimension ?

Déjà en dimension 4 ce travail n'est pas clair car l'idéal de définition du schéma L_4 n'est pas engendré par une suite régulière, et ainsi nous ne pourrions pas, *a priori* utiliser les méthodes que nous avons utilisées pour L_3 afin d'obtenir des résultats satisfaisants. Nous pourrions peut-être utiliser l'outil informatique (comme [Macaulay2]) pour essayer d'en apprendre plus sur la structure schématique de L_n mais déjà en dimension 4 cela demande de travailler avec un anneau de polynômes de 24 inconnues, avec 16 relations donc les calculs deviennent vite compliqués.

Concernant l'espace de module des prolongements $\mathcal{P}$, nous aimerions à l'avenir en connaître plus sur sa structure schématique :

- Quelles sont ses composantes connexes ?
- Quelles sont ses composantes irréductibles ?

De plus nous avons vu que $\mathcal{P}$ n'était pas de dimension finie en général, et que le groupe Γ des automorphismes de G_K agissait sur $\mathcal{P}$. On peut se demander :

- Est-ce que le quotient $\mathcal{P}/\Gamma$ est de dimension finie ?

Une stratégie pour en connaître plus est d'utiliser les outils que nous développons dans cette thèse, à savoir l'adhérence schématique plate et les dilatations. En particulier l'existence du morphisme rationnel entre l'espace de modules des sous-groupes de la fibre spéciale d'un prolongement et $\mathcal{P}$ amène des questions naturelles :

- Peut-on contrôler le défaut d'injectivité du morphisme $\widetilde{\mathcal{S}_d^{\mathrm{red}}} \dashrightarrow \mathcal{P}$?
- Ce morphisme peut-il s'étendre à un ouvert plus grand ? Peut-on connaître le plus grand ouvert sur lequel il s'étend ?
- S'il s'étend, est-il toujours égal à la dilatation ?

MODULI OF LIE p-ALGEBRAS

The aim of this chapter is to study height 1 group schemes, in families. For simplicity we write $\mathcal{G}_n := \mathcal{G}_n^1$ for the moduli stack of height 1 group schemes, finite locally free of rank n. This is of course by far the easiest case, because if we write $S \to \operatorname{Spec}(\mathbb{F}_p)$ for a base scheme, and $p\text{-}\mathcal{L}ie_n(S)$ for the category of n-dimensional restricted $\mathcal{O}_S$-Lie algebras, then the functor Lie gives us an equivalence:

$$\operatorname{Lie} : \mathcal{G}_n(S) \xrightarrow{\sim} p\text{-}\mathcal{L}ie_n(S).$$

We are thus reduced to studying the moduli of finite-dimensional Lie algebras and p-mappings on them. Our work is divided in two parts: in the first half of the paper we study the theoretical aspects, and in the second half we study in detail the three-dimensional case. In the first part we study a Lie algebra L over a scheme S, that is, a vector bundle equipped with a bracket satisfying the Jacobi condition. The difference of two p-mappings on L takes its values in the center $Z(L)$, which for this reason plays a key role. Our first main result is obtained after restriction to the flattening stratification $S^* \to S$ of the center, and is stated as follows (see Theorem 2.1.4 and Theorem 2.1.8).

Theorem A. *Let $L \to S$ be a Lie algebra vector bundle. Let us define the functor $X = X(L)$ of the p-mappings on L, i.e. $X(T) = \{p\text{-mappings on } L \times_S T\}$ for all S-schemes T. Let* $\operatorname{Frob} : S \to S$ *be the Frobenius morphism. Then, X is representable by an affine scheme, and is a formally principal homogeneous space under* $\mathrm{E} := \operatorname{Hom}(\operatorname{Frob}_S^* L, Z(L))$.
Now let us define the restrictable locus of L as follows:

$$S^{\text{res}} = S^{\text{res}}(L) : \{S\text{-schemes}\} \longrightarrow \text{Set}$$
$$T \longmapsto \begin{cases} \{\emptyset\} & \text{if } L_T \text{ is restrictable over } T \\ \emptyset & \text{otherwise.} \end{cases}$$

Then if we suppose $Z(L) \to S$ flat, the following two conditions are verified:

1. *S^{res} is representable by a closed subscheme of S.*
2. *$X \to S$ factors through S^{res} and $X \to S^{\text{res}}$ is an affine space under the vector bundle* $\mathrm{E} \times_S S^{\text{res}}$.

It follows in particular that if $Z(L)$ is flat over S, then $X \to S^{\text{res}}$ is smooth.

In the rest of the chapter, we will put our interest on the moduli stack $\mathcal{L}ie_n$ of n-dimensional Lie algebras, and especially on the case $n = 3$. For this, we will introduce the moduli space L_n of *based* Lie algebras locally free of rank n, with the natural action of GL_n on it, by change of basis. We can see that we have the quotient stack presentation $\mathcal{L}ie_n = [\mathrm{L}_n / \mathrm{GL}_n]$, so we are led to studying the orbits of the action of GL_n. You can find the classification on those isomorphism classes in Fulton and Harris' book [FH91], but in order to apply our theoretical results and to allow varying primes p, we reformulate in Subsection 3.1 the classification of 3-dimensional Lie algebras over algebraically closed fields in a characteristic-free way, giving representatives of the isomorphism classes defined over $\mathbb{Z}$ and $\mathbb{Z}[T]$. So let k be an algebraically closed field of characteristic $p > 0$. Let us denote by $\sim$ the equivalence relation on k, given by $x \sim x'$ if and only if $x' = x$ or $x' = x^{-1}$. Then any Lie algebras of dimension 3 over k is isomorphic to exactly one in the following table.

<table>
<tr><th colspan="2">Name</th><th>Structure</th><th>Orbit dimension</th><th>Center dimension</th><th colspan="2">Restrictable</th></tr>
<tr><td colspan="2">$\mathfrak{ab}_3$</td><td>abelian</td><td>0</td><td>3</td><td colspan="2">yes</td></tr>
<tr><td colspan="2">$\mathfrak{h}_3$</td><td>nilpotent</td><td>3</td><td>1</td><td colspan="2">yes</td></tr>
<tr><td colspan="2">$\mathfrak{r}$</td><td>solvable</td><td>5</td><td>0</td><td colspan="2">no</td></tr>
<tr><td colspan="2" rowspan="2">$\mathfrak{s}$</td><td rowspan="2">simple</td><td rowspan="2">6</td><td rowspan="2">0</td><td>$p \neq 2$</td><td>$p = 2$</td></tr>
<tr><td>yes</td><td>no</td></tr>
<tr><td rowspan="4">$\mathfrak{l}_t$</td><td>$\bar{t} \notin \mathbb{F}_p/\sim$</td><td>solvable</td><td>5</td><td>0</td><td colspan="2">no</td></tr>
<tr><td>$\bar{t} \in \mathbb{F}_p/\sim \setminus \{\bar{0}, \bar{1}\}$</td><td>solvable</td><td>5</td><td>0</td><td colspan="2">yes</td></tr>
<tr><td>$\bar{t} = \bar{0}$</td><td>solvable</td><td>5</td><td>1</td><td colspan="2">yes</td></tr>
<tr><td>$\bar{t} = \bar{1}$</td><td>solvable</td><td>3</td><td>0</td><td colspan="2">yes</td></tr>
</table>

Afterward in Subsection 3.2 we supplement the known results by giving more precise information on the scheme structure of the moduli space L_3, that we define over $\mathbb{Z}$. For this, we use liaison theory, as developed by Peskine and Szpiro in [PS74]; in fact L_3 turns out to be a typical case of a reducible scheme whose components are linked.

Theorem B. 1) *The functor* L_3 *is representable by an affine flat* $\mathbb{Z}$*-scheme of finite type.*

2) *The scheme* L_3 *has two relative irreducible components* $\mathrm{L}_3^{(1)}$ *and* $\mathrm{L}_3^{(2)}$ *which are both flat with Cohen-Macaulay integral geometric fibers of dimension* 6*.*

For the end, as we said before, we will come back to our equivalence between height 1 group schemes and restricted Lie algebras. Because the center of a Lie algebra plays a key role in our work, we extend the classical equivalence of categories between locally free Lie p-algebras of finite rank with finite locally free group schemes of height 1, showing that the centers of those objects correspond to each other in Proposition 4.1.2. For this reason, for $r \leq n$, let us denote by p-$\mathcal{L}ie_{n,r}(S)$ the category of n-dimensional restricted $\mathcal{O}_S$-Lie algebras, whose center is locally free of rank r, and with the same idea, let us denote by $\mathcal{G}_{n,r}(S)$ the category of finite locally free S-group schemes of order p^n, of height 1, whose center is locally free of rank p^r.

Theorem C. *Let S be a scheme of characteristic $p > 0$ and let $G \to S$ be a finite locally free group scheme of height* 1*. Let $Z(G)$ denote its center. Then*

$$Z(\mathrm{Lie}(G)) = \mathrm{Lie}(Z(G)).$$

Then the classical equivalence of categories

$$\mathrm{Lie} : \mathcal{G}_n(S) \xrightarrow{\sim} p\text{-}\mathcal{L}ie_n(S)$$

restricts to an equivalence

$$\mathrm{Lie} : \mathcal{G}_{n,r}(S) \xrightarrow{\sim} p\text{-}\mathcal{L}ie_{n,r}(S).$$

So using this, we can focus on the object p-$\mathcal{L}ie_{n,r}(S)$, and because we have the quotient stack presentation $\mathcal{L}ie_n = [\mathrm{L}_n / \mathrm{GL}_n]$, we can focus on L_n, and especially on $\mathrm{L}_n^{\mathrm{res}}$ the locally closed subscheme of L_n where the universal Lie algebra $\mathbb{L}_n \to \mathrm{L}_n$ is restrictable. In particular, if k is an algebraically closed field of characteristic $p > 0$, Theorem C and the previous results allow us to count the centerless finite locally free k-group schemes of order p^3, of height 1. This number is finite, equal to 1 if $p = 2$ and $(p+3)/2$ if $p \neq 2$ (See Proposition 4.1.3).

For the end, in the subsections 4.2, 4.3 and 4.4, we study the smoothness of the restrictable locus $\mathrm{L}_3^{\mathrm{res}} \subset \mathrm{L}_3$ of $\mathbb{L}_3$ in the different flattening strata of the center. For a better understanding of the following theorem, the reader can look at the pictures of Subsection 3.3.

Theorem D. *Let k be an algebraically closed field of characteristic $p > 0$. Let $\mathrm{L}_{3,r}^{\mathrm{res}} \to \mathrm{Spec}(k)$ be the locally closed subscheme of L_3 where the center $Z(\mathbb{L}_3)$ is locally free of rank r, and $\mathbb{L}_3$ is restrictable.*

1. *(i) If $p \neq 2$, the singular locus of $\mathrm{L}_{3,0}^{\mathrm{res}}$ is the orbit of $\mathfrak{l}_{-1}$. The singularity remains after intersection with $\mathrm{L}_3^{(1)}$ but $\mathrm{L}_{3,0}^{\mathrm{res}} \cap \mathrm{L}_3^{(2)}$ is smooth.*
   (ii) If $p = 2$, the scheme $\mathrm{L}_{3,0}^{\mathrm{res}}$ is smooth and remains smooth after intersection with any irreducible component.
2. *The singular locus of $\mathrm{L}_{3,1}^{\mathrm{res}}$ is the orbit of $\mathfrak{h}_3$. The singularity remains after intersection with $\mathrm{L}_3^{(2)}$ but $\mathrm{L}_{3,1}^{\mathrm{res}} \cap \mathrm{L}_3^{(1)}$ is smooth.*
3. *The scheme $\mathrm{L}_{3,2}^{\mathrm{res}}$ is empty.*
4. *The scheme $\mathrm{L}_{3,3}^{\mathrm{res}}$ is smooth and remains smooth after intersection with any irreducible component.*

It is well known that in Lie algebra theory, the characteristics $p = 2$ and $p = 3$ are special. In the previous result we see that the characteristic $p = 2$ appears as a special case, and the reader can see that the case $p = 3$ needs special care e.g. in the proof of Theorem 4.3.2.

Thanks to Theorem C, all the assertions of Theorem D hold also for $\mathcal{G}_{3,r}$, i.e. $\mathcal{G}_{3,r}$ splits in two irreducible components that we denote by $\mathcal{G}_{3,r}^{(1)}$ and $\mathcal{G}_{3,r}^{(2)}$; and we can say that if $p \neq 2$, $\mathcal{G}_{3,0}$ is singular, but becomes smooth if we intersect with $\mathcal{G}_{3,0}^{(2)}$, if $p = 2$ it is smooth. Moreover $\mathcal{G}_{3,1}$ is singular but becomes smooth when we intersect it with $\mathcal{G}_{3,1}^{(1)}$, $\mathcal{G}_{3,2}$ is empty and $\mathcal{G}_{3,3}^{p}$ is smooth. We refer to Corollary 4.4.1 for more details.

1 Preliminaries on Lie algebras

1.1 Definition and theory of Lie p-algebras over a ring

In this section, we recall basic notations and facts on Lie algebras and Lie p-algebras. We also recall Jacobson's theorems on existence and uniqueness of p-mappings for some Lie algebras over a commutative ring. The reader can find the proofs for Lie algebras over a field in Strade and Farnsteiner's book on Modular Lie algebras [SF88], and we verify easily that these proofs do not use the fact that the base ring is a field.

Let R be a base ring (commutative with unit). An R-Lie algebra is an R-module l endowed with an R-bilinear alternating map denoted by $[\cdot,\cdot] : l \otimes_R l \to l$ satisfying the Jacobi identity. If $R \to R'$ is a map of rings, there is an obvious structure of R'-Lie algebra on $l \otimes_R R'$. We denote by $\mathrm{End}(l)$ the R-module of R-linear endomorphisms of l, $\mathrm{ad} : l \to \mathrm{End}(l)$ the map $x \mapsto [x,\cdot]$ and $Z(l)$ the kernel of ad, called the *center* of l. If l is locally free of finite rank as a module, the formation of $\mathrm{End}(l)$ and ad commutes with base change, but the formation of the center does not in general.

Now let us assume that R is an $\mathbb{F}_p$-algebra, and write $\mathrm{Frob} : R \to R$ its Frobenius endomorphism.

1.1.1. Definition. We say that a mapping $(\cdot)^{[p]} : l \to l$ is a *p-mapping* if:

(AL1) for all $x \in l$, $\mathrm{ad}_{x^{[p]}} = (\mathrm{ad}_x)^p$

(AL2) for all $\lambda \in R$ and $x \in l$, $(\lambda x)^{[p]} = \lambda^p x^{[p]}$

(AL3) for all $x, y \in l$, $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$

$$\text{where for all } i,\ s_i(x,y) := -\frac{1}{i}\sum_u \mathrm{ad}_{u(1)}\mathrm{ad}_{u(2)}\ldots\mathrm{ad}_{u(p-1)}(y),$$

and u ranges through the maps from $\{1,\ldots,p-1\}$ to $\{x,y\}$ taking i times the value x.

These three conditions are called *Jacobson's identities*. Let us remark that this definition generalises the "Frobenius morphism", in the following sense: let A be an associative algebra. We can endow A with a Lie algebra structure, setting $[x,y] := xy - yx$. Then in this context, $x \mapsto x^p$ is a p-mapping.

For instance, we have

$$s_1(x,y) = -[y,[y,\dots,[x,y\underbrace{]]\dots]}_{p-1} \text{ and } s_{p-1}(x,y) = [x,[x,\dots,[x,y\underbrace{]]\dots]}_{p-1}.$$

1.1.2. Definition. A Lie algebra equipped with a p-mapping is called *Lie p-algebra* or we say that it is *restricted*. If a Lie algebra can be equipped with a p-mapping, we say that it is *restrictable*.

We also recall that a *p-morphism* between two Lie p-algebras is a morphism of Lie algebras that commutes with the p-mappings. A *p-ideal* is an ideal stable by the p-mappings. For example, the center $Z(l)$ is always a p-ideal, by the axiom (AL 1).

The next proposition shows that we can endow the image (under a Lie algebra morphism) of a Lie p-algebra with a natural p-mapping.

1.1.3. Proposition. *Let $(l_1, (\cdot)^{[p]})$ be a Lie p-algebra over R. Suppose that $f : l_1 \to l_2$ is a Lie algebra morphism such that $\ker(f)$ is a p-ideal of l_1. Then there exists exactly one p-mapping on $f(l_1)$ such that $f : l_1 \to f(l_1)$ is a p-morphism.*

Proof. See [SF88], Chapter 2, Section 2.1, Proposition 1.4. □

1.1.4. Theorem. *Let l be a Lie algebra over R.*

1. *Let γ_1 and γ_2 be two p-mappings on l. Then $\gamma_2 - \gamma_1 : l \to Z(l)$ is Frobenius-semi-linear.*
2. *Conversely, let $\phi : l \to Z(l)$ be a Frobenius-semi-linear map, and γ_1 a p-mapping on l. Then, $\gamma_1 + \phi : l \to l$ is also a p-mapping.*

Proof. See [SF88], Chapter 2, Section 2.2, Proposition 2.1. □

The following corollary is a rewording of the previous theorem. It will be useful for the following sections where we will present results on Lie p-algebras but in a geometric way.

1.1.5. Corollary. *Let l be a Lie algebra over R. We define*

$$\mathrm{E} := \mathrm{Hom}_{\mathrm{Frob}}(l, Z(l)) = \mathrm{Hom}_R(l \otimes_{R,\mathrm{Frob}} R, Z(l))$$

the set of Frobenius semi-linear maps from l to $Z(l)$ and let X denote the set of p-mappings on l. Then the map:

$$\begin{aligned} \mathrm{E} \times X &\to X \times X \\ (\phi, \gamma) &\mapsto (\phi + \gamma, \gamma) \end{aligned}$$

is bijective. □

In particular, the theorem says that if there exists a p-mapping on l, it is unique if and only if $\mathrm{E} = \{0\}$, i.e. if l is locally free of finite rank, the p-mapping is unique if and only if $Z(l) = \{0\}$.

The next proposition shows that the hypothesis (AL 1) is essential in the definition of a p-mapping, and gives an equivalent condition for a Lie algebra to be restrictable.

1.1.6. Theorem. ***(Jacobson)*** *Let l be a Lie algebra, free over R with basis $\{x_i\}_{i\in I}$. Let us assume that for all $i \in I$, there exists $y_i \in l$ such that $\mathrm{ad}_{x_i}^p = \mathrm{ad}_{y_i}$. Then, there exists a unique p-mapping $(\cdot)^{[p]} : l \to l$ such that for all $i \in I$, $x_i^{[p]} = y_i$.*

Proof. You can find the proof in [SF88], Chapter 2, Section 2.2, Theorem 2.3, but the initial version is due to Jacobson, in [J62], Chapter 5, Section 7, Theorem 11. □

1.1.7. Example. (Zassenhaus). You can have a look at [SF88], Chapter 1, Section 2.7, Theorem 7.9, or at Zassenhaus's article: [Z39] for more details. Let l be a free Lie algebra over R, with Killing form denoted by B. We suppose that B is *non-degenerate*, that is we suppose that the following map

$$\begin{aligned} l &\longrightarrow \mathrm{Hom}_R(l, R) \\ x &\longmapsto B(x, \cdot) \end{aligned}$$

is an isomorphism. Then there exists a unique p-mapping on l.

1.2 Vector bundles, quotient and image

In this section, S is a base scheme. We will study vector bundles equipped with a bracket, in order to study Lie algebras in families. We start by giving standard definitions and notations about vector bundles. We use the notation $\mathbb{O}_S$ for the ring scheme $\mathrm{Spec}(\mathcal{O}_S[X])$.

1.2.1. Definition. Throughout all this paper, we call a *generalized vector bundle* any scheme which is an $\mathbb{O}_S$-module, isomorphic to an $\mathbb{O}_S$-module of the form $\mathbb{V}(\mathcal{F}) := \mathbf{Spec}(\mathrm{Sym}(\mathcal{F}))$ with $\mathcal{F}$ any quasi-coherent $\mathcal{O}_S$-module.
We also call *vector bundles* those for which $\mathcal{F}$ is locally free of finite rank. In this case, we use the usual covariant equivalence for which the sheaf of sections of our scheme is $\mathcal{F}^\vee$.

1.2.2. Remark. Let $F = \mathbf{Spec}(\mathrm{Sym}(\mathcal{F}^\vee))$ be a vector bundle. Then $\mathcal{F}$ is the restriction of the functor of points of F to the small Zariski site of S, that is, to the open subschemes $U \hookrightarrow S$.

1.2.3. Definition. Let $f : E \to F$ be a morphism of generalized vector bundles over S. We define the *kernel* and the *image* of f as the fppf kernel sheaf of f and the fppf image sheaf of f, i.e. for all fppf covers $T \to S$, we have

$$\mathrm{im}(f)(T) = \Big\{ y \in F(T), \exists\, T' \to T \text{ fppf covering and } x' \in E(T') \text{ such that } f(x') = y_{|T'} \Big\}.$$

1.2.4. Remark. The image is not representable by a scheme in general, but its formation commutes with base change.

In the following, exact sequences of (generalized) vector bundles will be understood as exact sequences of fppf sheaves of modules.

1.2.5. Definition. Let $X \to S$ be a vector bundle and $Y \hookrightarrow X$ an $\mathbb{O}_S$-submodule of X. We say that Y is a *subbundle* of X if Y is a vector bundle and X/Y is also a vector bundle.

1.2.6. Remark. It is equivalent to be a subbundle of X and to be a locally direct factor of X.

1.2.7. Proposition. *Let $F \to S$ be a generalized vector bundle. Let us write $F = \mathbf{Spec}(\mathrm{Sym}(\mathcal{F}))$ for a given quasi-coherent $\mathcal{O}_S$-module $\mathcal{F}$. Then:*

1. *$F \to S$ is of finite presentation if and only if $\mathcal{F}$ is of finite presentation.*
2. *If $\mathcal{F}$ is of finite presentation, then*

$$F \to S \text{ is flat } \Leftrightarrow F \to S \text{ is smooth } \Leftrightarrow \mathcal{F} \text{ is locally free of finite rank.}$$

Proof. See Görtz and Wedhorn's book [GW20], Chapter 7, Proposition 7.41. □

For the following, it will be useful to characterize when an $\mathbb{O}_S$-submodule of a vector bundle is in fact a subbundle. In order to do this, we establish these two preliminary lemmas.

1.2.8. Lemma.

1. *Let R be a Noetherian ring. Then any surjective endomorphism $\alpha : R \to R$ is an automorphism.*
2. *Let R be a ring and $\alpha : R' \to R'$ a surjective R-algebra morphism. Then if $R \to R'$ is of finite presentation, α is an automorphism.*

Proof. 1. For a contradiction, let us assume that α is not injective: let $x \in \ker(\alpha), x \neq 0$ and $n \in \mathbb{N}$. Then α^n is surjective, so there exists $y \in R$ such that $x = \alpha^n(y)$. Thus, $\alpha^{n+1}(y) = 0$. Then $y \in \ker(\alpha^{n+1}) \setminus \ker(\alpha^n)$. Thus, the sequence $(\ker(\alpha^n))_{n \geq 0}$ is not stationary, then we get a contradiction.
2. Now we suppose that R is any ring and $R \to R'$ is of finite presentation. Then by standard arguments, there exists a subring $R_0 \subset R$ of finite type over $\mathbb{Z}$ and an R_0-algebra $R_0 \to R'_0$ of finite presentation such that $R' \simeq R'_0 \otimes_{R_0} R$. Then if $\alpha : R' \to R'$ is a surjective R-algebra morphism, we can write $\alpha = \alpha_0 \otimes_{R_0} \mathrm{id}_R : R'_0 \otimes R \to R'_0 \otimes R$ where $\alpha_0 : R'_0 \to R'_0$ is surjective. Then thanks to the previous point, α_0 is an automorphism, then so is α, as we wanted. □

1.2.9. Lemma. *Let $X \to S$ be a scheme and $G \to S$ a flat group scheme of finite presentation, acting on $X \to S$. Let $\pi : X \to Y$ be a faithfully flat S-morphism of finite presentation and G-invariant. Let us assume that the morphism*

$$\begin{aligned} G \times_S X &\to X \times_Y X \\ (g, x) &\mapsto (x, g \cdot x) \end{aligned}$$

is an isomorphism. Then, Y is the quotient of X by G in the category of fppf sheaves on S.

Proof. Let F be an fppf sheaf on S and $f : X \to F$ a G-invariant morphism. As $X \to Y$ is an fppf morphism and F is an fppf sheaf, the following sequence is exact:

$$F(Y) \xrightarrow{\pi^*} F(X) \rightrightarrows F(X \times_Y X)$$

and this sequence is isomorphic to this one:

$$F(Y) \xrightarrow{\pi^*} F(X) \underset{proj.}{\overset{act.}{\rightrightarrows}} F(G \times_S X).$$

And this proves the lemma. □

1.2.10. Proposition. *Let $E \to S$ be a vector bundle and $F \hookrightarrow E$ an $\mathbb{O}_S$-submodule of finite presentation. Then F is a subbundle of E if and only if $F \to S$ is flat.*

Proof. Let us assume F is a subbundle of E. Then by definition, $F \to S$ is flat. Conversely, let us suppose $F \to S$ is flat. Then, thanks to Proposition 1.2.7, we know that its sheaf of sections is locally free of finite rank. Then F is a vector bundle. We only need to show that E/F is also a vector bundle. Let us denote by $\mathcal{E}$ and by $\mathcal{F}$ the sheaves of sections of E and F. Then E and $\mathcal{E}$, and F and $\mathcal{F}$ determine each other. Moreover, for any $f : S' \to S$ base change and for any vector bundle $V \to S$, we have $(V \times_S S')_{|Zar} = f^*(V_{|Zar})$, and because a monomorphism of schemes remains a monomorphism after any base change, we know that the injection $\mathcal{F} \hookrightarrow \mathcal{E}$ remains injective after any base change. Then the cokernel $\mathcal{Y}$ of this injection is $\mathcal{O}_S$-flat. Because it is also of finite presentation, it is locally free of finite rank thanks to Proposition 1.2.7. Let us show now that $Y := \mathbf{Spec}(\mathrm{Sym}(\mathcal{Y}^\vee))$ is actually the quotient E/F. We have the following exact sequence:

$$0 \to \mathcal{F} \hookrightarrow \mathcal{E} \to \mathcal{Y} \to 0.$$

Dualizing this sequence, we obtain:

$$0 \to \mathcal{Y}^\vee \to \mathcal{E}^\vee \to \mathcal{F}^\vee \to 0.$$

As F is a subgroup of E, it acts on E by left translation. We then have the action morphism

$$\begin{aligned} F \times_S E &\to E \times_S E \\ (f, e) &\mapsto (f + e, e) \end{aligned}$$

given on the rings by:

$$\begin{aligned}\Phi : \mathrm{Sym}(\mathcal{E}^\vee) \otimes_{\mathrm{Sym}(\mathcal{O}_S^\vee)} \mathrm{Sym}(\mathcal{E}^\vee) &\to \mathrm{Sym}(\mathcal{F}^\vee) \otimes_{\mathrm{Sym}(\mathcal{O}_S^\vee)} \mathrm{Sym}(\mathcal{E}^\vee)\\ 1 \otimes X &\mapsto 1 \otimes X\\ X \otimes 1 &\mapsto \overline{X} \otimes 1 + 1 \otimes X\end{aligned}$$

for all $X \in \mathcal{E}^\vee = \mathrm{Sym}^1(\mathcal{E}^\vee)$. Using the definition we see that the elements of the form $X \otimes 1 - 1 \otimes X$ with $X \in \mathcal{Y}^\vee$ are in the kernel of Φ, then we obtain a factorized map

$$\tilde{\Phi} : \mathrm{Sym}(\mathcal{E}^\vee) \otimes_{\mathrm{Sym}(\mathcal{Y}^\vee)} \mathrm{Sym}(\mathcal{E}^\vee) \to \mathrm{Sym}(\mathcal{F}^\vee) \otimes_{\mathrm{Sym}(\mathcal{O}_S^\vee)} \mathrm{Sym}(\mathcal{E}^\vee).$$

Let us show that $\tilde{\Phi}$ is an isomorphism. First, one can see that the source and the target of $\tilde{\Phi}$ are sheaves of polynomial algebras, with the same number of variables, equal to $\mathrm{rk}(\mathcal{F}) + \mathrm{rk}(\mathcal{E})$. Moreover, $\tilde{\Phi}$ is surjective because

$$\tilde{\Phi}(1 \otimes X) = 1 \otimes X \text{ and}$$
$$\tilde{\Phi}(X \otimes 1 - 1 \otimes X) = \bar{X} \otimes 1.$$

Thus Lemma 1.2.8 2. shows that $\tilde{\Phi}$ is an isomorphism. Then we have an isomorphism

$$F \times_S E \xrightarrow{\sim} E \times_Y E.$$

Hence, using Lemma 1.2.9, we see that Y is the quotient of E by F in the category of fppf sheaves on S, so $E/F = Y = \mathbf{Spec}(\mathrm{Sym}(\mathcal{Y}^\vee))$ is a vector bundle and F is a subbundle of E. □

1.2.11. Proposition. *Let E_1 and E_2 be two generalized vector bundles. Let $f : E_1 \to E_2$ be a morphism of generalized vector bundles. If E_1 is of finite presentation and if E_2 is of finite type, then* $\ker(f)$ *is of finite presentation.*

Proof. By definition, we have:

$$\ker(f) = \mathbf{Spec}(\mathrm{Sym}(\mathcal{F}_1) \otimes_{\mathrm{Sym}(\mathcal{F}_2)} \mathcal{O}_S) = \mathbf{Spec}(\mathrm{Sym}(\mathcal{F}_1) \otimes_{\mathrm{Sym}(\mathcal{F}_2)} \mathrm{Sym}(\mathcal{F}_2)/(\mathcal{F}_2))$$

$$= \mathbf{Spec}(\mathrm{Sym}_{\mathrm{Sym}(\mathcal{F}_2)}(\mathcal{F}_1)/f^{\#}(\mathcal{F}_2)).$$

Then because $\mathcal{F}_1$ is of finite presentation and $\mathcal{F}_2$ is of finite type, $\ker(f)$ is of finite presentation. □

The next statement is a general result about images and kernels of morphisms of vector bundles, for which we could not find a proof in the literature. It gives conditions for the kernel and the image of a vector bundle morphism to be subbundles. For this result, we first recall that, for any morphism of schemes $f : X \to Y$, there exists a smallest closed subscheme of

Y that factorizes f. We denote it by $\mathrm{imsc}(f)$ and it is called *the schematic image of f*. See [GW20] Definition and Lemma 10.29.

1.2.12. Theorem. *Let $f : E := \mathbf{Spec}(\mathrm{Sym}(\mathcal{E}^\vee)) \to F := \mathbf{Spec}(\mathrm{Sym}(\mathcal{F}^\vee))$ be a morphism of S-vector bundles with kernel K and with image I. Then the following are equivalent:*

1) *$K \to S$ is flat.*
2) *$I \to S$ is representable by an S-scheme of finite presentation.*

Moreover, when these conditions are satisfied, we have:

(i) *K is a subbundle of E and I is a subbundle of F. Moreover, the induced morphism $E/K \to I$ is an isomorphism.*

(ii) *$I = \mathrm{imsc}(f)$.*

(iii) *The sheaf of sections of K is $\mathcal{K} := \ker(\mathcal{E} \to \mathcal{F})$, that of I is $\mathcal{I} := \mathrm{im}(\mathcal{E} \to \mathcal{F})$, and $\mathcal{E}/\mathcal{K} \simeq \mathcal{I}$.*
Moreover, the formation of K, I, and $\mathcal{K}, \mathcal{I}$ commute with base change.

Proof. 1) $\implies$ 2)
We denote by $\mathcal{K}$ the sheaf of sections of K. Because E and F are both of finite presentation, Proposition 1.2.11 tells us that K is of finite presentation, then we can apply Proposition 1.2.10 to say that K is a subbundle of E. Let us write the following exact sequence:

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{Q} \to 0$$

and let us denote $Y := \mathbf{Spec}(\mathrm{Sym}(\mathcal{Q}^\vee))$. Doing the same proof as in Proposition 1.2.10, we see that $\mathcal{Q}$ is locally free of finite rank, and Y is the quotient of E by K in the category of fppf sheaves on S. Then $I = Y = E/K$ is representable by an S-scheme of finite type, given by $I = \mathbf{Spec}(\mathrm{Sym}(\mathcal{Q}^\vee))$. Because $\mathcal{Q}$ is the cokernel of the injection $\mathcal{K} \hookrightarrow \mathcal{E}$, we can write $\mathcal{Q} \hookrightarrow \mathcal{F}$ so we get a surjection $\mathcal{F}^\vee \twoheadrightarrow \mathcal{Q}^\vee$ hence $I \hookrightarrow F$ is a closed immersion. But because I factorises f, by definition of the schematic image, we have $I \simeq \mathrm{imsc}(f)$.
2) $\implies$ 1)
Let us suppose I is representable by an S-scheme of finite presentation. In order to prove that $K \to S$ is flat, it is sufficient to prove that $E \to I$ is flat. Let $s \in S$. Then I_s is the image of $E_s \to F_s$ and K_s is its kernel. Because the formation of the kernel and of the image commutes with base change, we have an isomorphism of fppf sheaves

$$E_s/K_s \xrightarrow{\sim} I_s$$

then $E_s \to I_s$ is flat. Then, using the "critère de platitude par fibres" (see [EGA4], troisième partie, théorème 11.3.10), we obtain that $E \to I$ is flat. Moreover, the morphism $E \to I$ is surjective in the topological sense because it is surjective as a morphism of fppf sheaves, then $I \to S$ is flat.

Let us suppose now that these conditions are satisfied. Then looking at the proof of 1) $\Longrightarrow$ 2), we see that K and E/K are vector bundles on S. Using this same proof, we see that I is also a subbundle of F, and that $E/K \simeq I \simeq \mathrm{imsc}(f)$. The first part of (iii) is true because K is a subbundle of E and I is a subbundle of F. Then for the last assertion, we have to say that the formation of K commutes with base change because it is a kernel, then because K and $\mathcal{K}$ determine each other, we see that $\mathcal{K}$ commutes with base change. Finally, I commutes with base change because it is a quotient, and then $\mathcal{I}$ commutes with base change because it is determined by I. □

1.3 Lie algebra vector bundles

In the following, $\mathcal{L}$ is a Lie $\mathcal{O}_S$-algebra locally free of finite rank, whose bracket is denoted by $[\cdot,\cdot]$. We denote by $L := \mathbf{Spec}(\mathrm{Sym}(\mathcal{L}^\vee))$ the associated vector bundle, and $[\cdot,\cdot] : L \times L \to L$ the morphism of schemes we deduce from the bracket of $\mathcal{L}$, inducing a Lie S-algebra structure on L. We call these kinds of objects *Lie algebra vector bundles.*

We denote by $\mathcal{E}nd(\mathcal{L})$ the $\mathcal{O}_S$-module of $\mathcal{O}_S$-endomorphisms of $\mathcal{L}$, and $ad : \mathcal{L} \longrightarrow \mathcal{E}nd(\mathcal{L})$. We denote by $\mathrm{ad} : L \to \mathrm{End}(L) = \mathbf{Spec}(\mathrm{Sym}(\mathcal{E}nd(\mathcal{L})^\vee))$ the corresponding morphism of schemes, and for the end, we denote by $Z(L) := \ker(\mathrm{ad} : L \to \mathrm{End}(L))$ the *center* of L.

1.3.1. Remark. By definition, the formation of ad and $\mathrm{End}(L)$ commutes with base change, and because $\mathcal{L}$ is locally free of finite rank, the formation of $\mathcal{E}nd(\mathcal{L})$ does too.

1.3.2. Proposition. *The center $Z(L)$ of a Lie algebra vector bundle is of finite presentation.*

Proof. This is straightforward from Proposition 1.2.11. □

Then we see that we are in good conditions for using Proposition 1.2.7 with the center of a Lie algebra vector bundle.

Moreover, using Theorem 1.2.12 (iii), we know that if the center $Z(L) \to S$ is flat, then it is determined by its sheaf of Zariski sections, which is given by

$$\mathcal{Z}(\mathcal{L}) := \ker(ad : \mathcal{L} \to \mathcal{E}nd(\mathcal{L})),$$

and so we have

$$Z(L) = \mathbf{Spec}(\mathrm{Sym}(\mathcal{Z}(\mathcal{L})^\vee)).$$

1.3.3. Counter-example. The hypothesis "$Z(L) \to S$ flat" is essential. Here is a counter-example: let R be a ring and L be the Lie R-algebra with basis $\{x, y\}$ and bracket defined by $[x, y] = ax$ for some $a \in R$ such that $a \nmid 0$, that is, geometrically $L = \mathrm{Spec}(\mathrm{Sym}(Rx^* \oplus Ry^*))$. Let $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ be an open immersion. Hence R' is a flat R-algebra. Hence $a \nmid 0$ in R'. So using the previous notation, we have $\mathcal{Z}(\mathcal{L}) = 0$. But whenever R' is a $R/(a)$-algebra, we have $Z(L)(R') = L(R')$. Hence the center $Z(L)$ and its sheaf of Zariski sections do not determine each other.

1.3.4. Definition. Let $\mathcal{L}$ be an $\mathcal{O}_S$-module in Lie algebras. We define its *derived Lie algebra* $\mathcal{L}'$ as the image sheaf of $[\cdot,\cdot] : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$.

Let $L \to S$ be a Lie algebra generalised vector bundle. We define its *derived Lie algebra* L' as the fppf image sheaf of $[\cdot,\cdot] : L \otimes L \to L$.

In general, the derived Lie algebra is not representable. In fact, Theorem 1.2.12 tells us that it is representable if and only if the kernel of the bracket is flat. Moreover, in this situation, we have $L' = \mathrm{Spec}(\mathrm{Sym}(\mathcal{L}'^\vee))$.

2 The scheme of Lie p-algebra structures

2.1 The functor of p-mappings and the restrictable locus

From now on, S is a scheme of characteristic $p > 0$, and we globalize the definition of a p-mapping from Lie algebras to a definition on Lie algebra vector bundles as follows.

2.1.1. Definition. Let L be a Lie algebra generalised vector bundle. We say that a morphism of schemes $(\cdot)^{[p]} : L \to L$ is a *p-mapping* on L if for all S-schemes T, it is a p-mapping on $L(T)$.

2.1.2. Definition. Let $X \to S$ be an S-scheme, and let E be a generalised vector bundle over S. We say that X is a *formally principal homogeneous space under* E if E acts on X such that the action map

$$\begin{aligned} E \times_S X &\to X \times_S X \\ (e, x) &\mapsto (e \cdot x, x) \end{aligned}$$

is a scheme isomorphism.
Moreover, if E is a vector bundle, we say that X is a *formally affine space under* E. Moreover, if $X \to S$ has local sections, i.e. if $X \to S$ is a sheaf epimorphism for the fppf topology, we say that X is an *affine space under* E.

2.1.3. Remark. One can show that the second condition is equivalent to have local sections for the étale or for the Zariski topology. This is because $\mathrm{H}^1_{\mathrm{fppf}}(S, E) \simeq \mathrm{H}^1_{\mathrm{Zar}}(S, E)$. See Milnes's book on étale cohomology [Mi80], Chapter III, §3, Proposition 3.7.

Notations: Let X be a scheme of characteristic $p > 0$. We denote by $\mathrm{Frob}_X : X \to X$ or simply Frob the absolute Frobenius morphism of the scheme X.

Let $L \to S$ be a Lie algebra vector bundle. Let us denote by E the generalised vector bundle of Frobenius-semilinear morphisms between L and $Z(L)$:

$$\mathrm{E} := \mathrm{Hom}_{\mathrm{Frob}}(L, Z(L)) = \mathrm{Hom}(\mathrm{Frob}_S^* L, Z(L)) = (\mathrm{Frob}_S^* L)^\vee \otimes Z(L)$$

where the tensor product is taken in the category of vector bundles over S. If $Z(L)$ is a vector bundle, then so is E.

2.1.4. Theorem. *Let $L \to S$ be a Lie algebra vector bundle. Let us define a set-valued functor as follows:*

$$\begin{aligned} X : \{S\text{-}schemes\} &\longrightarrow \text{Set} \\ T &\longmapsto \Big\{p\text{-}mappings\ on\ L \times_S T\Big\}. \end{aligned}$$

Then, X is representable by an affine scheme, and is a formally principal homogeneous space under E*.*

Proof. Let $\mathcal{L}$ be the Zariski sheaf of sections of $L \to S$. Let us show that X is representable. Because the claim is local on the target, we can suppose $S = \mathrm{Spec}(R)$ affine, small enough so that $\mathcal{L}$ is free with basis $x_1, \dots, x_n$ on S, i.e.

$$\mathcal{L} = \mathcal{O}_S x_1 \oplus \cdots \oplus \mathcal{O}_S x_n \ ,\ x_i \in \mathcal{L}(S) = L(S) \text{ and } L = \mathrm{Spec}(\mathcal{O}_S[x_1^*, \dots, x_n^*]).$$

Let us define for all i the following morphism:

$$f_i : S \xrightarrow{x_i} L \xrightarrow{\mathrm{ad}} \mathrm{End}(L) \xrightarrow{\mathrm{p}} \mathrm{End}(L)$$

where $\mathrm{p} : \mathrm{End}(L) \to \mathrm{End}(L)$ maps an endomorphism to its p-power. Then, by definition of the fiber product, for all $T \to S$ and $i \in \{1, \dots, n\}$, we have

$$(L \times_{\mathrm{End}(L),(\mathrm{ad},f_i)} S)(T) = \{y \in L(T), \mathrm{ad}_y = (\mathrm{ad}_{x_i})^p_{|T}\}.$$

Then, by Jacobson's Theorem 1.1.6, the map

$$\begin{aligned} X &\to (L \times_{\mathrm{End}(L);(\mathrm{ad},f_1)} S) \times (L \times_{\mathrm{End}(L);(\mathrm{ad},f_2)} S) \times \cdots \times (L \times_{\mathrm{End}(L);(\mathrm{ad},f_n)} S) \\ \gamma &\mapsto (\gamma(x_1), \dots, \gamma(x_n)) \end{aligned}$$

is an isomorphism. This shows that X is representable. Let us show now that X is a formally principal homogeneous space under E. Let $T \to S$ be an S-scheme. We can write

$$\mathrm{E}(T) = \mathrm{Hom}_{\mathbb{G}_{a,T}-mod}(\mathrm{Frob}^* L \times T, Z(L) \times_S T) = \mathrm{Hom}_{\mathbb{G}_{a,T}-mod}(\mathrm{Frob}^* L \times_S T, Z(L \times_S T))$$

$$\text{and } X(T) = \Big\{p\text{-structures on } L \times_S T\Big\}.$$

Then the morphism

$$\begin{aligned} \mathrm{E} \times_S X &\to X \times_S X \\ (\phi, \gamma) &\mapsto (\phi + \gamma, \gamma) \end{aligned}$$

is well-defined and is an isomorphism thanks to Corollary 1.1.5. □

2.1.5. Remark. If we suppose moreover that $Z(L) \to S$ flat, then E is a vector bundle, so X is a formally affine space under E.

For the next theorem, we recall that a scheme $X \to S$ is said to be *essentially free* if we can find a cover of S by affine opens S_i, and for all i an affine and faithfully flat S_i-scheme S'_i, and a cover $(X'_{i,j})_j$ of $X'_i := X \times_S S'_i$ by affine opens X'_{ij} such that for all (i, j), the ring of functions of X'_{ij} is a free module on the ring of S'_i.

2.1.6. Theorem. *Let S be a scheme. Let $Z \to S$ be essentially free and let $Y \hookrightarrow Z$ be a closed subscheme of Z. Then, the Weil restriction defined by*

$$\begin{aligned} \Pi_{Z/S}(Y) : \{S\text{-}schemes\} &\longrightarrow \mathrm{Set} \\ T &\longmapsto \begin{cases} \{\emptyset\} \text{ if } Z_T = Y_T \\ \emptyset \text{ otherwise} \end{cases} \end{aligned}$$

is representable by a closed subscheme of S.

Proof. See [SGA3] Tome 2, exposé VIII, Théorème 6.4. □

2.1.7. Lemma. *Let*

$$0 \to K \to E \xrightarrow{\pi} F \to 0$$

be an exact sequence of vector bundles (i.e. seen as fppf sheaves) on a scheme S. Then, π is surjective, Zariski-locally on S.

Proof. By hypothesis, $E \to F$ is a K_F-torsor for the fppf topology. Let $f : S \to F$ be a section on F. Let $E \times_F S$ be the fiber product made with the section f. Then by base change, $E \times_F S$ is a $K_F \times_F S$-torsor for the fppf topology. But $K_F \times_F S = K$ and

$$\mathrm{H}^1_{fppf}(S, K) = \mathrm{H}^1_{Zar}(S, K)$$

because K is a vector bundle over S (see [Mi80] for more details). Then $E \times_F S$ is a K-torsor over S, for the Zariski topology. Then there exists a covering of open immersions $g : S' \to S$

and $h : S' \to E \times_F S$ such that this diagram commutes:

$$\begin{array}{ccc} & & S' \, . \\ & \overset{h}{\swarrow} & \downarrow g \\ E \times_F S & \longrightarrow & S \\ \mathrm{pr}_1 \downarrow & & \downarrow f \\ E & \underset{\pi}{\longrightarrow} & F \end{array}$$

Then the Zariski section we are looking for is given by $\mathrm{pr}_1 \circ h : S' \to E$. □

2.1.8. Theorem. *Let $L \to S$ be a Lie algebra vector bundle whose center $Z(L) \to S$ is flat. Let us recall the notation $\mathrm{E} := \mathrm{Hom}_{\mathrm{Frob}}(L, Z(L))$. Let $X \to S$ be the functor of p-mappings on L defined above, and let $S^{\mathrm{res}} = S^{\mathrm{res}}(L)$ be defined as:*

$$S^{\mathrm{res}} : \{S\text{-}schemes\} \longrightarrow \mathrm{Set}$$
$$T \longmapsto \begin{cases} \{\emptyset\} \text{ if } L_T \text{ is Zar-loc. restrictable over } T \\ \emptyset \text{ otherwise.} \end{cases}$$

Then the following two conditions are verified:

1. *S^{res} is representable by a closed subscheme of S.*
2. *$X \to S$ factors through S^{res} and $X \to S^{\mathrm{res}}$ is an affine space under the vector bundle $\mathrm{E} \times_S S^{\mathrm{res}}$.*

2.1.9. Remark.

- The functor S^{res} could have been defined as the unique sub-functor of S such that

$$L_T \text{ is Zar-loc. restrictable} \Leftrightarrow T \to S \text{ can be factorized by } S^{\mathrm{res}}.$$

  Indeed $T \to S$ can be factorized by S^{res} if and only if $S^{\mathrm{res}}(T) \neq \emptyset$ if and only if L_T is Zar-loc. restrictable. Let F be a subfunctor of S such that L_T is Zar-loc. restrictable if and only if $T \to S$ can be factorized by F. But for all T, $F(T) \subset \mathrm{Hom}_S(T, S) = \{*\}$. Hence by definition, S^{res} is the only subfunctor of S satisfying the property above.
- We could have defined S^{res} to be the locus where a Lie algebra is fppf-loc. restrictable, because this is less restrictive, but the following results will show that those conditions are the same.
- By Yoneda, we can see $X(X) = \mathrm{Hom}_S(X, X) \neq \emptyset$ because $\mathrm{id} \in X(X)$. But by definition, $\mathrm{id} \in X(X)$ corresponds to a p-mapping on L_X. Then L_X is Zar-loc. restrictable and we call this mapping the *universal p-mapping on L_X*.

Proof. 1. Let I be the image of ad. Because $Z(L)$ is flat, I is a subbundle of $\mathrm{End}(L)$ by

Theorem 1.2.12 (i). Let $\rho : I \to \mathrm{End}(L)$ be the p-th power map, restricted to I. Let $W = W(L)$ be the subfunctor of S defined by:

$$W : \{S\text{-schemes}\} \longrightarrow \mathrm{Set}$$
$$T \longmapsto \begin{cases} \{\emptyset\} \text{ if } I_T \text{ is stable by } \rho \\ \emptyset \text{ otherwise.} \end{cases}$$

Let us show that W is representable by a closed subscheme of S. Let $T \to S$ be an S-scheme. Then, I_T is ρ-stable if and only if $\rho^{-1}(I_T) \xrightarrow{\sim} I_T$. But $I \hookrightarrow \mathrm{End}(L)$ is closed thanks to Theorem 1.2.12 (ii), and closed immersion are stable by base change. Then $\rho^{-1}(I)$ is a closed subscheme of I. We know that $I \to S$ is essentially free because it is a vector bundle, then using Theorem 2.1.6 with $\rho^{-1}(I) \hookrightarrow I$, we see that W is a closed subscheme of S.

Let us now show that $W = S^{\mathrm{res}}$. First, let us show $S^{\mathrm{res}} \subset W$. Let $T \to S$. If $S^{\mathrm{res}}(T) = \emptyset$, there is nothing to prove. Let us suppose $S^{\mathrm{res}}(T) \neq \emptyset$. Then by definition L_T is Zar-loc. restrictable over T, hence there exists a p-mapping on L_T, locally on T for the Zariski topology. We denote this p-mapping by γ. We want to show that I_T is stable by ρ. That means we want to show that there exists a map $\sigma : I_T \to I_T$ such that the following diagram commutes:

$$\begin{array}{ccc} I_T & \xrightarrow{\rho_T} & \mathrm{End}(L_T) \\ & \searrow^{\sigma} & \uparrow i \\ & & I_T. \end{array}$$

Thanks to Theorem 1.2.12, we know that $I_T = L_T/Z(L_T)$. But $Z(L_T) \subset L_T$ is an ideal of L_T, and thanks to (AL 1), it is stable by any p-mapping, so it is stable by γ. Then γ induces a p-mapping that we can write $\sigma : L_T/Z(L_T) \to L_T/Z(L_T)$ by Proposition 1.1.3. If we denote by $\pi : L_T \to L_T/Z(L_T)$ the quotient morphism, we have a commutative diagram, where $\mathrm{p} : \mathrm{End}(L_T) \to \mathrm{End}(L_T)$ is the p-power:

$$\begin{array}{ccccc} L_T & \xrightarrow{\pi} & L_T/Z(L_T) = I_T & \xhookrightarrow{i} & \mathrm{End}(L_T) \\ \downarrow \gamma & & \downarrow \sigma & & \downarrow \mathrm{p} \\ L_T & \xrightarrow[\pi]{} & L_T/Z(L_T) = I_T & \xhookrightarrow[i]{} & \mathrm{End}(L_T) \end{array}$$

The big square in this diagram is commutative thanks to axiom (AL 1). As $\rho = p \circ i$, we can calculate

$$\rho \circ \pi = \mathrm{p} \circ i \circ \pi = i \circ \pi \circ \gamma = i \circ \sigma \circ \pi.$$

But π is an epimorphism in the category of schemes, then we obtain $\rho = i \circ \sigma$, i.e ρ factors via i as we wanted. Then, $S^{\mathrm{res}} \subset W$.

Conversely, let $T \to S$ be such that I_T is stable by ρ. Let us show that $X(T)$ is nonempty, locally for the Zariski topology on T. As everything is local on S, and Z_T is locally a direct factor in L_T, we can assume T is affine, small enough such that $L = \mathbf{Spec}(\mathcal{O}_T[x_1^*, \ldots, x_n^*])$. Then we have the exact sequence

$$0 \to Z(L) \to L \xrightarrow{\text{ad}} I \to 0$$

and Lemma 2.1.7 says that ad is surjective, Zariski locally on T. Then, thanks to Jacobson's Theorem 1.1.6, we know that we have existence of a p-mapping, Zariski locally on T. Then $S^{\text{res}} = W$, so S^{res} is representable by a closed subscheme of S.

2. Let $T \to S$ be an S-scheme. Let $\gamma \in X(T)$. Then by definition, L_T is restrictable so $S^{\text{res}}(T) = \{\emptyset\}$. Then we define this map:

$$X(T) \to S^{\text{res}}(T)$$
$$\gamma \mapsto \emptyset$$

that factorizes $X \to S$. Thanks to Theorem 2.1.4, and because $X \times_S X = X \times_{S^{\text{res}}} X$, we know that

$$\mathrm{E}_{S^{\text{res}}} \times_{S^{\text{res}}} X \simeq X \times_{S^{\text{res}}} X.$$

We need to show that $X \to S^{\text{res}}$ is a sheaf epimorphism for the fppf topology. It suffices to show that for all $T \to S$ such that L_T is Zar-loc. restrictable, we can find an fppf morphism $T' \to T$ such that there exists a p-mapping on $L_{T'}$. We just have to take for T' the Zariski covering on which L_T possesses a p-mapping. □

2.1.10. Corollary. *With the same hypothesis, if $Z(L) = \{0\}$, then $X \simeq S^{\text{res}}$, so $X \to S$ is a closed immersion.*

2.2 A case of existence of local p-mapping

In general it is not easy to decide if a given finite-dimensional Lie algebra or a Lie algebra vector bundle admits a p-mapping. Here is a brief review of the easiest cases we have already seen, where such existence is known to hold:

1. Associative Lie algebras, with the Frobenius map.
2. Lie algebras of group schemes.
3. Lie algebras whose Killing form is nondegenerate (Zassenhaus).
4. Somewhat opposite to 3. is the abelian case, where $\gamma = 0$ is a p-mapping.

The last case corresponds to the situation where the derived Lie algebra has rank 0. In the rest of the section, we will extend that case to the mildly non-abelian case where the derived Lie algebra has rank 1.

2.2.1. Theorem. *Let $L \to S$ be a Lie algebra vector bundle, such that L' is a locally free subbundle of rank 1. Then, there exist p-mappings, Zariski-locally on L. So in this case $S^{\mathrm{res}} = S$.*

Proof. In order to prove this, we can suppose L is free, such that L' is free, given by $L' = \mathbb{G}_a \cdot v$. Then the bracket is given on the functor of points by:

$$\begin{aligned} [\cdot,\cdot] : L \times L &\to L \\ (x, y) &\mapsto f(x, y)v \end{aligned}$$

where f is a bilinear alternating form.
Let us write $\{x_1, \cdots, x_n\}$ for a basis of L. For any $i \in [1, \cdots, n]$, we write

$$y_i := f(x_i, v)^{p-1} x_i.$$

Then for any $y \in L$ we can write :

$$\mathrm{ad}_{y_i}(y) = f(f(x_i, v)^{p-1} x_i, y)v = f(x_i, v)^{p-1} f(x_i, y)v.$$

Moreover, we can write

$$\mathrm{ad}_{x_i}(\mathrm{ad}_{x_i}(y)) = \mathrm{ad}_{x_i}(f(x_i, y)v) = f(x_i, y)\,\mathrm{ad}_{x_i}(v) = f(x_i, y)f(x_i, v)v.$$

Then by induction, we find

$$\mathrm{ad}^p_{x_i}(y) = f(x_i, y)f(x_i, v)^{p-1}v.$$

Thus, (AL 1) is checked on the basis $\{x_1, \cdots, x_n\}$. Therefore, thanks to Jacobson's theorem 1.1.6, we know that there is a p-mapping on L.

□

3 The moduli space of Lie p-algebras of rank 3

In the remaining sections, we illustrate the previous results in the case of three-dimensional Lie algebras. As stated in the introduction, let us denote by $\mathcal{L}ie_n$ the moduli stack of n-dimensional Lie algebras, and L_n the moduli space of *based* Lie algebras. Then we have the quotient stack presentation $\mathcal{L}ie_n = [\mathrm{L}_n / \mathrm{GL}_n]$ where GL_n acts by change of basis, by this action for any S-scheme T:

$$\begin{aligned} \mathrm{GL}_n(T) \times \mathrm{L}_n(T) &\to \mathrm{L}_n(T) \\ (M, [\cdot,\cdot]_T) &\mapsto [\cdot,\cdot]'_T := \Big(v \otimes w \mapsto M^{-1}[Mv, Mw]_T\Big). \end{aligned}$$

Hence we are led to studying the GL_n-equivariant geometry of L_n. In the following we will focus on the case $n = 3$. For a fixed prime p, we are interested in the moduli stack $p\text{-}\mathcal{L}ie_3$ of restricted Lie algebras. For this, we use the morphism $\pi : \ p\text{-}\mathcal{L}ie_3 \to \mathcal{L}ie_3$ to the moduli stack of three-dimensional Lie algebras. Thanks to Theorem 2.1.8, after passing to the flattening stratification of the center of the universal Lie algebra, the map π is an affine bundle, so before studying $p\text{-}\mathcal{L}ie_3$, we will focus on $\mathcal{L}ie_3$, i.e. on L_3.

For our purposes, it is important to obtain a description available in all characteristics. Even better, by defining L_3 as a functor over $\mathbb{Z}$ and proving its representability we gain insight into its scheme structure and the way the fibers vary. For all this section, we denote by $\mathrm{L}_{3,k}$ the base change of L_3 with a field k. Here is a summary of our main results:

3.0.1. Theorem.

1) *The functor* L_3 *is representable by an affine flat* $\mathbb{Z}$*-scheme of finite type.*
2) *The scheme* L_3 *has two relative irreducible components* $\mathrm{L}_3^{(1)}$ *and* $\mathrm{L}_3^{(2)}$ *which are both flat with Cohen-Macaulay integral geometric fibers of dimension* 6*.*

In 2) it is noteworthy that the component we call $\mathrm{L}_3^{(1)}$ is very simple: it is isomorphic to 6-dimensional affine space $\mathbb{A}^6_{\mathbb{Z}}$. This is crucial because it turns out that the other component $\mathrm{L}_3^{(2)}$ is *linked* to it in the sense of liaison theory as developed by Peskine and Szpiro in [PS74], which provides powerful tools to deduce its properties.

Here we use the terminology "relative irreducible components" in the sense that $\mathrm{L}_3^{(1)}$ and $\mathrm{L}_3^{(2)}$ are flat of finite presentation over $\mathbb{Z}$, and that for all algebraically closed fields k, $\mathrm{L}_{3,k}^{(1)}$ and $\mathrm{L}_{3,k}^{(2)}$ are the irreducible components of $\mathrm{L}_{3,k}$. We use this terminology because we are over the ring of integers $\mathbb{Z}$ then it makes more sense. For more details the reader can have a look at [Ro11], where the definition is given in 2.1.1, with a small (but not important for us) difference. Following the notation of *loc. cit.*, we will show in the following that $\mathrm{Irr}(\mathrm{L}_3/\mathbb{Z}) = \mathrm{Spec}(\mathbb{Z}) \amalg \mathrm{Spec}(\mathbb{Z})$.

3.1 Classification over an algebraically closed field

To begin with, we recall the classification of isomorphism classes of three-dimensional Lie algebras over any algebraically closed field k. That is, the description of the (GL_3-orbits of) geometric points of the moduli space $\mathrm{L}_{3,k}$. Historically, the isomorphism classes of complex and real three-dimensional Lie algebras were classified as early as 1898 in Bianchi's paper [Bi98]. After the development of the algebraic theory of Lie algebras, the topic appeared in the lecture notes of Jacobson's course [J62]. From this moment the focus shifted to the algebraic variety structure of the set L_n of n-dimensional Lie algebras in work of Vergne [V66], Carles [Carl79], Carles and Diakité [CD84], Kirillov and Neretin [KN84] and others. There, emphasis was put on low dimensions. Note that this bibliographic selection is by no means complete. Here, in order to allow varying primes p, we need to reformulate the classification of 3-dimensional Lie algebras over algebraically closed fields in a characteristic-free way.

3.1.1. Some Lie algebras: four discrete ones, and a family.

We introduce the five Lie algebras involved in the classification in a way that allows a characteristic-free statement. Notationally speaking, if l is a Lie algebra over a ring R, free of rank 3 with basis $\{x, y, z\}$ and bracket defined by $[x, y] = ax + by + cz$, $[x, z] = dx + ey + fz$, $[y, z] = gx + hy + iz$ for some coefficients $a, \ldots, i \in R$, then we say that

$$\text{“}\textit{the Lie algebra structure of } l \textit{ is given by the matrix } \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}.\text{”}$$

Moreover, for any Lie algebra l and any $v \in l$, the map ad_v is linear, so we will always represent this linear map by its matrix in the base $\{x, y, z\}$.

The first four Lie algebras are defined over the ring of integers $R = \mathbb{Z}$:

(1) the *abelian Lie algebra* $\mathfrak{ab}_3$ with structure given by the zero matrix,

(2) the *Heisenberg Lie algebra* $\mathfrak{h}_3$ with structure matrix $\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$,

(3) the *Lie algebra* $\mathfrak{r}$, with structure matrix $\left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)$,

(4) the *simple Lie algebra* $\mathfrak{s}$ with structure matrix $\left(\begin{smallmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right)$.

The fifth Lie algebra is a family defined over the polynomial ring $R = \mathbb{Z}[T]$:

(5) the Lie algebra $\mathfrak{l}_T$ is defined by the structure matrix $\left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & T & 0 \end{smallmatrix}\right)$.

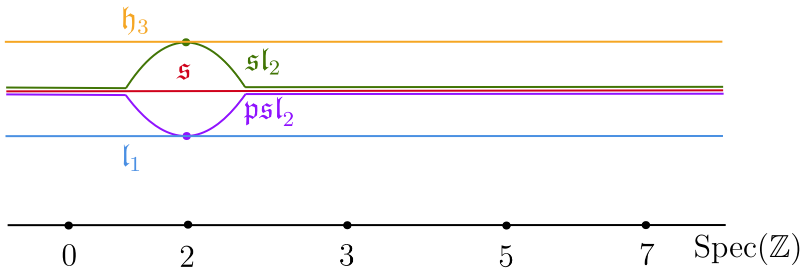
3.1.2. More about the simple Lie algebra.

The reader wondering about the place of $\mathfrak{sl}_2$ and $\mathfrak{psl}_2$ in the picture will find the following explanations useful. Let us write $\{X, Y, H\}$ and $\{X', Y', Z'\}$ for the classical bases of $\mathfrak{sl}_2$ and $\mathfrak{psl}_2$, and $\{x, y, z\}$ for that of $\mathfrak{s}$. We can write a sequence of morphisms of $\mathbb{Z}$-Lie algebras:

$$\mathfrak{sl}_2 \xrightarrow{\pi} \mathfrak{psl}_2 \xrightarrow{f} \mathfrak{s} \xhookrightarrow{\mathrm{ad}} \mathfrak{gl}_3$$

with π and f given by $X \mapsto X' \mapsto 2x$, $Y \mapsto Y' \mapsto y$, $H \mapsto 2Z' \mapsto 2z$. The morphism f is an isomorphism over $\mathbb{Z}[1/2]$, but a contraction onto the subalgebra generated by y in the fiber at the prime $p = 2$. For any algebraically closed field k, the Lie algebra $\mathfrak{s} \otimes k$ is the only simple three-dimensional Lie algebra over k because in characteristic $p \neq 2$ we have $\mathfrak{s} \otimes k \simeq \mathfrak{sl}_2 \otimes k$, while if $p = 2$ the algebra $\mathfrak{s} \otimes k$ is known as $W(1, \underline{2})'$, the derived algebra of the *Jacobson-Witt algebra*. See [SF88], § 4.2 for more on $W(n, \underline{m})$, and especially Strade's paper [S07], Theorem 3.2 for the case of characteristic 2. In characteristic $p = 2$, the algebra $\mathfrak{sl}_2$ happens to be isomorphic with the Heisenberg algebra $\mathfrak{h}_3$. Moreover, again when 2 is invertible the morphism π is an isomorphism so $\mathfrak{psl}_2$ is isomorphic to $\mathfrak{sl}_2$ i.e. to $\mathfrak{s}$. But in characteristic 2, using the adjoint representation of $\mathfrak{psl}_2$ in $\mathfrak{gl}_3$ we see the bracket is given by the one denoted by $\mathfrak{l}_1$ is the above classification. The following picture gives a summary of the situation. Note that in characteristic 2, the Lie algebra $\mathfrak{sl}_2$ is restrictable not simple while the Lie algebra $\mathfrak{s}$ is simple not restrictable.

Figure II.1 – Representation of different fibers of Lie algebras over Spec($\mathbb{Z}$)



It can be surprising to find that the group U_3 of upper-triangular unipotent matrices of size 3 and the reductive group SL_2 have the same Lie algebra in characteristic 2. But those Lie algebras are not isomorphic as restricted Lie algebras. Indeed, let k be a field of characteristic 2. Seeing SL_2 as a subgroup of GL_2, we can see $\mathfrak{sl}_2$ as a 2-subalgebra of $M_2(k)$, where the 2-mapping on $M_2(k)$ is the square map. Then we obtain that the 2-mapping on $\mathfrak{sl}_2$ is the one given on a basis $\{x, y, z\}$ by $x, y \mapsto 0$ and $z \mapsto z$. Doing the same for U_3, i.e. seeing the group U_3 as a subgroup of GL_3, we obtain that the 2-mapping on $\mathfrak{h}_3$ is the one sending the basis $\{x, y, z\}$ on 0, and so the 2-mapping is given on any vector by

$$ax + by + cz \mapsto bcx.$$

We can see that there is no element sent to itself, so the two mappings are different.

3.1.3. More about the family $\mathfrak{l}_T$. The Lie algebra $\mathfrak{l}_0$ has center of dimension 1 and a 1-dimensional derived Lie algebra $\mathfrak{g}'_0 = \mathrm{Span}(y)$. Now let us suppose $t \in k$ for some field k and $t \neq 0$. Then the Lie algebra $\mathfrak{l}_t$ has a trivial center and 2-dimensional derived Lie algebra $\mathfrak{g}'_t = \mathrm{Span}(y, z)$. The adjoint action $\mathrm{ad} : \mathfrak{l}_t \to \mathrm{End}(\mathfrak{g}'_t)$ factors through $\mathfrak{l}_t^{\mathrm{ab}} := \mathfrak{l}_t/\mathfrak{g}'_t$ which is free of rank 1. Any generator of $\mathfrak{l}_t^{\mathrm{ab}}$ is of the form ux for some unit $u \in k^\times$ and acts on $\mathfrak{g}'_t$ with eigenvalues $\{u, ut\}$. We see that the ratio of eigenvalues is well-defined up to inversion: that is, the class of t modulo the equivalence relation $t \sim t^{-1}$ is independent of u and thus intrinsic to $\mathfrak{l}_t$. In this way we see that for every field k and elements $t, t' \in k^\times$ we have: $\mathfrak{l}_t \simeq \mathfrak{l}_{t'}$ if and only if $t' \in \{t, t^{-1}\}$.

Here is the main theorem of this subsection:

3.1.4. Theorem. *Let k be an algebraically closed field and denote by p its characteristic. Then any Lie algebra of dimension 3 over k is isomorphic to exactly one in the following table.*

<table>
<tr><th colspan="2">Name</th><th>Structure</th><th>Orbit dimension</th><th>Center dimension</th><th colspan="2">Restrictable</th></tr>
<tr><td colspan="2">$\mathfrak{ab}_3$</td><td>abelian</td><td>0</td><td>3</td><td colspan="2">yes</td></tr>
<tr><td colspan="2">$\mathfrak{h}_3$</td><td>nilpotent</td><td>3</td><td>1</td><td colspan="2">yes</td></tr>
<tr><td colspan="2">$\mathfrak{r}$</td><td>solvable</td><td>5</td><td>0</td><td colspan="2">no</td></tr>
<tr><td colspan="2" rowspan="2">$\mathfrak{s}$</td><td rowspan="2">simple</td><td rowspan="2">6</td><td rowspan="2">0</td><td>$p \neq 2$</td><td>$p = 2$</td></tr>
<tr><td>yes</td><td>no</td></tr>
<tr><td rowspan="4">$\mathfrak{l}_t$</td><td>$\bar{t} \notin \mathbb{F}_p/\sim$</td><td>solvable</td><td>5</td><td>0</td><td colspan="2">no</td></tr>
<tr><td>$\bar{t} \in \mathbb{F}_p/\sim \setminus \{\bar{0}, \bar{1}\}$</td><td>solvable</td><td>5</td><td>0</td><td colspan="2">yes</td></tr>
<tr><td>$\bar{t} = \bar{0}$</td><td>solvable</td><td>5</td><td>1</td><td colspan="2">yes</td></tr>
<tr><td>$\bar{t} = \bar{1}$</td><td>solvable</td><td>3</td><td>0</td><td colspan="2">yes</td></tr>
</table>

Let us remark that it is almost the same classification as in characteristic 0, the difference is that the Lie algebra $\mathfrak{s}$ is changed by $\mathfrak{sl}_2$, but those Lie algebras are isomorphic when $p \neq 2$.

We split the proof in three parts: first of all we list the isomorphism classes (3.1.5), then we compute the dimensions of the orbits (3.1.7) and finally we determine the restrictable Lie algebras (3.1.8).

3.1.5. Proof of the statement on isomorphism classes. In order to have the list of the different orbits, we are following the proof in Fulton and Harris's book [FH91], Chapter 10. In this chapter the proof is divided in three parts, depending on the dimension of the derived Lie algebra. In this book though, the classification is done over the ring of complex numbers. The reader can verify that the proof can be generalised to any field of characteristic $\neq 2$, and up to a change of basis for the Lie algebra whose Lie structure is given by the matrix

$$\begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix},$$

we find the classification we claim in the theorem (indeed changing X into $2X$ and H into $2H$, we find the Lie algebra $\mathfrak{s}$).

Now let us suppose $\mathrm{char}(k) = 2$. The reader can verify that the proof done in [FH91] can still be generalised until the *loc. cit.* §10.4, where the authors consider Lie algebras with derived Lie algebra of rank 3. Indeed, in this part, they use an argument that is no longer true in characteristic 2: a certain endomorphism denoted by ad_H has three eigenvalues: $0, \alpha$ and $-\alpha$, and because $\alpha \neq 0$, these three eigenvalues are different, then this endomorphism is diagonalizable. So now let us transform this argument in our case. So let $\mathfrak{g}$ be a Lie algebra over k, with derived Lie algebra of rank 3. Let us do the same proof as done in *loc. cit.* §10.4 until this argument.

Then, changing the eigenvector X of ad_H for the eigenvalue α into αX, and changing H in $\alpha^{-1}H$, we find that ad_H has 0 and 1 as eigenvalues. If ad_H is diagonalizable, we can apply the proof of [FH91]. Otherwise, we can apply the Jordan–Chevalley decomposition to ad_H and so we can suppose there is a basis $\{X, Y, H\}$ of $\mathfrak{g}$ such that $[H, X] = X$ and $[H, Y] = X + Y$. Then thanks to the Jacobi condition, we know that

$$[H, [X, Y]] = [X, [H, Y]] + [Y, [H, X]] = [X, Y] + [X, Y] = 0.$$

Then $[X, Y] = \beta H$ with $\beta \neq 0$ because the derived Lie algebra of $\mathfrak{g}$ is of dimension 3. Changing X into aX and Y into aY where $a^2 = \beta^{-1}$, we can suppose $\beta = 1$ and using the matrix notation, we can suppose the bracket of $\mathfrak{g}$ is given by

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

in the basis $\{X, Y, H\}$. Using the basis $x = X, y = X + Y + H$ and $z = X + H$, we obtain

$$\begin{cases} [x, y] = [X, Y] + [X, H] = H + X = z \\ [x, z] = [X, H] = X = x \\ [y, z] = [X, H] + [Y, X] + [Y, H] + [H, X] = [X, Y] + [Y, H] = H + X + Y = y. \end{cases}$$

Hence we finally find the Lie algebra structure of $\mathfrak{s}$, so we find our classification. □

3.1.6. Remark. Here we use the terminology of [FH91] for the Lie algebras $\mathfrak{l}_t$, in particular for the Lie algebra $\mathfrak{l}_{-1}$. Actually you can find in the literature (for example in [KN84]) the terminology m(2) for this one. This name is due to the fact it is the Lie algebra of the group M(2) of euclidean motions of the plane.

3.1.7. Proof of the statement on the dimension of the orbits.

From now on, we use the notation $o(l)$ for the orbit of a Lie algebra l under the group GL_3. In order to find the dimension of the orbits, we can calculate the dimension of the stabilizer, and use the orbit-stabilizer relation. Let l be a Lie algebra over k, i.e. $l \in \mathrm{L}_{3,k}(k)$. Then the orbit of l is the image of this k-morphism:

$$\begin{aligned} \mathrm{GL}_3(k) &\longrightarrow \mathrm{L}_{3,k}(k) \\ A &\mapsto A \cdot l. \end{aligned}$$

Let $A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \in \mathrm{GL}_3(k)$ be a matrix in the stabilizer of $o(l)$. Then, we write

$$[Av, Aw] = A[v, w]$$

for the elements of the basis and we can find the equations for the stabilizer. For example let us fix a t in some field k and let us do it for $\mathfrak{l}_t$. We obtain theses conditions:

$$\begin{cases} (a_{1,1}a_{2,2} - a_{2,1}a_{1,2})y + t(a_{1,1}a_{3,2} - a_{3,1}a_{1,2})z & = a_{1,2}x + a_{2,2}y + a_{3,2}z, \\ (a_{1,1}a_{2,3} - a_{1,3}a_{2,1})y + t(a_{1,1}a_{3,3} - a_{3,1}a_{1,3})z & = ta_{1,3}x + ta_{2,3}y + ta_{3,3}z \\ (a_{1,2}a_{2,3} - a_{2,2}a_{1,3})y + t(a_{1,2}a_{3,3} - a_{1,3}a_{3,2})z & = 0 \end{cases}$$

Then for instance if $t = 0$, the conditions of the stabilizer are now:

$$\begin{cases} a_{1,2} = a_{3,2} = 0,\ a_{1,1}a_{2,2} = a_{2,2} \\ a_{1,1}a_{2,3} - a_{1,3}a_{2,1} = 0 \\ a_{2,2}a_{1,3} = 0 \end{cases}.$$

But $\det(A) = a_{2,2}a_{3,3} \neq 0$ then $a_{1,3} = 0$ and $a_{1,1} = 1$, so $a_{2,3} = 0$. Hence

$$\mathrm{Stab}(o(\mathfrak{l}_0)) = \left\{ A \in \mathrm{GL}_3(k), A = \begin{pmatrix} 1 & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 \\ a_{3,1} & 0 & a_{3,3} \end{pmatrix} \right\}.$$

Then $\dim(\mathrm{Stab}(o(\mathfrak{l}_0))) = 4$, so $\dim(o(\mathfrak{l}_0)) = 5$. Now let us suppose $t \neq 0$ and $t \neq 1$. Doing the same type of calculation, we obtain again:

$$\mathrm{Stab}(o(\mathfrak{l}_t)) = \left\{ A \in \mathrm{GL}_3(k), A = \begin{pmatrix} 1 & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 \\ a_{3,1} & 0 & a_{3,3} \end{pmatrix} \right\}.$$

Then $\dim(\mathrm{Stab}(o(\mathfrak{l}_t))) = 4$ and $\dim(o(\mathfrak{l}_t)) = 5$.

We can do the same calculations for the other orbits in order to find the announced dimensions. The details are left to the reader. □

3.1.8. Proof of the statement on the restricted orbits. Now we can have a look at the restrictable orbits. Let us suppose for this section that $\mathrm{char}(k) = p > 0$.

1. On the abelian Lie algebra, $\gamma \equiv 0$ is a p-mapping.
2. The Lie algebra $\mathfrak{h}_3 = \mathrm{Lie}(\mathrm{U}_3)$ is algebraic, hence restrictable.
3. The Lie algebra $\mathfrak{s}$ is restrictable if $\mathrm{char}(k) \neq 2$, because then $\mathfrak{s} \simeq \mathfrak{sl}_2$ so it is algebraic. But if $\mathrm{char}(k) = 2$, $\mathfrak{s}$ is not restrictable: one can see that ad_x^2 is not a linear combination

of ad_x, ad_y and ad_z, then the condition (AL 1) can not be verified.

4. Let $l := \mathfrak{r}$ with basis $\{x, y, z\}$. We have

$$\mathrm{ad}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \ \mathrm{ad}_y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \mathrm{ad}_z = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Then we have

$$(\mathrm{ad}_x)^p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence $(\mathrm{ad}_x)^p$ is not a linear combination of ad_x, ad_y and ad_z, so we conclude that $\mathfrak{r}$ is not restrictable.

5. For the end let $t \in k$ and let us have a look at the Lie algebra $\mathfrak{l}_t$ with basis $\{x, y, z\}$. We have

$$\mathrm{ad}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}; \ \mathrm{ad}_y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \mathrm{ad}_z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -t & 0 & 0 \end{pmatrix}.$$

Then we have

$$(\mathrm{ad}_x)^p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^p \end{pmatrix} \text{ and } (\mathrm{ad}_y)^p = (\mathrm{ad}_z)^p \equiv 0.$$

Then using Theorem 1.1.6 (Jacobson's theorem), and the definition of a restrictable Lie algebra, we know that l is restrictable if and only if $t^p = t$ i.e. if and only if $t \in \mathbb{F}_p$. □

3.1.9. Example. Thanks to this classification, we can illustrate Theorem 2.1.8. Indeed, let k be an algebraically closed field of characteristic $p > 0$. Let $\mathfrak{l}_T := \mathrm{Spec}(k[T]) \to \mathrm{L}_3$, given on the rings by $a, c, d, e, g, h, i \mapsto 0$, $b \mapsto 1$ and $f \mapsto T$. Let us calculate $\mathrm{L}_3^{\mathrm{res}} \times \mathfrak{l}_T$. Thanks to what we have done before, we know

$$\mathrm{L}_3^{\mathrm{res}} \times \mathfrak{l}_T = \mathrm{Spec}\, k[T]/(T^p - T).$$

Then we see that $\mathrm{L}_3^{\mathrm{res}} \times \mathfrak{l}_T$ is closed in $\mathrm{Spec}(k[T])$.

3.1.10. First consequences for the topology of $\mathrm{L}_{3,k}$.

To finish this subsection, we derive the first topological description of the irreducible components of the moduli space that the classification just given affords. Finer information can only be obtained with the more advanced algebraic tools of liaison theory presented in Subsection 3.2. First, note that:

— the points corresponding to the Lie algebras $\mathfrak{ab}_3$ and $\mathfrak{h}_3$ are in the closure of the orbit of the simple algebra $\mathfrak{s}$;

— the point corresponding to the Lie algebras $\mathfrak{r}$ is in the closure of the orbit of the 1-parameter algebra $\mathfrak{l}_T$ (to see this, let k be a field and let $t \in k$, and consider the Lie algebra defined by the structure matrix $\left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & t & 0 \end{smallmatrix}\right)$. For $t \neq 1$, the structure constants of this algebra in the basis $\{x, y, y + (t-1)z\}$ are those of $\mathfrak{l}_t$ and when $t \to 1$ the limit of this family is $\mathfrak{r}$).

Therefore, in order to single out the irreducible components of $\mathrm{L}_{3,k}$ it is enough to look at $o(\mathfrak{s})$ and $o(\mathfrak{l}_T)$. We consider their orbit morphisms:

$$\mathrm{ev}_{\mathfrak{s}} : \mathrm{GL}_3 \times \mathrm{Spec}(\mathbb{Z}) \longrightarrow \mathrm{L}_3 \quad , \quad \mathrm{ev}_{\mathfrak{l}_T} : \mathrm{GL}_3 \times \mathrm{Spec}(\mathbb{Z}[T]) \longrightarrow \mathrm{L}_3.$$

We obtain the following result.

3.1.11. Lemma. *In each geometric fiber over a point* $\mathrm{Spec}(k) \to \mathrm{Spec}(\mathbb{Z})$*, the following hold:* $\mathrm{ev}_{\mathfrak{s}}$ *and* $\mathrm{ev}_{\mathfrak{l}_T}$ *have 6-dimensional image, their sum* $\mathrm{GL}_{3,k} \amalg \mathrm{GL}_{3,k[T]} \to \mathrm{L}_{3,k}$ *is dominant, and* $\mathrm{L}_{3,k}$ *has pure dimension* 6 *with two irreducible components.*

Proof. Everything takes place in the fiber over $\mathrm{Spec}(k) \to \mathrm{Spec}(\mathbb{Z})$ so for simplicity we omit k from the notation. The stabilizer of $\mathfrak{s}$ has dimension 3, hence its orbit (the image of $\mathrm{ev}_{\mathfrak{s}}$) has dimension 6. For the orbit of $\mathfrak{l}_T$ we may as well remove the value $t = 1$ without changing the dimension. Then the stabilizer of $\mathfrak{l}_T$ is flat, of dimension 2 over $\mathrm{Spec}(k[T, (T-1)^{-1}])$, hence it has dimension 3 over k, and again the orbit (the image of $\mathrm{ev}_{\mathfrak{l}_T}$) has dimension 6. The fact that $\mathrm{GL}_{3,k} \amalg \mathrm{GL}_{3,k[T]} \to \mathrm{L}_{3,k}$ is dominant follows from the fact that the remaining orbits lie in the closure of those two, as we indicated before the lemma. Finally since both images of $\mathrm{ev}_{\mathfrak{s}}$ and $\mathrm{ev}_{\mathfrak{l}_T}$ are distinct, irreducible, of dimension 6, their closures are the irreducible components of $\mathrm{L}_{3,k}$. □

3.2 Schematic description of the moduli space L_3

Let us now focus on the schematic structure of the moduli space of three-dimensional Lie algebras. We first prove the representability of the functor L_3 over the ring of integers.

3.2.1. Definition. The *moduli space of based Lie algebras of rank three* is the following functor:

$$\begin{aligned} \mathrm{L}_3 : \mathrm{Sch} &\longrightarrow \mathrm{Set} \\ T &\longmapsto \Big\{ [\cdot,\cdot] : \mathcal{O}_T^3 \otimes \mathcal{O}_T^3 \to \mathcal{O}_T^3 \ ; \text{ where } [\cdot,\cdot] \text{ is a Lie bracket} \Big\}. \end{aligned}$$

3.2.2. Proposition. *This functor is representable by a closed subscheme of* $\mathbb{A}^9_{\mathbb{Z}}$*, given by*

$$\mathrm{Spec}\Big(\mathbb{Z}[a,b,c,d,e,f,g,h,i] \Big/ \langle ah + di - fg - bg, ie + bd - fh - ae, hc + dc - af - bi \rangle\Big).$$

Proof. Let T be a scheme, and let $\{x, y, z\}$ be a $\mathcal{O}_T(T)$-basis of $\mathcal{O}_T(T)^3$.

Let us write $(a, b, c, d, e, f, g, h, i) \in \mathcal{O}_T(T)^9$ for the coefficients of the Lie bracket $[\cdot,\cdot]$, where

$$[x, y] = ax + by + cz \ , \ [x, z] = dx + ey + fz \text{ and } [y, z] = gx + hy + iz.$$

Then by definition, we have:

$$\begin{aligned} \mathrm{L}_3(T) &= \Big\{[\cdot,\cdot] : \mathcal{O}_T^3 \times \mathcal{O}_T^3 \to \mathcal{O}_T^3 \text{ , where } [\cdot,\cdot] \text{ is a Lie bracket}\Big\} \\ &\simeq \Big\{(a, \ldots, i) \in \mathcal{O}_T(T)^9 \ ; \ ah + di - fg - bg = ie + bd - fh - ae = hc + dc - af - bi = 0\Big\}. \end{aligned}$$

One can easily verify that the conditions on the 9-tuple correspond to the Jacobi condition. □

Notations: From now on, we will use the following notations:

- $Q_1 := ah + di - fg - bg$, $Q_2 := ie + bd - fh - ae$ and $Q_3 := hc + dc - af - bi$ and $R_3 := \mathbb{Z}[a, b, c, d, e, f, g, h, i]\big/(Q_1, Q_2, Q_3)$, hence $\mathrm{L}_3 = \mathrm{Spec}(R_3)$. For any ring A, we write $R_{3,A}$ for $R_3 \otimes A$.
- Let us remark that the Jacobi condition can be written as

$$\begin{cases} Q_1 = ah + di - fg - bg = 0 \\ Q_2 = ie + bd - fh - ae = 0 \\ Q_3 = hc + dc - af - bi = 0 \end{cases} \Leftrightarrow \begin{cases} (a-i)h + (b+f)(-g) + (d+h)i &= 0 \\ (a-i)(-e) + (b+f)(-h) + (d+h)b &= 0 \\ (a-i)(-f) + (b+f)(-i) + (d+h)c &= 0. \end{cases}$$

  Then let us denote

$$M := \begin{pmatrix} h & -g & i \\ -e & -h & b \\ -f & -i & c \end{pmatrix} \text{ and } X := \begin{pmatrix} L_1 := a - i \\ L_2 := b + f \\ L_3 := d + h \end{pmatrix}.$$

  Then, the Jacobi condition is verified if and only if $MX = 0$.
  Now we can remark that a product matrix-vector vanishes when, either the vector is the zero vector, or the matrix has a non-trivial vector in its kernel, and this means that its determinant vanishes.
- For these reasons, we finally set

$$L := (L_1, L_2, L_3),\ I := (Q_1, Q_2, Q_3) \text{ and } J = (Q_1, Q_2, Q_3, \det(M)) = I + (\det(M)),$$

  and we will see that the two irreducible components are given, as schemes, by the ideals L and J, and we will give a more precise description of them. When it is clear from the context, we will still write I, J and L for those ideals seen in $R_{3,A}$ for any ring A.

3.2.3. Description of the irreducible components.

3.2.4. Theorem. *The affine scheme* L_3 *can be decomposed in two irreducible components: the* *first one is*

$$\mathrm{L}_3^{(1)} := \mathrm{Spec}\left(\mathbb{Z}[a,\dots,i]/L\right) \simeq \mathbb{A}^6$$

and the second one is

$$\mathrm{L}_3^{(2)} := \mathrm{Spec}\left(\mathbb{Z}[a,\dots,i]/J\right).$$

These irreducible components are linked to each other, they are both Cohen-Macaulay, flat over $\mathbb{Z}$ *with integral geometric fibers of dimension* 6*.*

Let A be any regular ring (for the following we will use $A = \mathbb{Z}$, $A = \mathbb{Q}$ or $A = \mathbb{F}_p$). We have $R_{3,A}/\left(L \otimes A\right) \simeq A[a,b,c,d,e,g]$ so $L \otimes A$ is prime in $\mathrm{L}_{3,A}$. Let us show that the ideal L describes an irreducible component of $\mathrm{L}_{3,A}$. Let us denote $D := A[a,\dots i]$.

3.2.5. Lemma. *The ideal L is minimal in D among the prime ideals containing I.*

Proof. Let $\mathbf{p} \in \mathrm{Spec}(D)$ be such that $I \subset \mathbf{p} \subset L$. First of all, because we have

$$M \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix},$$

we obtain

$$\det(M) \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = (\mathrm{com}(M))^{\mathrm{t}} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}.$$

So we can write

$$(*) \left\{ \begin{array}{l} \det(M)L_1 = (-hc+bi)Q_1 + (gc-i^2)Q_2 + (-gb+ih)Q_3 \\ \det(M)L_2 = (ec-bf)Q_1 + (hc+if)Q_2 + (-ie-bh)Q_3 \\ \det(M)L_3 = (ei-hf)Q_1 + (fg+ih)Q_2 + (-h^2-ge)Q_3 \end{array} \right. .$$

But $\det(M) = -ch^2 + gbf + ei^2 - gec + hbi - ihf$, then

$$\overline{\det(M)} = -cd^2 - gb^2 + ea^2 - gec - dba - adb \neq 0 \in D/L.$$

Then $\det(M) \notin L$ so $\det(M) \notin \mathbf{p}$. Thanks to $(*)$, this means that $L_1, L_2, L_3 \in \mathbf{p}$, i.e. $\mathbf{p} = L$. So L is a minimal prime among the prime ideals containing I. □

So now we need to show that J also describes schematically an irreducible component of $\mathrm{L}_{3,A}$. In order to do this, we use liaison theory. Let us recall that for any ideal I_1 and I_2 of a ring R, we write $[I_1 : I_2] := \{x \in R, xI_2 \subset I_1\}$.

3.2.6. Definition. Let J and L be two ideals in a ring R. We say that J and L are *linked* in R by an ideal I if $L = [I : J]$ and $J = [I : L]$.

3.2.7. Lemma. *The sequences* (L_1, L_2, L_3) *and* (Q_1, Q_2, Q_3) *are regular in* A.

Proof. It is trivial for (L_1, L_2, L_3). For (Q_1, Q_2, Q_3), let us remark that for any ring R, any polynomial in $R[X]$ whose leading coefficient is regular, is regular. But, the variable g appears only in Q_1, the variable e appears only in Q_2 and the variable c appears only in Q_3. Moreover, all of them appear with a regular coefficient. Then let us set $C := B[a, b, d, f, h, i]$. Then,

— $Q_1 \in C[c, e][g]$ seen as a polynomial in g has a regular leading coefficient, hence is regular

— $Q_2 \in (C[g]/(Q_1))[c, e]$ seen as a polynomial in e has a regular leading coefficient, hence is regular

— $Q_3 \in (C[g, e]/(Q_1, Q_2))[c]$ seen as a polynomial in c has a regular leading coefficient, hence is regular.

□

3.2.8. Corollary. *Let us denote by* M^{t} *the multiplication by the matrix* $M^{\mathrm{t}} : D^3 \to D^3$. *The two regular sequences* $\{L_1, L_2, L_3\}$ *and* $\{Q_1, Q_2, Q_3\}$ *define two Koszul complexes denoted by* $K_\bullet[L_1, L_2, L_3]$ *and* $K_\bullet[Q_1, Q_2, Q_3]$, *and we have a morphism of Koszul complexes between them:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D & \xrightarrow{d_3^Q} & D[2]^3 & \xrightarrow{d_2^Q} & D[4]^3 & \xrightarrow{d_1^Q} & D[6] & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \det(M^{\mathrm{t}})} & & \big\downarrow{\scriptstyle \wedge^2 M^{\mathrm{t}}} & & \big\downarrow{\scriptstyle M^{\mathrm{t}}} & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & D[3] & \xrightarrow{d_3^L} & D[4]^3 & \xrightarrow{d_2^L} & D[5]^3 & \xrightarrow{d_1^L} & D[6] & \longrightarrow & 0.
\end{array}$$

Here: $D[n]$ *is the graded ring* D *where we shift the graduation* n *times, in order to have a morphism of graded rings (i.e. a polynomial of degree* d *in* D *is seen in the* $(d-n)$*-th graduation of* $D[n]$*).*

Proof. This diagram comes from the definition of the Koszul complex (see for example Eisenbud's book [E95], Section 17, Subsection 17.2) and the functoriality of the Koszul complex: indeed we have $\wedge^0(D^3) = \wedge^3(D^3) = D$ and $\wedge^1(D^3) = \wedge^2(D^3) = D^3$, and the morphism id is just the morphism $\wedge^0 M^{\mathrm{t}}$, the morphism M^{t} is $\wedge^1 M^{\mathrm{t}}$ and the morphism $\det(M^{\mathrm{t}})$ is $\wedge^3 M^{\mathrm{t}}$. □

3.2.9. Remark. In the following, we will not need the graduation of our complex, so we will be writing and using it without specifying the graduation.

3.2.10. Corollary. *A projective resolution of* $D/[I : L]$ *can be obtained by taking the mapping cone of the map of Koszul complexes* $(M^{\mathrm{t}})^\vee : K_\bullet[L_1, L_2, L_3]^\vee \to K_\bullet[Q_1, Q_2, Q_3]^\vee$.

Proof. This is straightforward from Proposition 2.6 in [PS74]. □

3.2.11. Corollary. *The ideal* $[I : L]$ *is perfect of height* 3. *Moreover,*

$$[I : L] = I + \det(M) = J.$$

Proof. Using the notations of Corollary 3.2.8, we can see that the mapping cone of M^{t} is the following complex:

$$0 \to D \oplus 0 \xrightarrow{u} D^3 \oplus D \to D^3 \oplus D^3 \to D \oplus D^3 \xrightarrow{f} 0 \oplus D \to 0$$

where the morphism f is defined by:

$$\begin{aligned} f : D \oplus D^3 &\to D \\ (x, y, z, t) &\mapsto x + yL_1 + zL_2 + tL_3. \end{aligned}$$

Then, dualizing the complex, we obtain:

$$0 \to (D)^\vee \xrightarrow{f^\vee} (D \oplus D^3)^\vee \to (D^3 \oplus D^3)^\vee \to (D^3 \oplus D)^\vee \to (D)^\vee \to 0$$

where the morphism $f^\vee$ is defined by:

$$\begin{aligned} f^\vee : (D)^\vee &\to (D \oplus D^3)^\vee \\ \phi &\mapsto \big((x, y, z, t) \mapsto \phi(x + yL_1 + zL_2 + tL_3)\big). \end{aligned}$$

Replacing the morphism $f^\vee$ with its image, and showing this image is projective, we manage to reduce the length of this resolution. Indeed, let us denote by H the kernel of f:

$$H := \{(x, y, z, t) \in D^4,\ x + yL_1 + zL_2 + tL_3 = 0\}.$$

Then let us show

$$\mathrm{im}(f^\vee) = (D^4/H)^\vee \simeq (D^3)^\vee = \{\psi \in (D^4)^\vee,\ \psi_{|H} \equiv 0\}.$$

Let ψ be a form on D^4 such that $\psi(H) = 0$. Then for all $y, z, t \in D$,

$$\psi(-yL_1 - zL_2 - tL_3, y, z, t) = 0.$$

Let $(x, y, z, t) \in D^4$. Then

$$\begin{aligned} \psi(x, y, z, t) &= \psi(x + yL_1 + zL_2 + tL_3, 0, 0, 0) + \psi(-yL_1 - zL_2 - tL_3, y, z, t) \\ &= \psi(x + yL_1 + zL_2 + tL_3, 0, 0, 0). \end{aligned}$$

Let us set $\phi : D \to D$, $x \mapsto \psi(x, 0, 0, 0)$. Then we obtain $\psi = f^\vee(\phi)$. The other inclusion is trivial.

Then, $\mathrm{im}(f)$ is free over D, so the projective resolution given by Corollary 3.2.10 can be changed into this one:

$$0 \to \mathrm{im}(f)^\vee \to (D^3 \oplus D^3)^\vee \to (D^3 \oplus D)^\vee \to (D)^\vee \to 0$$

which is a projective resolution of length 3 of $D/[I : L]$. Then $\mathrm{projdim}([I : L]) \leq 3$. But because $I \subset [I : L]$, we know that $\mathrm{grade}([I : L]) \geq 3$. Hence $[I : L]$ is perfect of grade 3.

Now in order to show $[I : L] = I + \det(M)$, we will see that the resolution found above is actually a resolution of $D/(I + \det(M))$. Let us calculate the cokernel of the dual of this map:

$$\begin{aligned} u : D &\to D^3 \oplus D \\ 1 &\mapsto (-Q_3, Q_2, -Q_1, \det(M)). \end{aligned}$$

Then the dual map is given by

$$\begin{aligned} u^\vee : (D^3 \oplus D)^\vee &\to D^\vee \\ \phi &\mapsto (1 \mapsto \phi(-Q_3, Q_2, -Q_1, \det(M))). \end{aligned}$$

Then the cokernel of this morphism is given by $D/(I + \det(M))$. So by uniqueness of the cokernel, we have $[I : L] = I + \det(M) = J$. □

In order to show that the ideals L and J are linked, it remains to show that $L = [I : J]$. It is one of the purposes of the following proposition, which is the main result of liaison theory that we will be using in this chapter. It will give us powerful tools to understand the ideal J thanks to the ideal L. For more convenience, let us denote by $\underline{L} := L/I$ and $\underline{J} := J/I$ the two quotient ideals in the quotient ring $R_{3,A}$. We have seen that $\underline{J}$ is the annihilator of $\underline{L}$ in $R_{3,A}$.

3.2.12. Proposition. *The ideal $\underline{L}$ is the annihilator of $\underline{J}$, and $R_{3,A}/\underline{J}$ has Cohen-Macaulay geometric fibers of dimension* 6*.*

Proof. Let $\mathbf{p} \in \mathrm{Spec}(R_{3,A})$ and let us denote $R := (R_{3,A})_{\mathbf{p}}$. Let us denote $I_1 := (\underline{L})_{\mathbf{p}}$ and $I_2 := (\underline{J})_{\mathbf{p}}$. We would like to apply Proposition 1.3 from [PS74]. Let us show that we are in good conditions:

1. R is a Gorenstein local ring: indeed, $A[a, \dots, i]$ is regular hence Gorenstein, but because $I = (Q_1, Q_2, Q_3)$ is a regular sequence, then $A[a, \dots, i]/I$ is also Gorenstein. Hence R is Gorenstein as a localisation of a Gorenstein ring.
2. $I_2 = \mathrm{ann}(I_1)$: indeed $I_2 = (\underline{J})_{\mathbf{p}} = (\mathrm{ann}(\underline{L}))_{\mathbf{p}} = \mathrm{ann}((\underline{L})_{\mathbf{p}})$ because R is Noetherian.
3. $\dim(R) = \dim(R/I_1)$ because as R is Gorenstein hence Cohen-Macaulay, so we can apply Proposition 2.15 d) in Chapter 8, Section 8.2.2 in Liu's book [L02], using I_1 as prime ideal which has height 0 (because L is a minimal prime thanks to Lemma 3.2.5).

4. R/I_1 is regular hence Cohen-Macaulay. Then using Proposition 1.3 in [PS74], we obtain that
$$(\underline{L})_{\mathbf{p}} = [0 : (\underline{J})_{\mathbf{p}}] = [0 : \underline{J}]_{\mathbf{p}}.$$
Because we obtain this result for all $p \in \mathrm{Spec}(R)$ and because we already know the inclusion $\underline{L} \subset [0 : \underline{J}]$, then we have the equality not only locally but globally
$$\underline{L} = [0 : \underline{J}].$$
Hence L is an associate prime of I, and L is the annihilator of J in $R_{3,A}$.

The end of the lemma follows from Proposition 1.3 in [PS74]. □

Then now we know that the ideals J and L are linked, and thanks to this the previous proposition says that because D/L is Cohen-Macaulay, then D/J is Cohen-Macaulay as well. Thanks to this, we will prove that, for any algebraically closed field k, the ideal J is prime in $R_{3,k}$, then it describes schematically the second irreducible component of $\mathrm{L}_{3,k}$. We need this preliminary lemma first.

3.2.13. Lemma. *Let k be a field. The scheme $\mathrm{L}_{3,k}^{(2)} = \mathrm{Spec}(k[a,\dots,i]/J)$ has a smooth point.*

Proof. Actually, we will find a $\mathbb{Z}$-point of L_3 along which L_3 is smooth. Let $t \in \mathbb{Z}$ and let $\mathfrak{l}_t : \mathrm{Spec}(\mathbb{Z}) \to \mathrm{Spec}(\mathbb{Z}[a,\dots,i]/J)$ given on the rings by $a,c,d,e,g,h,i \mapsto 0, b \mapsto 1$ and $f \mapsto t$. Let us recall that $\det(M) = -ch^2 + gbf + ei^2 - gec + hbi - ihf$.
Then,
$$\mathfrak{l}_t^*\left(\Omega^1_{\mathrm{L}_3^{(2)}}\right) = \mathfrak{l}_t^*\left(\frac{\mathbb{Z}\cdot \mathrm{d}a \oplus \cdots \oplus \mathbb{Z}\cdot \mathrm{d}i}{\mathrm{d}Q_1, \mathrm{d}Q_2, \mathrm{d}Q_3, \mathrm{d}\det(M)}\right)$$
and we have the following equalities:

$$\begin{aligned}
\mathrm{d}Q_1 &= a(\mathrm{d}h) + h(\mathrm{d}a) + d(\mathrm{d}i) + i(\mathrm{d}d) - f(\mathrm{d}g) - g(\mathrm{d}f) - b(\mathrm{d}g) - g(\mathrm{d}b)\\
\mathrm{d}Q_2 &= i(\mathrm{d}e) + e(\mathrm{d}i) + b(\mathrm{d}d) + d(\mathrm{d}b) - f(\mathrm{d}h) - h(\mathrm{d}f) - a(\mathrm{d}e) - e(\mathrm{d}a)\\
\mathrm{d}Q_3 &= h(\mathrm{d}c) + c(\mathrm{d}h) + d(\mathrm{d}c) + c(\mathrm{d}d) - a(\mathrm{d}f) - f(\mathrm{d}a) - b(\mathrm{d}i) - i(\mathrm{d}b)\\
\mathrm{d}\det(M) &= (fg + hi)\mathrm{d}b + (-eg - h^2)\mathrm{d}c + (-cg + i^2)\mathrm{d}e + (bg - hi)\mathrm{d}f\\
&\quad + (-ce + bf)\mathrm{d}g + (-2ch + bi - fi)\mathrm{d}h + (bh - fh + 2ei)\mathrm{d}i.
\end{aligned}$$

Then,
$$\mathfrak{l}_t^*\left(\Omega^1_{\mathrm{L}_3^{(2)}}\right) = \frac{\mathbb{Z}\cdot \mathrm{d}a \oplus \cdots \oplus \mathbb{Z}\cdot \mathrm{d}i}{((t+1)\mathrm{d}g, \mathrm{d}d - t\mathrm{d}h, \mathrm{d}i + t\mathrm{d}f, t\mathrm{d}g)} = \frac{\mathbb{Z}\cdot \mathrm{d}a \oplus \cdots \oplus \mathbb{Z}\cdot \mathrm{d}i}{(\mathrm{d}g, \mathrm{d}d - t\mathrm{d}h, \mathrm{d}i + t\mathrm{d}f)}$$
so it is a free $\mathbb{Z}$-module of rank 6. But thanks to Proposition 3.2.12, we know that $\mathrm{L}_3^{(2)}$ has dimension 6, thus this $\mathbb{Z}$-point is smooth and the proof is done. □

3.2.14. Corollary. *The ideal J is prime in $R_{3,k}$ for any algebraically closed field k, and the scheme* $\mathrm{Spec}(k[a, \cdots, i]/J)$ *is integral.*

Proof. Let k be an algebraically closed field. We have proved in the Lemma 3.1.11 that

$$V(J) \ = \ \overline{o(\mathfrak{l}_T)} \subset R_{3,k}$$

which is irreducible. Moreover, we saw in Lemma 3.2.13 that it has a smooth point. But because we know from Proposition 3.2.12 that it is Cohen-Macaulay, then without associated points, so because it is generically reduced, it is reduced. Hence $V(J)$ is integral and J is prime. □

3.2.15. Proposition. *In the polynomial ring $A[a, \ldots, i]$, we have the equality*

$$I = J \cap L.$$

Proof. Thanks to Corollary 3.5 in Section 3, Subsection 3.2 of [E95], it is sufficient to show that $(J \cap L)_{\mathbf{p}} \subset I_{\mathbf{p}}$ for all primes $\mathbf{p}$ associated to I. Thanks to Proposition 3.2.12, we know that L is an associated prime of I.

Now let $\mathbf{p}$ be such a prime ideal. As $(J \cap L)_{\mathbf{p}} \subset J_{\mathbf{p}} \cap L_{\mathbf{p}}$, it is sufficient to prove that $J_{\mathbf{p}} \cap L_{\mathbf{p}} \subset I_{\mathbf{p}}$.

— If $\mathbf{p} = L$, we will show that $I_{\mathbf{p}} = L_{\mathbf{p}}$. Then let $\dfrac{a}{b} \in L_{\mathbf{p}}$, with $a \in L$ and $b \notin L$. Then, because $\det(M) \notin L$, $\dfrac{a}{b} = \dfrac{a \det(M)}{b \det(M)} \in I_{\mathbf{p}}$ because we showed in Lemma 3.2.5 that $\det(M)L_1 \in I$. Then $I_{\mathbf{p}} = L_{\mathbf{p}}$, so $J_{\mathbf{p}} \cap L_{\mathbf{p}} \subset I_{\mathbf{p}}$.

— If $\mathbf{p} \neq L$, we will show that $I_{\mathbf{p}} = J_{\mathbf{p}}$. Let $\dfrac{\det(M)}{b} \in J_{\mathbf{p}}$. As $L \neq \mathbf{p}$ and $\mathbf{p}$ is minimal, we have $L \not\subset \mathbf{p}$, so we can suppose $L_1 \notin \mathbf{p}$. Then because $\det(M)L_1 \in I$ and $\dfrac{\det(M)}{b} = \dfrac{\det(M)L_1}{bL_1}$, we have the equality $I_{\mathbf{p}} = J_{\mathbf{p}}$. Hence $J_{\mathbf{p}} \cap L_{\mathbf{p}} \subset I_{\mathbf{p}}$.

□

3.2.16. Corollary. *The ideal J is minimal among the prime ideals containing I.*

Proof. Let $\mathbf{p} \in \mathrm{Spec}(A[a, \ldots, i])$ such that $I \subset \mathbf{p} \subset J$. If $\det(M) \in \mathbf{p}$, then $J = \mathbf{p}$. Otherwise, we have $L \subset \mathbf{p} \subset J$, hence $I = J \cap L = L$ which is impossible. Then $\mathbf{p} = J$ and J is a minimal prime among the ones containing I. □

Now we have our two relative irreducible components denoted by $\mathrm{L}_{3,A}^{(1)}$ and $\mathrm{L}_{3,A}^{(2)}$, we still have to prove the flatness of $\mathrm{L}_3^{(2)}$ over the ring of integer $\mathbb{Z}$. We need a preliminary lemma first.

3.2.17. Lemma. *Let R be any commutative ring with unit, and $d \geq 1$. Let*

$$0 \to P_{d+1} \to P_d \to \cdots \to P_1 \to M \to 0$$

be an exact sequence of R-modules such that all $P_1, \ldots, P_d$ *are R-flat and we suppose that this exact sequence is still exact after any base change* $R \to R/I$ *where* I *is an ideal of* R. *Then,* M *is also R-flat.*

Moreover, if R *is Noetherian, we only have to verify the condition for all* $R \to R/I$ *where* I *is a prime ideal.*

Proof. We do an induction on the integer d. If $d = 1$, then this is classic. If $d > 1$, let

$$0 \to P_{d+2} \xrightarrow{\phi_{d+2}} P_{d+1} \xrightarrow{\phi_{d+1}} P_d \to \cdots \to P_1 \to M \to 0$$

be an exact sequence with $d+3$ terms, which is like in the statement. Let us define

$$C := \operatorname{coker}(\phi_{d+2}) = P_{d+1}/\operatorname{im}(\phi_{d+2}) = P_{d+1}/\ker(\phi_{d+1}) = \operatorname{im}(\phi_{d+1}) = \ker(\phi_d).$$

Because the inclusion $P_{d+2} \xrightarrow{\phi_{d+2}} P_{d+1}$ is universally injective by hypothesis, the cokernel C is flat over R. Then the following exact sequence:

$$0 \to C \xrightarrow{\phi_d} P_d \to \cdots \to P_1 \to M \to 0$$

is an exact sequence of $d+2$ terms which is like in the statement. Then by induction, we conclude that M is R-flat.

Let us now prove the result when R is Noetherien. Because the localisation morphism

$$R \to \prod_{p \in \operatorname{Spec}(R)} R_p$$

is faithfully flat, we can suppose that R is local. This case is well-known (see for example [RG71], 3.1.6). □

3.2.18. Corollary. *The scheme* $\mathrm{L}_3^{(2)}$ *is flat over* $\operatorname{Spec}(\mathbb{Z})$.

Proof. Let us take $D = \mathbb{Z}[a, \cdots, i]$ and let us take the corresponding resolution of $Z[a, \cdots, i]/J$ found in Corollary 3.2.10. This gives a resolution of flat $\mathbb{Z}$-modules

$$0 \to P_{d+1} \to P_d \to \cdots \to P_1 \to \mathbb{Z}[a, \cdots, i]/J \to 0.$$

Let I be a prime ideal of $\mathbb{Z}$. The sequence above is still exact after any base change of the form $\mathbb{Z} \to \mathbb{Z}/I$ because the base change is given by the resolution of Corollary 3.2.10, where we take $D = \mathbb{Z}/I[a \cdots, i]$. Then we can apply the previous Lemma 3.2.17. □

3.2.19. Proposition. *The entire scheme* L_3 *is flat over* $\mathbb{Z}$, *and the ideal* I *is radical.*

Proof. Because we have proved $I = L \cap J$ in Proposition 3.2.15, the following map is injective:

$$\begin{aligned} \mathbb{Z}[a,\dots,i]/I &\hookrightarrow \mathbb{Z}[a,\dots,i]/L \times \mathbb{Z}[a,\dots,i]/J \\ \overline{P} &\mapsto (\overline{P},\overline{P}). \end{aligned}$$

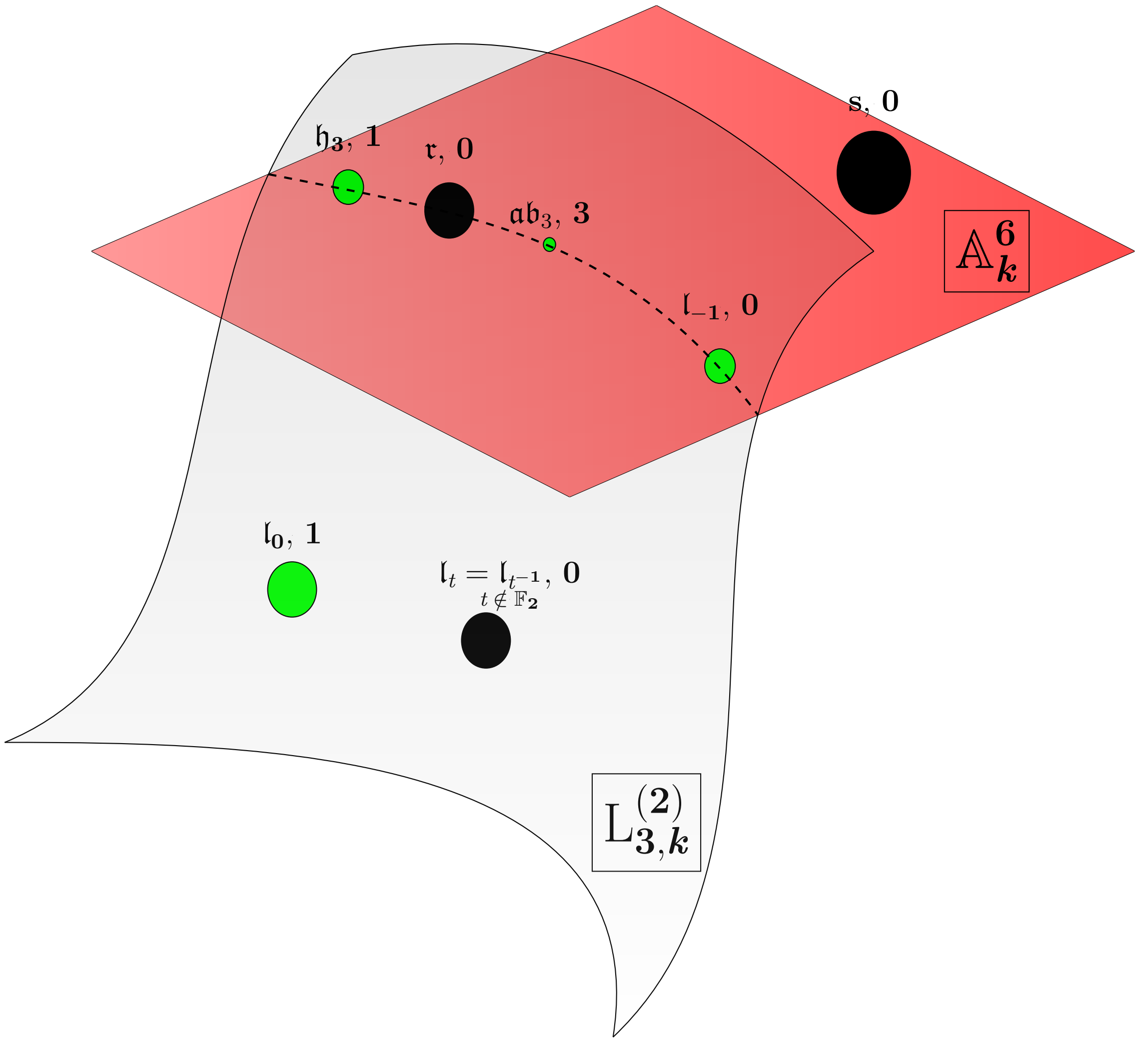
But both of the rings that appear on the right-hand side are flat over $\mathbb{Z}$, then without torsion, so $\mathbb{Z}[a,\dots,i]/I$ is without $\mathbb{Z}$-torsion, then it is $\mathbb{Z}$-flat.

Moreover, because L and J are both radical and $I = J \cap L$, then I is radical. $\square$

3.3 Summary: picture of a geometric fiber of our moduli space

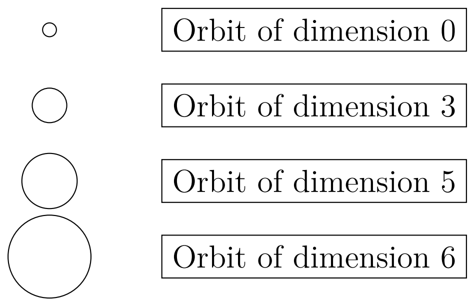
3.3.1. In characteristic $p = 2$.

Let k be an algebraically closed field of characteristic $p = 2$. Here is a picture representing the two irreducible components of $L_{3,k}$. The points correspond to the different orbits on it, and we specify the restrictable ones. We also write on it the dimension of the center of those Lie algebras.



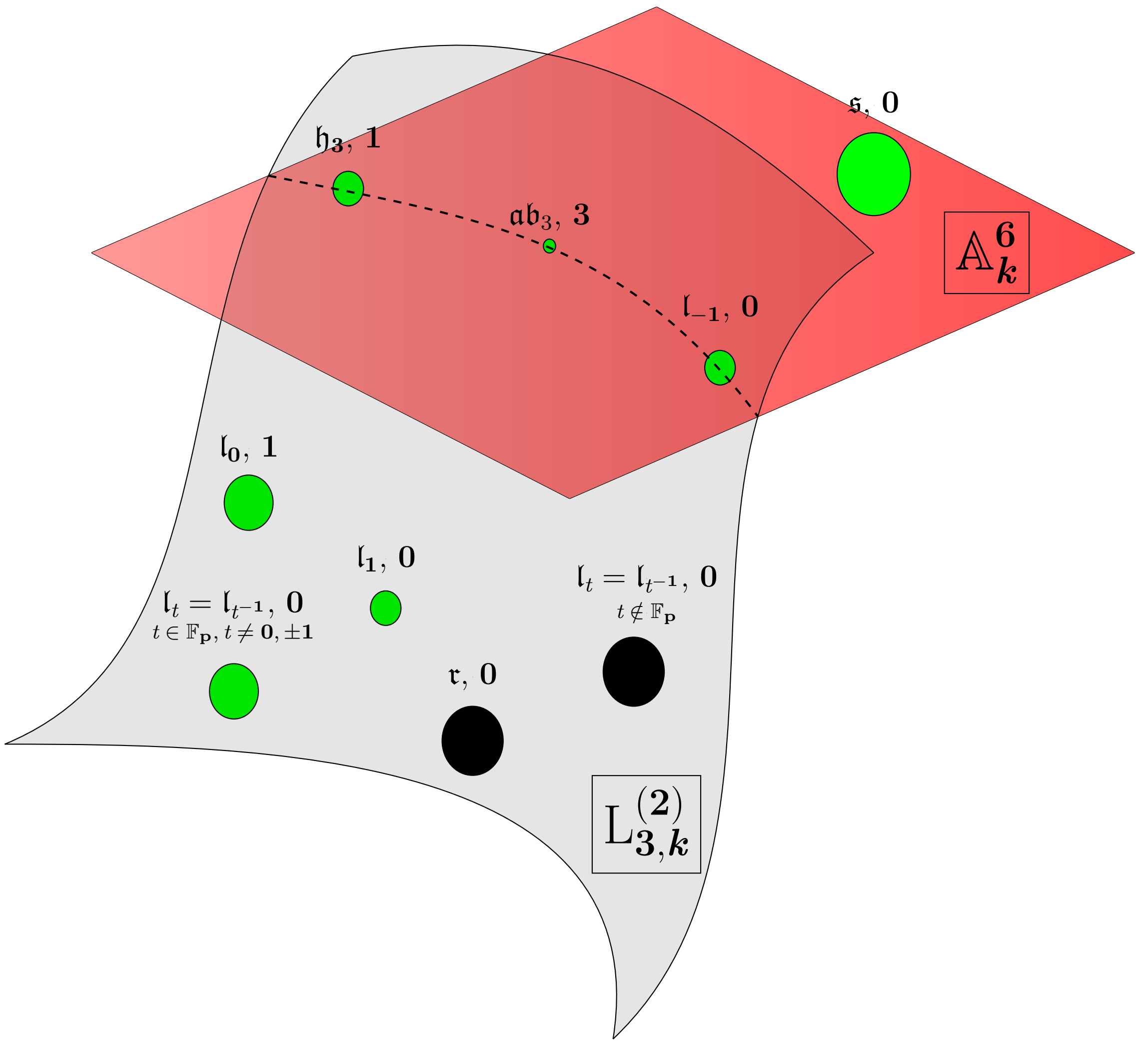
<u>**Caption**</u>:

- Restrictable orbit.
- Non-restrictable orbit



3.3.2. In characteristic $p \neq 2$.

Let k be an algebraically closed field of characteristic $p > 2$. As on the previous page, here is a picture representing the two irreducible components of $\mathrm{L}_{3,k}$ with the orbits on it, and the dimension of the center of those Lie algebras.



Caption:

- Restrictable orbit.
- Non-restrictable orbit

Orbit of dimension 0

Orbit of dimension 3

Orbit of dimension 5

Orbit of dimension 6

4 Smoothness of $\mathrm{L}_3^{\mathrm{res}}$ on the flattening stratification of the center

We did not study all the equations of the singular locus of L_3, but using [Macaulay2], we can see that the singular locus of $\mathrm{L}_{3,\mathbb{Q}}^{(2)}$ over $\mathbb{Q}$ is given by an ideal, whose radical is $I_2(M)+L$, where $I_2(M)$ is the ideal generated by the two-minors of the matrix M, the one introduced in Subsection 3.2. In order to study the singular locus over $\mathbb{Z}$, we prefer to carry out explicit tangent space computations.

In all this chapter, let us denote by $\mathbb{L}_n := \mathbb{A}^n_{\mathrm{L}_n}$ the universal Lie algebra of rank n over L_n. Then in the following, we will study the smoothness of the restricted locus $\mathrm{L}_3^{\mathrm{res}} \hookrightarrow \mathrm{L}_3$ of the universal Lie algebra $\mathbb{L}_3 \to \mathrm{L}_3$. As said before, we know from Theorem 2.1.8 that it is interesting to study it after passing to the flattening stratification of the center. So for all this section, let us denote $k = \mathbb{F}_p$. All the schemes are understood as k-schemes.

Thanks to the theory of Fitting ideals (the reader can look at [SP23, Tag 0C3C] for more details), we can have an explicit description of the different strata. We write $Z(\mathbb{L}_n)$ the center of the universal Lie algebra. Let

$$\mathrm{L}_n =: Z_{-1} \supset Z_0 \supset Z_1 \supset \dots$$

be the closed subschemes defined by the Fitting ideals of $Z(\mathbb{L}_n)$. Then, for $r \geq 0$, let us define $\mathrm{L}_{n,r} := Z_{r-1} \setminus Z_r$ to be the locally closed subscheme of L_n where $Z(\mathbb{L}_n)$ is locally free of rank r. Actually in the following, we will not need to calculate explicitly the flattening stratification. We use the notation $\mathrm{L}_{n,r}^{\mathrm{res}}$ for the locally closed subscheme of L_n where the center $Z(\mathbb{L}_n)$ is locally free of rank r, and $\mathbb{L}_n$ is restrictable, i.e. $\mathrm{L}_{n,r}^{\mathrm{res}} := \mathrm{L}_{n,r} \cap \mathrm{L}_n^{\mathrm{res}}$.

4.1 Correspondence between the centers of the group and the Lie algebra

In the following, we will extend the classical equivalence of categories between locally free Lie p-algebras of finite rank with finite locally free group schemes of height 1, (see [SGA3], exposé VII$_A$, section 7) showing that the centers of those objects correspond to each other. This is remarkable because the centers are not flat in general. In order to do this we will use the functor denoted by Spec^* in [SGA3], Tome 1, exposé VII$_A$, §3.1.2. We need first a preliminary lemma. For all this section, let S be a scheme of characteristic $p > 0$. We will use the exponential notation which is explained in [DG70], Chapitre III, §4, n°3.

4.1.1. Lemma. *Let $G \to S$ a group scheme. Let R be any ring. Then the following morphism:*

$$\begin{aligned} \mathrm{Lie}(G)(R) &\xrightarrow{\exp} G(R[\alpha,\beta]/(\alpha^2,\beta^2)) \\ x &\longmapsto \exp(\alpha\beta x) \end{aligned}$$

is injective.

Proof. Let us write $R(\alpha,\beta) := R[\alpha,\beta]/(\alpha^2,\beta^2)$ and let $f : \mathrm{Spec}(R(\alpha,\beta)) \to \mathrm{Spec}(R[\epsilon]/(\epsilon^2))$ be the scheme morphism coming from this injective ring morphism: $R[\epsilon]/(\epsilon^2) \hookrightarrow R(\alpha,\beta)$, $\epsilon \mapsto \alpha\beta$. Then f is surjective as a topological map, so because $f^{\#}$ is injective, f is an epimorphism in the category of schemes. Then this gives an injective morphism $G(R(\epsilon)) \hookrightarrow G(R(\alpha,\beta))$ which gives by restriction to the Lie algebras an injective morphism

$$\mathrm{Lie}(G)(R) \hookrightarrow G(R(\epsilon)) \hookrightarrow G(R(\alpha,\beta)).$$

□

4.1.2. Proposition. *Let $G \to S$ be a finite locally free group scheme of height* 1*. Let $Z(G)$ denote its center. Then*

$$Z(\mathrm{Lie}(G)) = \mathrm{Lie}(Z(G)).$$

Proof. For more convenience, let us write $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{z} := Z(\mathfrak{g})$. When $\mathfrak{l}$ is a Lie p-algebra, we use the notation $G_p(\mathfrak{l}) := \mathrm{Spec}^*(U_p(\mathfrak{l}))$ where $U_p(\mathfrak{l})$ is the universal restricted enveloping algebra of $\mathfrak{l}$ and where the notation Spec^* comes from [SGA3], exposé VII_A, §3.1.2. and is defined for any S-scheme $T \to S$ by:

$$G_p(\mathfrak{l})(T) := \mathrm{Spec}^*(U_p(\mathfrak{l}))(T) = \{x \in U_p(\mathfrak{l}) \otimes \mathcal{O}_T(T), \epsilon(x) = 1 \text{ and } \Delta(x) = x \otimes x\}.$$

Let us show $\mathfrak{z} \subset \mathrm{Lie}(Z(G))$. The inclusion $\mathfrak{z} \subset \mathfrak{g}$ gives a bialgebra inclusion of universal restricted enveloping algebras $U_p(\mathfrak{z}) \subset U_p(\mathfrak{g})$, and looking at the definition, we see that this gives an inclusion of functors:

$$G_p(\mathfrak{z}) \subset G_p(\mathfrak{g}) = G$$

where the last equality is because G is of height 1. But actually, this subfunctor takes its values in the center of G: indeed, because $\mathfrak{z}$ is an abelian Lie algebra, the bialgebra $U_p(\mathfrak{z})$ is commutative (because for all $x, y \in \mathfrak{z}$, we have $x \otimes y - y \otimes x = [x,y] = 0$ in $U_p(\mathfrak{z})$). Moreover by definition, we have for any S-scheme $T \to S$,

$$G(T) = G_p(\mathfrak{g})(T) = \{x \in U_p(\mathfrak{g}) \otimes \mathcal{O}_T(T), \epsilon(x) = 1 \text{ and } \Delta(x) = x \otimes x\}$$

where the group law of $G(T)$ is given by $(x,y) \mapsto x \otimes y$. But because the algebra $U_p(\mathfrak{z})$ is abelian, then if $x \in G_p(\mathfrak{z})(T)$, then $x \in Z(G_p(\mathfrak{g}))$, i.e.

$$G_p(\mathfrak{z}) \subset Z(G).$$

Applying the functor Lie we obtain

$$\mathrm{Lie}(G_p(\mathfrak{z})) \subset \mathrm{Lie}(Z(G))$$

but looking at [SGA3], exposé VII_A, §3.2.3, we know that $\mathrm{Lie}(G_p(\mathfrak{z})) = \mathrm{Prim}(\mathbb{W}(U_p(\mathfrak{z})))$ and by definition, $\mathfrak{z} \subset \mathrm{Prim}(\mathbb{W}(U_p(\mathfrak{z})))$ so we have the inclusion

$$\mathfrak{z} \subset \mathrm{Lie}(Z(G)).$$

Now let us show $\mathrm{Lie}(Z(G)) \subset \mathfrak{z}$. Let $f : Z(G) \hookrightarrow G$ be the closed immersion. It is a monomorphism then it is injective on the functor of points. Let R be any ring and let us denote by $R(\alpha, \beta) := R[\alpha, \beta]/(\alpha^2, \beta^2)$. Then we know from [DG70] Chapitre II, §4, n°3, 3.7 (3), that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Lie}(Z(G))(R) & \xrightarrow{\exp} & Z(G)(R(\alpha,\beta)) \\ \downarrow{\scriptstyle \mathrm{Lie}(f_R)} & & \downarrow{\scriptstyle f_{R(\alpha,\beta)}} \\ \mathrm{Lie}(G)(R) & \xrightarrow[\exp]{} & G(R(\alpha,\beta)) \end{array}$$

and the composed map $\mathrm{Lie}(Z(G))(R) \to G(R(\alpha, \beta))$ is injective. Moreover, if $x \in \mathrm{Lie}(Z(G))(R)$, then $\exp(\alpha x) \in Z(G)(R) \subset Z(G)(R(\alpha, \beta))$ hence for all $y \in \mathrm{Lie}(G)(R)$,

$$1 = \exp(\alpha x) \exp(\beta y) \exp(-\alpha x) \exp(-\beta y) = \exp(\alpha\beta[x, y])$$

where the last equality comes from [DG70], Chapitre II, §4, n°4, 4.2 (6), and where $[x, y]$ is the bracket on $\mathrm{Lie}(G)(R)$. But $x \mapsto \exp(\alpha\beta x)$ is injective thanks to Lemma 4.1.1, then we obtain $[x, y] = 0$ for all $y \in \mathrm{Lie}(G)(R)$ then $x \in Z(\mathrm{Lie}(G))(R)$. □

Thanks to this result, we can count the number of centerless finite locally free group schemes of order p^3 of height 1 on an algebraically closed field:

4.1.3. Proposition. *Let k be an algebraically closed field of characteristic $p > 0$. Up to isomorphism,*

- *if $p = 2$, there is only 1 such group scheme.*
- *if $p \neq 2$, there are $(p + 3)/2$ such group schemes.*

Proof. It suffices to count the centerless restrictable Lie algebras of rank 3, classified in 3.1.8. Indeed, because they are centerless, they have only one structure of Lie p-algebra so there is only one algebraic group scheme corresponding to it. For $p \neq 2$, it is useful to remember that, with our notations, the Lie algebras $\mathfrak{l}_t$ and $\mathfrak{l}_{t^{-1}}$ are in same orbit when $t \neq 0$. □

This extended equivalence allows us to use the properties of $\mathrm{L}_n^{\mathrm{res}}$ to deduce properties on the moduli space of finite locally free group schemes killed by Frobenius. That is, let S be a scheme of characteristic $p > 0$, and for $r \leq n$, let us recall the notations $p\text{-}\mathcal{L}ie_{n,r}(S)$ for the category of n-dimensional restrictable $\mathcal{O}_S$-Lie algebras whose center is locally free of rank r, and $\mathcal{G}_{n,r}(S)$ the category of finite locally free group schemes of order p^n, of height 1, whose center is locally free of rank p^r. With these notations and using the previous results, we know that the functor Lie gives us an equivalence of categories:

$$\mathrm{Lie} : \mathcal{G}_{n,r}(S) \xrightarrow{\sim} p\text{-}\mathcal{L}ie_{n,r}(S).$$

Moreover, because GL_n is smooth, the quotient map $\mathrm{L}_n \to \mathcal{L}ie_n$ is smooth, so studying the smoothness of L_n is equivalent to study the one of $\mathcal{L}ie_n$. Let us denote by $\mathrm{L}_n^p := X(\mathbb{L}_n)$ the set of p-mappings on $\mathbb{L}_n$. Then the quotient map $\mathrm{L}_n^p \to p\text{-}\mathcal{L}ie_n$ is smooth, and

$$\mathrm{L}_n^p \xrightarrow{\text{Forgetful}} \mathrm{L}_n^{\mathrm{res}}$$

is an affine fibration, and if for $r \leq n$, we denote by $\mathrm{L}_{n,r}^p := \mathrm{L}_n^p \cap \mathrm{L}_{n,r}$, we know that

$$\mathrm{L}_{n,r}^p \xrightarrow{\text{Forgetful}} \mathrm{L}_{n,r}^{\mathrm{res}}$$

is smooth for all $r \leq n$. This is the reason why in the following, we will study the smoothness of $\mathrm{L}_{3,r}^{\mathrm{res}}$ for $r \leq 3$.

4.2 In the stratum $\mathrm{L}_{3,0}$

4.2.1. Study of $\mathrm{L}_{3,0}^{\mathrm{res}}$ in the whole scheme L_3.

Thanks to the results we have established before, we can imagine all the k-points which are in the orbit of $\mathfrak{l}_{-1}$ are singular in $\mathrm{L}_{3,0}^{\mathrm{res}}$ because this orbit is in the intersection of two irreducible components. Actually, thanks to a calculation of tangent space, we will see that they are the only singular ones.

4.2.2. Theorem. *If* $\mathrm{char}(k) \neq 2$*, the singular locus of* $\mathrm{L}_{3,0}^{\mathrm{res}}$ *is the orbit of* $\mathfrak{l}_{-1}$*. If* $\mathrm{char}(k) = 2$*, the scheme* $\mathrm{L}_{3,0}^{\mathrm{res}}$ *is smooth.*

Proof. - If $p \neq 2$. We see in the classification Theorem 3.1.4 that the points of $\mathrm{L}_{3,0}^{\mathrm{res}}$ are the points which are in the orbit of $\mathfrak{l}_t$ with $t \in \mathbb{F}_p$ and $t \neq 0$, and the points in the orbit of $\mathfrak{s}$. Let us start with $\mathfrak{l}_t$, i.e. let us denote for $t \in \mathbb{F}_p^*$ as before the k-point $\mathfrak{l}_t := \mathrm{Spec}(k) \to \mathrm{L}_{3,k}$. We need to calculate the local ring of this point. We will show that $\mathcal{O}_{\mathrm{L}_{3,0}^{\mathrm{res}},\mathfrak{l}_t} = \mathcal{O}_{\mathrm{L}_3^{\mathrm{res}},\mathfrak{l}_t}$ is smooth.

Let us compute $T_{\mathrm{L}_3^{\mathrm{res}},\mathfrak{l}_t}$. Let us denote by N the $\mathbb{F}_p[\varepsilon]$-module $\mathbb{F}_p[\varepsilon]x \oplus \mathbb{F}_p[\varepsilon]y \oplus \mathbb{F}_p[\varepsilon]z$. Let us recall the definition of the tangent vector of a scheme X at a point x:

$$T_x(X) = \{\phi : \mathrm{Spec}(k[\epsilon]) \to X,\ \mathrm{im}(\phi) = \{x\}\}.$$

Then we can write

$$T_{\mathrm{L}_3^{\mathrm{res}},\mathfrak{l}_t} = \Big\{\text{Structures of Lie algebra on } N \text{, restrictable, such that } N \otimes \mathbb{F}_p = \mathfrak{l}_t\Big\}.$$

Let us use again the matrix notation for a Lie algebra structure over N. We denote it by

$$\mathfrak{l}_{t,\varepsilon} := \begin{pmatrix} a\varepsilon & d\varepsilon & g\varepsilon \\ 1+b\varepsilon & e\varepsilon & h\varepsilon \\ c\varepsilon & t+f\varepsilon & i\varepsilon \end{pmatrix}.$$

First of all, $\mathfrak{l}_{t,\varepsilon}$ is a Lie algebra structure if and only if its coefficients satisfy the conditions denoted by Q_1, Q_2 and Q_3 above, that is if and only if

$$\begin{cases} (1+t)g = 0 \\ d - th = 0 \\ ta + i = 0. \end{cases}$$

Then, $\mathfrak{l}_{t,\varepsilon}$ is in $T_{\mathrm{L}_3^{\mathrm{res}},\mathfrak{l}_t}$ if and only if it is restrictable. In order to see the conditions to be restrictable we will calculate ad_x^p, ad_y^p and ad_z^p for any p prime. Let us denote

$$\beta := 1 + t + \cdots + t^{p-1} = \begin{cases} 0 & \text{if } t = 1 \\ 1 & \text{if } t \neq 1. \end{cases}$$

Now, using the matrix notation in the basis $\{x, y, z\}$, we have $\mathrm{ad}_x = \begin{pmatrix} 0 & a\varepsilon & d\varepsilon \\ 0 & 1+b\varepsilon & e\varepsilon \\ 0 & c\varepsilon & t+f\varepsilon \end{pmatrix}$ then for all p prime $\mathrm{ad}_x^p = \begin{pmatrix} 0 & a\varepsilon & d\varepsilon \\ 0 & 1 & \beta e\varepsilon \\ 0 & \beta c\varepsilon & t \end{pmatrix}$.

Likewise, $\mathrm{ad}_y = \begin{pmatrix} -a\varepsilon & 0 & g\varepsilon \\ -1-b\varepsilon & 0 & h\varepsilon \\ -c\varepsilon & 0 & i\varepsilon \end{pmatrix}$ hence $\mathrm{ad}_y^2 = \begin{pmatrix} 0 & 0 & 0 \\ a\varepsilon & 0 & -g\varepsilon \\ 0 & 0 & 0 \end{pmatrix}$ and for all $p > 2$, $\mathrm{ad}_y^p \equiv 0$.

Likewise, $\mathrm{ad}_z = \begin{pmatrix} -d\varepsilon & -g\varepsilon & 0 \\ -e\varepsilon & -h\varepsilon & 0 \\ -t-f\varepsilon & -i\varepsilon & 0 \end{pmatrix}$, hence $\mathrm{ad}_z^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ td\varepsilon & tg\varepsilon & 0 \end{pmatrix}$ and for all $p > 2$,

$\mathrm{ad}_z^p \equiv 0$.
Then ad_x^p is a linear combination of ad_x, ad_y and ad_z if and only if it exists $\lambda = \lambda_0 + \lambda_1\varepsilon$, $\mu = \mu_0 + \mu_1\varepsilon$ and $\nu = \nu_0 + \nu_1\varepsilon$ such that:

$$\begin{pmatrix} 0 & a\varepsilon & d\varepsilon \\ 0 & 1 & \beta e\varepsilon \\ 0 & \beta c\varepsilon & t \end{pmatrix} = \begin{pmatrix} (-\mu_0 a - \nu_0 d)\varepsilon & (\lambda_0 a - \nu_0 g)\varepsilon & (\lambda_0 d + \mu_0 g)\varepsilon \\ -\mu_0 - (b\mu_0 + \mu_1 + \nu_0 e)\varepsilon & \lambda_0 + (b\lambda_0 + \lambda_1 - \nu_0 h)\varepsilon & (\lambda_0 e + \mu_0 h)\varepsilon \\ -t\nu_0 - (\mu_0 c + s\nu_0 + t\nu_1)\varepsilon & (\lambda_0 c - \nu_0 i)\varepsilon & t\lambda_0 + (f\lambda_0 + t\lambda_1 + \mu_0 i)\varepsilon \end{pmatrix}.$$

Then ad_x^p is a linear combination of ad_x, ad_y and ad_z if and only if

$$\begin{cases} bt = f \\ c = \beta c \\ e = \beta e. \end{cases}$$

Because $\mathrm{ad}_y^p \equiv \mathrm{ad}_z^p \equiv 0$, they are always linear combination of ad_x, ad_y and ad_z.
Hence we obtain the following conditions:

$$(*)\begin{cases} (1+t)g = d - th = i + ta = 0 \\ bt - f = c - \beta c = e - \beta e = 0. \end{cases}$$

So we have to distinguish different cases. First let us suppose $t = -1$. Then the conditions $(*)$ are equivalent to:

$$\begin{cases} d + h = 0 \\ a - i = 0 \\ b + f = 0. \end{cases}$$

Hence $\dim(T_{\mathrm{L}_3^{\mathrm{res}},\mathfrak{l}_{-1}}) = 6$. But $\dim(o(\mathfrak{l}_{-1})) = 5$ from Theorem 3.1.4, hence the local ring of $\mathfrak{l}_{-1}$ is singular.
Let us suppose $t = 1$. Then the conditions $(*)$ are equivalent to

$$\begin{cases} g = d - h = a + i = 0 \\ b - f = c = e = 0 \end{cases}$$

so $\dim(T_{\mathrm{L}_3^{\mathrm{res}},\mathfrak{l}_1}) = 3 = \dim(o(\mathfrak{l}_1))$. Then the point $\mathfrak{l}_1$ is smooth.
Then let us suppose $t \neq 1$ and $t \neq -1$. Then the conditions $(*)$ are equivalent to:

$$\begin{cases} g = 0 \\ d - th = i + ta = bt - f = 0 \end{cases}$$

Hence $\dim(T_{\mathrm{L}_3^{\mathrm{res}},\mathfrak{l}_t}) = 5$. But $\dim(o(\mathfrak{l}_t)) = 5$, hence the local ring of $\mathfrak{l}_t$ is regular.

Doing the same calculations for $\mathfrak{s} \in \mathrm{L}_{3,0}^{\mathrm{res}}$, we obtain these conditions:

$$\begin{cases} a - i = 0 \\ b + f = 0 \\ d + h = 0. \end{cases}$$

Hence $\dim(T_{\mathrm{L}_3^{\mathrm{res}},\mathfrak{s}}) = 6$. But $\dim(o(\mathfrak{s})) = 6$, hence the local ring of $\mathfrak{s}$ is regular.

- Let us suppose $p = 2$. Then the only point in $\mathrm{L}_{3,0}^{\mathrm{res}}$ is $\mathfrak{l}_1$. So using the same notations as before, we see in this case, the conditions are equivalent to

$$\begin{cases} d - h = a + i = b - f = 0 \\ c = e = g = 0 \end{cases}$$

so $\dim(T_{\mathrm{L}_3^{\mathrm{res}},\mathfrak{l}_1}) = 3$. But $\dim(o(\mathfrak{l}_1)) = 3$, hence the local ring of $\mathfrak{l}_1 = \mathfrak{l}_{-1}$ is regular.

□

4.2.3. Study of $\mathrm{L}_{3,0}^{\mathrm{res}}$ in the first irreducible component. We start by establishing a result on the scheme structure of $\mathrm{L}_{3,0}^{\mathrm{res}}$ in the first irreducible component, in the case we choose a field k of characteristic $p \neq 2$.

4.2.4. Proposition. *The scheme* $\mathrm{L}_{3,0} \cap \mathrm{L}_3^{(1)}$ *is reduced. Moreover, if* $\mathrm{char}(k) \neq 2$*,*

$$\mathrm{L}_{3,0}^{\mathrm{res}} \cap \mathrm{L}_3^{(1)} \simeq \mathrm{L}_{3,0} \cap \mathrm{L}_3^{(1)} \text{ as schemes.}$$

Proof. Because $\mathrm{L}_{3,0}$ is open in L_3, $\mathrm{L}_{3,0} \cap \mathrm{L}_3^{(1)}$ is open in the reduced irreducible component $\mathrm{L}_3^{(1)}$, then it is reduced. Moreover, using the classification of the Lie algebras we have done above, we can write:

$$|\mathrm{L}_{3,0}^{\mathrm{res}} \cap \mathrm{L}_3^{(1)}| = \bigcup_{R_k \to k = \overline{k}} (\mathrm{L}_{3,0}^{\mathrm{res}} \cap \mathrm{L}_3^{(1)})(k) = \bigcup_{R_k \to k = \overline{k}} \mathrm{L}_3^{\mathrm{res}}(k) \times_{S(k)} \mathrm{L}_{3,0}(k) \times_{S(k)} \mathrm{L}_3^{(1)}(k)$$

$$= \bigcup_{R_k \to k = \overline{k}} \mathrm{L}_{3,0}(k) \times_{S(k)} \mathrm{L}_3^{(1)}(k) = |\mathrm{L}_{3,0} \cap \mathrm{L}_3^{(1)}|.$$

Then $\mathrm{L}_{3,0}^{\mathrm{res}} \cap \mathrm{L}_3^{(1)}$ is a closed subscheme of the reduced scheme $\mathrm{L}_{3,0} \cap \mathrm{L}_3^{(1)}$ with the same underlying set. Then they are equal as schemes. □

Now we study the k-points of this intersection of schemes. We have to do exactly the same calculus we have done in the previous subsection, but we have to change the conditions Q_1, Q_2 and Q_3 for the conditions L_1, L_2 and L_3. Then we find:

4.2.5. Proposition. *If* $\mathrm{char}(k) \neq 2$*, in* $\mathrm{L}_{3,0}^{\mathrm{res}} \cap \mathrm{L}_3^{(1)}$*, the* k*-points* $\mathfrak{l}_{-1}$ *is singular, and* $\mathfrak{s}$ *is regular. If* $\mathrm{char}(k) = 2$*, the scheme* $\mathrm{L}_{3,0}^{\mathrm{res}} \cap \mathrm{L}_3^{(1)}$ *is smooth.*

Proof. - Let us suppose $p \neq 2$. We first look at the point $\mathfrak{s}$. We obtain, as before, these conditions:

$$\begin{cases} a - i = 0 \\ b + f = 0 \\ d + h = 0. \end{cases}$$

Hence $\dim(T_{\mathrm{L}_3^{\mathrm{res}},\mathfrak{s}}) = 6$, so the local ring of $\mathfrak{s}$ is regular.

Let us do the same for the point $\mathfrak{l}_{-1}$. Doing the same calculations we obtain $\dim(T_{\mathrm{L}_3^{\mathrm{res}},\mathfrak{l}_{-1}}) = 6$, so the local ring of $\mathfrak{l}_{-1}$ is singular.

- If $p = 2$, we have $\dim(T_{\mathrm{L}_3^{\mathrm{res}},\mathfrak{l}_1}) = 3$ so $\mathfrak{l}_1$ is regular.

□

4.2.6. Study of $\mathrm{L}_{3,0}^{\mathrm{res}}$ in the second irreducible component.

4.2.7. Theorem. *In the second irreducible component, all the k-points of $\mathrm{L}_{3,0}^{\mathrm{res}} \cap \mathrm{L}_3^{(2)}$ are smooth.*

Proof. We can do the same proof as before, we just need to add the condition $\det(M) = 0$. That is, if we keep the same notations as before, we need to add to the system $(*)$ the condition $gt = 0$. Hence the new system is given by

$$\begin{cases} g = d - th = i + ta = 0 \\ bt - f = c - \beta c = e - \beta e = 0. \end{cases}$$

So in this case, we obtain

$$\dim(T_{\mathrm{L}_3^{\mathrm{res}},\mathfrak{l}_t}) = \begin{cases} 3 & \text{if } t = 1 \\ 5 & \text{if } t \neq 1. \end{cases}$$

□

4.2.8. Remark. By a simple computation, we can see that any deformation of Lie algebras which are in the stratum $\mathrm{L}_{3,0}$ is centerless without any condition. It is because the stratum $\mathrm{L}_{3,0}$ is open in L_3.

4.3 In the stratum $\mathrm{L}_{3,1}$

Let us do the same calculations for the points of $\mathrm{L}_{3,1}$.

4.3.1. Study of $\mathrm{L}_{3,1}^{\mathrm{res}}$ in L_3.

4.3.2. Proposition. *The k-point $\mathfrak{h}_3$ is singular in $\mathrm{L}_{3,1}^{\mathrm{res}}$, and $\mathfrak{l}_0$ is smooth.*

Proof. - For the point $\mathfrak{h}_3$, as in the previous section, let us denote by $\mathfrak{h}_{3,\varepsilon}$ a deformation of the Lie algebra $\mathfrak{h}_3$:

$$\mathfrak{h}_{3,\varepsilon} := \begin{pmatrix} a\varepsilon & d\varepsilon & 1 + g\varepsilon \\ b\varepsilon & e\varepsilon & h\varepsilon \\ c\varepsilon & f\varepsilon & i\varepsilon \end{pmatrix}.$$

Then, $\mathfrak{h}_{3,\varepsilon}$ gives the constants of structure of a Lie algebra if and only if $b+f=0$. Moreover, $\mathfrak{h}_{3\varepsilon}$ is restrictable if and only if:

- if $p=2$: $b=c=e=0$
- if $p=3$: there is no condition
- if $p>3$: there is no condition.

For the end, the center $Z(\mathfrak{h}_{3,\varepsilon})$ is locally free of rank 1 if and only if:

— if $p=2$: there is no condition

— if $p=3$: $b=c=e=0$

— if $p>3$: $b=c=e=0$

So to conclude we use the fact that $\dim(o(\mathfrak{h}_3))=3$.

- For the point $\mathfrak{l}_0$ let us do the same. Then using the same notations, $\mathfrak{l}_{0,\varepsilon}$ gives the constants of structure of a Lie algebra if and only if $d=g=i=0$. Moreover, $\mathfrak{l}_{0,\varepsilon}$ is restrictable if and only if $f=0$. For the end, the center $Z(\mathfrak{l}_{0,\varepsilon})$ is always locally free of rank 1. So we conclude using the fact that $\dim(o(\mathfrak{l}_0))=5$.

□

4.3.3. Study of $\mathrm{L}_{3,1}^{\mathrm{res}}$ in $\mathrm{L}_3^{(1)}$.

4.3.4. Proposition. *The k-point $\mathfrak{h}_3$ is smooth in $\mathrm{L}_{3,1}^{\mathrm{res}} \cap \mathrm{L}_3^{(1)}$.*

Proof. We have to add to the conditions found before the conditions $a=i$ and $d=-h$. □

4.3.5. Study of $\mathrm{L}_{3,1}^{\mathrm{res}}$ in $\mathrm{L}_3^{(2)}$.

4.3.6. Proposition. *The k-point $\mathfrak{h}_3$ is singular in $\mathrm{L}_{3,1}^{\mathrm{res}} \cap \mathrm{L}_3^{(2)}$ and the point $\mathfrak{l}_0$ is smooth.*

Proof. For both of those points, the condition $\det(M)=0$ is always satisfied for any deformation. □

4.4 In the stratum $\mathrm{L}_{3,3}$

This case is really simple because the condition "to be in the stratum $\mathrm{L}_{3,3}$" implies, using the same notations as in the previous subsections, that all the coefficient of the matrix $\mathfrak{ab}_{3,\varepsilon}$ are 0. Then, $\dim(T_{\mathrm{L}_{3,3}^{\mathrm{res}},\mathfrak{ab}_3})=0$, in the whole scheme and in the irreducible components. Then the point $\mathfrak{ab}_3$ is smooth seen in $\mathrm{L}_{3,3}^{\mathrm{res}}$.

As stated in the introduction, we can apply the previous results of smoothness to the moduli space $\mathcal{G}_{3,r}$, and this gives the following result:

4.4.1. Corollary. *The scheme $\mathcal{G}_{3,r}$ splits in two irreducible components that we denote by $\mathcal{G}_{3,r}^{(1)}$ and $\mathcal{G}_{3,r}^{(2)}$, and we have:*

- *If $p\neq 2$, $\mathcal{G}_{3,0}$ is singular, but becomes smooth after intersection with $\mathcal{G}_{3,0}^{(2)}$, if $p=2$, $\mathcal{G}_{3,0}$ is smooth.*

- $\mathcal{G}_{3,1}$ *is singular but becomes smooth when we intersect it with* $\mathcal{G}_{3,1}^{(1)}$.
- $\mathcal{G}_{3,2}$ *is empty and* $\mathcal{G}_{3,3}$ *is smooth.*

Proof. We have an equivalence of categories given by the functor Lie : $\mathcal{G}_{3,r} \to p\text{-}\mathcal{L}ie_{3,r}$. Moreover, the quotient morphism $\mathrm{L}_{3,r}^{p} \to p\text{-}\mathcal{L}ie_{3,r}$ is smooth. Then, because $\mathrm{L}_{3,r}^{p} \to \mathrm{L}_{3,r}^{\mathrm{res}}$ is smooth, we can apply the results of the subsections 4.2, 4.3 and 4.4. $\square$

Moduli of prolongations of finite group schemes

Let us first recall the definition of height. Over any base scheme S of characteristic $p > 0$, we say that a group scheme $G \to S$ is *of height* $\leq n$ if the n-th Frobenius morphism vanishes. If $S = \mathrm{Spec}(k)$ with k a field, we say in this case that the group is exactly *of height* n. For the end, over any based scheme S, we say that G is *of height* n if it is of height n in any fiber. As explained in the introduction, we are interested in the stack p-$\mathcal{L}ie$ of Lie p-algebras, and the stack $\mathcal{G}^r$ of finite locally free group schemes of height r. Such stacks are non separated, and this can be measured in different manners: for example we could study the fibers of the diagonal, or the lack of uniqueness in the valuative criterion for separatedness. We use the second way, which leads to the study of *models* of group schemes, that in the following we will call *prolongations*. Here is the set-up. Let k be a field. Let us fix a discrete valuation ring $R := k[\![t]\!]$, with fraction field $K := k(\!(t)\!) = k[\![t]\!][t^{-1}]$. For all this chapter, we fix a finite group scheme $G_K \to \mathrm{Spec}(K)$. We will study the space of all prolongations $\mathcal{P}$ of G_K, i.e. the space of finite flat group schemes G defined over $A[\![t]\!]$, where A is a k-algebra, endowed with an $A(\!(t)\!)$-isomorphism from $G_K \otimes A(\!(t)\!)$ to $G \otimes A(\!(t)\!)$. In the following, we will generalise results that have been obtained studying prolongations over a DVR, like in [Ra74] or in [BLR90]. We bring a new point of view in the following by studying these objects in *families*. Then in the first section we lay the foundations of the space of prolongations of a fixed finite group scheme G_K defined over $\mathrm{Spec}(K)$.

We start by proving that the functor we are looking at is an algebraic object: actually it is not representable by a scheme, but by an inductive limit of schemes, see Theorem 1.2.4. We develop two important tools to study $\mathcal{P}$: the dilatations of subgroup schemes of the special fiber of prolongations of G_K (see Subsection 2.3), and the action of the (Weil restriction of the) sheaf of automorphisms of G_K (see Subsection 2.4). In the following the focus will be done on the dilatations, for which we encounter two types of problems. The first one is that, taking a prolongation G defined over $A[\![t]\!]$, it can happen that the set of prolongation maps $G' \to G$ taking the special fiber into the chosen subgroup scheme of the special fiber is in fact empty. The second is that when trying to construct the dilatation, we have to ensure flatness, which we can do only for very specific rings (like low-dimensional regular ones). At least dilatations are well-known on a DVR (e.g. on $k[\![t]\!]$), and the work done in [BLR90] will allow us to study in details the k-points of the ind-scheme $\mathcal{P}$, and we will be able to represent $\mathcal{P}(k)$ as the set of vertices of a certain tree. We also succeed in constructing dilatations over any ring of the form $A[\![t]\!]$ with A regular of dimension at most 1.

1 Construction of the moduli space of prolongations

1.1 Prolongations

1.1.1. Notation.

- As stated above, in all this chapter we fix a finite group scheme G_K over $\mathrm{Spec}(K)$. In the literature, you can also find the notation G_K for the generic fiber of a certain group G. In order to avoid any ambiguity, for any group scheme G defined over R, we will write $G \otimes K := G \times_R \mathrm{Spec}(K)$ for its *generic fiber*. More generally, if A is a k-algebra, and if G is defined over $A[\![t]\!]$, we define the *generic fiber of* G and write $G \otimes A(\!(t)\!)$ the group scheme $G \times_{A[\![t]\!]} \mathrm{Spec}(A(\!(t)\!))$.
- Moreover, for any group $G \to \mathrm{Spec}(A[\![t]\!])$, we call *special fiber of* G the A-group scheme $G \times_{A[\![t]\!]} \mathrm{Spec}(A)$, where the $A[\![t]\!]$-module structure of A is given by $A[\![t]\!] \to A$, $t \mapsto 0$. It will be denoted by $G_{|t=0}$.

1.1.2. Definition. Let A be any k-algebra.

- A *prolongation of* G_K *above* A is a pair (G, i) where G is a finite locally free group scheme over $A[\![t]\!]$, and i is an $A(\!(t)\!)$-isomorphism of groups $i : G_K \otimes A(\!(t)\!) \xrightarrow{\sim} G \otimes A(\!(t)\!)$.
- A morphism between two prolongations (G_1, i_1) and (G_2, i_2), called a *prolongation map*, is a morphism of $A[\![t]\!]$-group schemes $f : G_1 \to G_2$ such that $f \otimes A(\!(t)\!) = i_2 \circ i_1^{-1}$.
- We denote by $\mathcal{P}_{G_K}(A)$ the set of prolongations of G_K above A, up to isomorphisms. When there will be no ambiguity on the group scheme G_K, we will just write $\mathcal{P}(A)$ instead of $\mathcal{P}_{G_K}(A)$.

1.1.3. Remark.

- In the literature, we can also find the terminology *model* instead of *prolongation*. Here we choose this word for reasons of euphony, the radical "mod-" (as in model, module, moduli) being a little invasive.
- By flatness (and because $t \nmid 0 \in A[\![t]\!]$), the generic fiber of a prolongation G is schematically dense in G.
- For any prolongation G of a group scheme G_K, we have $G \otimes A(\!(t)\!) \simeq G_K \otimes A(\!(t)\!)$. Then, thanks to the previous remark, if there exists a prolongation map between two prolongations, it is unique.
- Thanks to the previous remark, the category of prolongations is equivalent to a partially ordered set, and the order relation will be studied below.
- It is important to emphasize that, in the datum of a prolongation of a group scheme,

the given isomorphism on the generic fiber is equivalent to the datum of an inclusion

$$\mathcal{O}_G \subset \mathcal{O}_{G_K} \otimes A(\!(t)\!)$$

that turns $\mathcal{O}_G$ into a lattice of $A(\!(t)\!)^{\mathrm{rk}(G)}$ (see Definition 1.2.11).

In the following, we will sometimes use Cartier duality in order to take advantage of the symmetry it provides. But here, we do not suppose that the groups we are working with are commutative: indeed in particular we would like our setting to include the group schemes of rank p^3 and the corresponding Lie algebras that appear in Chapter II, Section 3 (e.g. SL_2), which are not commutative in general. Then, in order to use Cartier duality freely, we will also define the space of prolongations of a Hopf algebra (not necessarily commutative or co-commutative).
So for the rest of this chapter, let us fix $\mathcal{H}_K$ a finite Hopf algebra over K. The definition of a prolongation of a Hopf algebra can be stated as follows:

1.1.4. Definition. Let A be any k-algebra.

- A *prolongation of* $\mathcal{H}_K$ *above* A is a pair $(\mathcal{H}, i)$ where $\mathcal{H}$ is a Hopf algebra, finite locally free over $A[\![t]\!]$ and i is an $A(\!(t)\!)$-isomorphism of Hopf algebras $i : \mathcal{H} \otimes A(\!(t)\!) \xrightarrow{\sim} \mathcal{H}_K \otimes A(\!(t)\!)$.
- A morphism between two prolongations $(\mathcal{H}_1, i_1)$ and $(\mathcal{H}_2, i_2)$, called a *prolongation map*, is a Hopf algebra morphism $f : \mathcal{H}_2 \to \mathcal{H}_1$ such that $f \otimes A(\!(t)\!) = i_1^{-1} \circ i_2$.
- We denote by $\mathcal{P}_{\mathcal{H}_K}(A)$ the set of prolongations of $\mathcal{H}_K$ above A, up to isomorphisms.

We need to verify that the prolongations of a Hopf algebra are generalisations of prolongations of group schemes, in the sense that these objects coincide in the commutative case.

1.1.5. Proposition. *Let us suppose $\mathcal{H}_K$ is commutative. Let A be a k-algebra, and let us take $\mathcal{H} \in \mathcal{P}_{\mathcal{H}_K}(A)$. In this case $\mathcal{H}$ is also commutative, and the functor* Spec *gives an equivalence of categories:*

$$\mathcal{P}_{\mathcal{H}_K}(A) \xrightarrow{\sim} \mathcal{P}_{\mathrm{Spec}(\mathcal{H}_K)}(A)$$
$$\mathcal{H} \longmapsto \mathrm{Spec}(\mathcal{H}).$$

Proof. Let us write $\mathcal{H}_{A(\!(t)\!)} := \mathcal{H} \otimes_{A[\![t]\!]} A(\!(t)\!)$ for the image of $\mathcal{H}$ by the tensor product functor. By flatness, the canonical morphism $i : \mathcal{H} \otimes_{A[\![t]\!]} \mathcal{H} \hookrightarrow \mathcal{H}_{A(\!(t)\!)} \otimes_{A(\!(t)\!)} \mathcal{H}_{A(\!(t)\!)}$ is injective. Let $x \otimes y$ and $y \otimes x \in \mathcal{H} \otimes_{A[\![t]\!]} \mathcal{H}$. Then because $\mathcal{H}_{A(\!(t)\!)}$ is commutative, $i(x \otimes y) = i(y \otimes x)$, then by injectivity $x \otimes y = y \otimes x$. ☐

1.1.6. Remark. Doing the same proof, we see that if $\mathcal{H}_K$ is co-commutative, then so is $\mathcal{H}$ for any prolongation $\mathcal{H} \in \mathcal{P}_{\mathcal{H}_K}(A)$.

1.1.7. Theorem. *(Cartier duality) Let A be any k-algebra. The linear dual functor gives a* *natural and involutive isomorphism:*

$$\mathcal{P}_{\mathcal{H}_K}(A) \xrightarrow{\sim} \mathcal{P}_{\mathcal{H}_K^*}(A)$$
$$\mathcal{H} \longmapsto \mathcal{H}^*.$$

Proof. The reader can verify that the generic fiber of the dual is the dual of the generic fiber. Then this functor is well defined, and because all Hopf algebras $\mathcal{H}$ of $\mathcal{P}_{\mathcal{H}_K}(A)$ are finite locally free by definition, they verify the ordinary Cartier duality $\mathcal{H} \xrightarrow{\sim} \mathcal{H}^{**}$ so it is involutive. □

Let us see some basic properties of prolongations.

1.1.8. Proposition. *Let G_1 and $G_2 \in \mathcal{P}_{G_K}(A)$ be two prolongations of G_K. Let $f : G_1 \to G_2$ be a prolongation map. Then f is an epimorphism in the category of separated $A[\![t]\!]$-schemes. Moreover, because there is at most one prolongation map between two prolongations, in particular any prolongation morphism is an epimorphism and a monomorphism in the category of prolongations.*

Proof. Let $Z \to \operatorname{Spec}(A[\![t]\!])$ be any separated scheme and $v_1, v_2 : G_2 \to Z$ be two morphisms such that $v_1 \circ f = v_2 \circ f$. Let us write $U := G_2 \otimes A(\!(t)\!)$. Then U is schematically dense in G_2, and $U \subset \operatorname{im}(|f|)$. So we have $v_{1|U} = v_{2|U}$. But because Z is separated, we obtain $v_1 = v_2$. □

Let us remark that we can also define the moduli space of prolongations of a morphism:

1.1.9. Definition. Let H_K and G_K be two K-group schemes. Let $f_K : H_K \to G_K$ be a K-morphism. Let A be a k-algebra. Then a *prolongation of f_K above A* is a morphism $f : H \to G$ where $H \in \mathcal{P}_{H_K}(A)$ and $G \in \mathcal{P}_{G_K}(A)$, such that the following diagram is commutative:

$$\begin{array}{ccc} H & \xrightarrow{\quad f \quad} & G \\ \uparrow & & \uparrow \\ H_K \otimes A(\!(t)\!) & \xrightarrow{f_K} & G_K \otimes A(\!(t)\!). \end{array}$$

1.1.10. Remark. Like before, if there exists a morphism $f : H \to G$ which is a prolongation of a morphism f_K, this one is unique.

1.2 Representability of the functor of prolongations

The functor of all the prolongations of a finite group scheme G_K (or of a finite-dimensional Hopf algebra, a finite-dimensional Lie p-algebra vector bundle,...) is not a scheme in general (we will see examples in section 3.2). But thanks to the affine Grassmannian, we will see that it is at least an *ind-scheme* over $\operatorname{Spec}(k)$. Here, our definition of *ind-scheme* is the one you can find in [G10], Definition 2.3: let us call a *k-space* any functor from the category of k-schemes which is a sheaf for the fpqc-topology. An ind-scheme is a k-space which is the inductive limit (in the category of k-spaces) of an inductive system of schemes, indexed by $\mathbb{N}$, and where all the transition maps are closed immersions. We call an ind-scheme X of *ind-finite type*, or *ind-projective*, etc... if we can write $X = \varinjlim X_n$ where each X_n is of finite type, projective, etc... In the following, for any ring A, we write Aff_A for the site whose underlying category is the category of affine schemes over $\mathrm{Spec}(A)$, and the topology is the fpqc one.

1.2.1. Definition.

- Let G_K be a finite group scheme over $\mathrm{Spec}(K)$. We call *moduli presheaf of prolongations of* G_K the presheaf $\mathcal{P}_{G_K} : \mathrm{Aff}_k \to \mathrm{Set}$, which sends a k-algebra A to the set of all prolongations of G_K above A, and which sends a morphism $A \to B$ to the map $(\mathcal{P}(A) \to \mathcal{P}(B)), G \mapsto G \otimes_{A[\![t]\!]} B[\![t]\!] := \mathrm{Spf}(\mathcal{O}_G \hat{\otimes}_A B)$, where the "$\hat{\otimes}$" denotes the t-adically completed tensor product.
- Likewise, let $\mathcal{H}_K$ be a finite K-Hopf algebra. We call *moduli presheaf of prolongations of* $\mathcal{H}_K$ the presheaf $\mathcal{P}_{\mathcal{H}_K} : \mathrm{Aff}_k \to \mathrm{Set}$ which sends a k-algebra A to the set of all prolongations of $\mathcal{H}_K$ defined over $A[\![t]\!]$.
- Finally, for any finite group schemes H_K and G_K over $\mathrm{Spec}(K)$, for any K- morphism $f_K : H_K \to G_K$, we call *moduli presheaf of prolongations of* f_K the presheaf: $\mathcal{P}_{f_K} : \mathrm{Aff}_k \to \mathrm{Set}$ which send a k-algebra A to the set of all morphisms which are prolongations of f_K.

1.2.2. Remark. (Extending $\mathcal{P}$ to the category of ind-schemes, of ind-(finite type)).
In later constructions, it will be convenient to handle objects like the universal prolongation, which lives over the ind-scheme $\mathcal{P}$. With this in mind, we explain in this remark how to extend the functor $\mathcal{P}$ to the category of all k-schemes of finite type (not only the affine ones), and then to the category of ind-schemes, of ind-finite type over k. First of all, we need to associate to any k-scheme S an object that we will denote by $S[\![t]\!]$, which is a generalisation of the scheme $\mathrm{Spec}(A[\![t]\!])$ when $S = \mathrm{Spec}(A)$, and we will need to make sense of the definition of a "generic fiber". This is for this reason that the hypothesis "of finite type" appears. Then we associate to any k-scheme S a formal scheme defined by:

$$S[\![t]\!] := S \times_{\mathrm{Spf}(k)} \mathrm{Spf}(R)$$

where S and $\mathrm{Spec}(k)$ are seen as discrete formal schemes, and the fiber product is the one of the category of formal schemes, so it is defined on affine schemes by

$$\mathrm{Spec}(B) \times_{\mathrm{Spf}(k)} \mathrm{Spf}(R) := \mathrm{Spf}(B \hat{\otimes}_k R).$$

Then a prolongation of a group scheme G_K over a scheme S will be a finite, locally free t-adic formal group scheme

$$\mathfrak{G} \to S[\![t]\!].$$

Now we need to adapt the definition of the generic fiber of such a scheme. In order to do this, we will use the tools of the *rigid analytic geometry* as introduced by Tate in the 60s. We want to define a functor from the category of formal R-schemes to the category of rigid K-spaces, which then will be interpreted as associating to a formal R-scheme $\mathfrak{X}$ its "generic fiber" that we will denote by $\mathfrak{X}_{\mathrm{rig}}$. On affine topologically of finite type formal schemes, we define this functor by

$$\mathrm{rig} : \mathfrak{X} = \mathrm{Spf}(A) \mapsto \mathfrak{X}_{\mathrm{rig}} = \mathrm{Spm}(A \otimes_R K).$$

Then one can show that this construction can be generalised to all formal R-schemes topologically of finite type, by gluing. Then we obtain:

1.2.3. Proposition. *The functor $A \mapsto A \otimes_R K$ on R-algebras of topologically finite type gives rise to a functor $\mathfrak{X} \mapsto \mathfrak{X}_{\mathrm{rig}}$, from the category of formal R-schemes that are locally of topologically finite type, to the category of rigid K-spaces. It is defined locally as above.*

Proof. See [Bo14], 7.4/3. □

This functor is compatible with the classical generic fiber, in the sense that for any formal R-scheme $\mathfrak{X} = \hat{X}$ which is the completion of a R-scheme X, then

$$\mathfrak{X}_{\mathrm{rig}} = (X \otimes K)^{\mathrm{an}}$$

where for any K-scheme Y of finite type, we denote by Y^{an} the analytification of Y.

Then now we can extend $\mathcal{P}$ to all k-schemes of finite type. Indeed let S be such a scheme. Let us write $\mathrm{ForSch}(S[\![t]\!])$ for the category of formal schemes over $S[\![t]\!]$. We define

$$\mathcal{P}(S) := \left\{ \begin{array}{c} \mathfrak{G} \in \mathrm{ForSch}(S[\![t]\!]),\ \text{finite locally free group} \\ \text{with a group isomorphism } i : (G_K)^{\mathrm{an}} \xrightarrow{\sim} \mathfrak{G}_{\mathrm{rig}}. \end{array} \right\}$$

We need to see that this definition is the one we were looking for, in the sense that it is a generalisation of the set $\mathcal{P}(A)$ when $S = \mathrm{Spec}(A)$ is affine. This is given by Grothendieck's existence theorem of algebraic coherent sheaves (see [EGA3], première partie, §5, 5.1.4). Because when $S = \mathrm{Spec}(A)$, then $S[\![t]\!] = \mathrm{Spf}(A[\![t]\!])$, this theorem implies that the t-adic completion functor

$$\left\{ \begin{array}{c} \text{Finite locally free group} \\ \text{scheme over } \mathrm{Spec}(A[\![t]\!]) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Finite locally free } t\text{-adic formal} \\ \text{group schemes over } \mathrm{Spf}(A[\![t]\!]) \end{array} \right\}$$

is an equivalence.

Now we can also extend the functor $\mathcal{P}$ to the category of ind-schemes, of ind-(finite type) as follows: let

$$X = \varinjlim_{i \in I} X_i$$

be a presentation of X with schemes of finite type. We define

$$\mathcal{P}(X) := \varprojlim_{i\in I}\mathcal{P}(X_i).$$

This definition does not depend on the chosen presentation: indeed let us take another presentation

$$X = \varinjlim_{j\in J} X'_j$$

where each X'_j is of finite type over $\mathrm{Spec}(k)$. Let $j \in J$ and let us consider the inclusion $X'_j \hookrightarrow X$. Because X'_j is of finite type, it is quasi-compact, so there exists $i \in I$ such that $X'_j \subset X_i$ thanks to Lemma 2.4 in [G10]. Likewise, for all $i \in I$ there exists $j \in J$ such that $X_i \subset X'_j$. Then we obtain

$$\varprojlim_{i\in I}\mathcal{P}(X_i) = \varprojlim_{j\in J}\mathcal{P}(X'_j).$$

□

The rest of this subsection is devoted to the proof of the representability of $\mathcal{P}$. Actually, we prove the representability of $\mathcal{P}_{\mathcal{H}_K}$ where $\mathcal{H}_K$ is a finite K-Hopf algebra (not necessarily commutative) instead, because we will need it in the following in order to apply Cartier duality, and because it is more general.

1.2.4. Theorem. *The functor $\mathcal{P}_{\mathcal{H}_K}$ is representable by an ind-scheme, ind-projective on* $\mathrm{Spec}(k)$.

In order to prove this, we will need to recall and introduce some notations and results about the affine Grassmannian. Our main reference is [G10]. Let V be a k-vector space of dimension n.

1.2.5. Definition. Let $r \leq n$. Let us write $\mathrm{Gr}(V, r)$ for the following functor:

$$\begin{aligned}\mathrm{Gr}(V,r) : \mathrm{Aff}_k &\longrightarrow \mathrm{Set}\\ A &\longmapsto \{U \subset V \otimes A \text{ sub-}A\text{-module s.t. } (V\otimes A)/U \text{ is loc. free of rank } n-r\}.\end{aligned}$$

1.2.6. Proposition. *For all $r \in \mathbb{N}$, the functor $\mathrm{Gr}(V,r)$ is representable by a projective scheme over k.*

Proof. See Proposition 8.14 and Proposition 8.23 in [GW20]. □

1.2.7. Definition. For any vector $v \in V$, we write $\mathrm{Gr}(V,r)(v)$ for the subfunctor of $\mathrm{Gr}(V,r)$ comprising the $U \subset V \otimes A$ which contain $v \otimes 1$.

1.2.8. Proposition. *Let $v \in V$. Then the monomorphism $\mathrm{Gr}(V,r)(v) \subset \mathrm{Gr}(V,r)$ is representable by a closed immersion, so the functor $\mathrm{Gr}(V,r)(v)$ is representable by a projective scheme over k.*

Proof. Le A be a k-algebra and let us consider a morphism $\mathrm{Spec}(A) \to \mathrm{Gr}(V, r)$. This morphism corresponds to a module $U \in \mathrm{Gr}(V, r)(A)$. Let us write $X := \mathrm{Gr}(V, r)(v) \times_{\mathrm{Gr}(V,r)} \mathrm{Spec}(A)$. We need to show that the morphism

$$X \to \mathrm{Spec}(A)$$

is a closed immersion. Let A' be an A-algebra. Then $X(A')$ is non empty if and only if the vector $v \otimes 1 \in U \otimes A'$ if and only if $v \in U$. But by hypothesis, $U \otimes A'$ is a locally direct factor of $V \otimes A'$, and because the property "$v \otimes 1 \in U \otimes A$" is local on A, we can localise and suppose that U is a direct factor of $V \otimes A$. So let us write W such that $U \oplus W = V \otimes A'$. Then $U/(v) \oplus W = V/(v) \otimes A'$ and so $U/(v)$ is a locally direct factor of $V/(v) \otimes A'$. Then we obtain an isomorphism

$$\mathrm{Gr}(V, r)(v)(A') \simeq \mathrm{Gr}(V/(v), r)(A'),$$

so the inclusion we are looking for is representable by the closed immersion of ind-projective schemes:

$$\mathrm{Gr}(V/(v), r)(v) \hookrightarrow \mathrm{Gr}(V, r).$$

□

We recall here the definition of the functor of *lattices*, because we will need it to prove that $\mathcal{P}$ is representbale by an ind-scheme.

1.2.9. Notation. Let $N \in \mathbb{N}$.

- Let us write $E_N := t^{-N}R^n/t^N R^n$. We also write

$$\mathrm{Gr}(E_N) := \coprod_{0 \le r \le 2N} \mathrm{Gr}(E_N, r).$$

- We write $\mathrm{Gr}_t(E_N, r)$ for the closed subscheme of $\mathrm{Gr}(E_N, r)$ which parameterizes the submodules which are stable by t. Similarly, we write

$$\mathrm{Gr}_t(E_N) := \coprod_{0 \le r \le 2N} \mathrm{Gr}_t(E_N, r).$$

1.2.10. Definition. For $N \in \mathbb{N}$, let us write $\mathcal{L}att_n^N$ for the following functor:

$$\mathcal{L}att_n^N : \mathrm{Aff}_k \to \mathrm{Set}$$
$$A \mapsto \left\{ \begin{array}{l} M \subset A(\!(t)\!)^n, \text{ sub-}A[\![t]\!] \text{ module}, t^N A[\![t]\!]^n \subset M \subset t^{-N} A[\![t]\!]^n \\ \text{and } t^{-N}A[\![t]\!]^n/M \text{ is projective of finite rank over } A \end{array} \right\}.$$

We also write $\mathcal{L}att_n := \varinjlim_N \mathcal{L}att_n^N$.

1.2.11. Definition. For any k-algebra A, we call *lattice of* $A(\!(t)\!)^n$ an element of $\mathcal{L}att_n(A)$.

Let us recall a classical lemma, which shows that the projective property of the lattices is local on R. This will help us for technical reasons in representability proofs like the one of Proposition 1.2.16, Lemma 2.1.4 or also Theorem 2.2.8.

1.2.12. Lemma. *Let k be a field and let R be a k-algebra. Let $\mathcal{L} \subset R((t))^n$ be an $R[\![t]\!]$-submodule. Then the following are equivalent:*

- *The submodule $\mathcal{L}$ is a lattice.*
- *The submodule $\mathcal{L}$ is a projective $R[\![t]\!]$-module and $\mathcal{L} \otimes_{R[\![t]\!]} R((t)) = R((t))^n$.*
- *Zariski-locally on R, $\mathcal{L}$ is a free $R[\![t]\!]$-module of rank n (i.e. there exist $f_1, \cdots, f_n \in R$ such that $(f_1, \cdots, f_n) = R$ and for all i, $\mathcal{L} \otimes_{R[\![t]\!]} R_{f_i}[\![t]\!]$ is a free $R_{f_i}[\![t]\!]$- module of rank n) and $\mathcal{L} \otimes_{R[\![t]\!]} R((t)) = R((t))^n$.*
- *Fpqc-locally on R, $\mathcal{L}$ is a free $R[\![t]\!]$-module of rank n (i.e. there exists a faithfully flat ring homomorphism $R \to R'$ such that $\mathcal{L} \otimes_{R[\![t]\!]} R'((t))$ is a free $R'((t))$-module) and $\mathcal{L} \otimes_{R[\![t]\!]} R((t)) = R((t))^n$.*

Proof. See [G10], Lemma 2.11. □

In our case, this means that the ring of functions of a prolongation of G_K is free after localising on R instead of $R[\![t]\!]$.

1.2.13. Proposition. *For any $N \in \mathbb{N}$, the functor $\mathcal{L}att_n^N$ is representable by a projective scheme over k, then $\mathcal{L}att_n$ is an ind-scheme, ind-projective.*

Proof. See [G10], part 2.3. □

1.2.14. Lemma. *Let $v \in K^n$. Let $\mathcal{L}att_n(v) \subset \mathcal{L}att_n$ be the subfunctor comprising the lattices of $A((t))^n$ which contain v. Then, $\mathcal{L}att_n(v) \subset \mathcal{L}att_n$ is representable by a closed immersion of k-ind-schemes.*

Proof. It suffices to show that $\mathcal{L}att_n^N(v) := \mathcal{L}att_n(v) \cap \mathcal{L}att_n^N$ is representable by a closed immersion of k-schemes, for N large enough. Let us choose $N \in \mathbb{N}$ such that $v \in t^{-N}R^n$, and let $M \in \mathcal{L}att_n^N(A)$ for a k-algebra A. Then the image of v in $t^{-N}A[\![t]\!]^n$ is in M if and only if its image $\overline{v} \in t^{-N}A[\![t]\!]^n/t^N A[\![t]\!]^n$ is in $M/t^N A[\![t]\!]^n$. Then thanks to the bijection

$$\mathcal{L}att_n^N(A) \xrightarrow{\sim} \mathrm{Gr}_t(E_N)(A)$$

we obtain

$$\mathcal{L}att_n^N(v)(A) \xrightarrow{\sim} (\mathrm{Gr}_t(E_N) \cap \mathrm{Gr}(E_N)(\overline{v}))(A).$$

We conclude with Proposition 1.2.8. □

Proof of Theorem 1.2.4. Let n be the rank of $\mathcal{H}_K$, and let us fix an isomorphism of K-vector spaces: $\mathcal{H}_K \xrightarrow{\sim} K^n$. For any k-algebra A, the set $\mathcal{P}_{\mathcal{H}_K}(A)$ can be identified by flatness with the subset of $\mathcal{L}att_n(A)$ of lattices $M \in K^n$ such that the structure morphisms (multiplication, comultiplication, unit, counit) of the Hopf algebra $\mathcal{H}_K \otimes_K A((t))$ stabilise M. Let us write m for the multiplication of $\mathcal{H}_K$, seen as an element of the free module $\mathcal{H}_K^* \otimes \mathcal{H}_K^* \otimes \mathcal{H}_K$, where we write $\mathcal{H}_K^*$ for the K-linear dual of $\mathcal{H}_K$. The subset of $\mathcal{L}att_n(A)$ of the lattices $M \subset A((t))^n$ such that $m \otimes 1$ stabilises M can be identified with the A-points of the closed sub-ind-scheme of $\mathcal{L}att_n$, inverse image of $\mathcal{L}att_{n^3}(m) \subset \mathcal{L}att_{n^3}$ by

$$\begin{aligned} \mathcal{L}att_n &\to \mathcal{L}att_{n^3} \\ M &\mapsto M^* \otimes M^* \otimes M. \end{aligned}$$

Using the same reasoning for the three other structure morphisms of the Hopf algebra, we obtain that $\mathcal{P}_{\mathcal{H}_K}$ is the intersection of four closed sub-ind-schemes in $\mathcal{L}att_n$. □

1.2.15. Remark. Thanks to Proposition 1.1.5, we obtain that $\mathcal{P}_{G_K}$ is representable by an ind-scheme, ind-projective over $\mathrm{Spec}(k)$.

Likewise, the moduli space of prolongations of a morphism f_K is also representable by an ind-scheme:

1.2.16. Proposition. *Let $f_K : H_K \to G_K$ be a K-morphism. Then the morphism*

$$\begin{aligned} i : \mathcal{P}_{f_K} &\hookrightarrow \mathcal{P}_{H_K} \times \mathcal{P}_{G_K} \\ \left(H \xrightarrow{f} G\right) &\longmapsto (H, G) \end{aligned}$$

is representable by a closed immersion of finite presentation.

Proof. Let A be a k-algebra and let us consider a morphism $\mathrm{Spec}(A) \to \mathcal{P}_{H_K} \times \mathcal{P}_{G_K}$. Let us write $H \in \mathcal{P}_{H_K}(A)$ and $G \in \mathcal{P}_{G_K}(A)$ the corresponding prolongations of respectively H_K and G_K. Then the fiber product X of the later morphism and i is given by:

$$\begin{aligned} X : \{\mathrm{Aff}_A\} &\to \mathrm{Set} \\ A' &\mapsto \begin{cases} \{\emptyset\} \text{ if there exists a morphism } f : H \to G \text{ which is a prolongation of } f_K \\ \emptyset \text{ otherwise.} \end{cases} \end{aligned}$$

Let us show that X is representable by a closed subscheme of $\mathrm{Spec}(A)$. Because this is local on $\mathrm{Spec}(A)$ and thanks to Lemma 1.2.12, we can suppose that $\mathrm{Spec}(A)$ is small enough so that $\mathcal{O}_G$ and $\mathcal{O}_H$ are free $A[\![t]\!]$-modules, say $\mathcal{O}_H = \bigoplus_{j=1}^{n} A[\![t]\!]e_j$ and $\mathcal{O}_G = \bigoplus_{i=1}^{m} A[\![t]\!]f_i$. Then there exists a morphism $f : H \to G$ prolongation of f_K if and only if there exists a morphism

$f^{\#} : \bigoplus_{i=1}^{m} A[\![t]\!] f_i \to \bigoplus_{j=1}^{n} A[\![t]\!] e_j$ such that the following diagram commutes:

$$\begin{array}{ccc} \bigoplus_{j=1}^{n} A[\![t]\!] e_j & \overset{f^{\#}}{\dashleftarrow} & \bigoplus_{i=1}^{m} A[\![t]\!] f_i \\ \big\downarrow & & \big\downarrow \\ \bigoplus_{j=1}^{n} A(\!(t)\!) e_j & \underset{f_K^{\#}}{\longleftarrow} & \bigoplus_{i=1}^{m} A(\!(t)\!) f_i . \end{array}$$

Now for all $i \in [1, m]$ let us write

$$f_K^{\#}(f_i) = \sum_{j=1}^{n} a_{i,j} e_j$$

with for all $i, j \in [1, n] \times [1, m]$,

$$a_{i,j} = \sum_{k=-N_j}^{+\infty} a_{i,j,k} t^k.$$

Hence we see that there exists such a morphism $f^{\#}$ if and only if $a_{i,j,k} = 0$ for all $i, j \in [1, n] \times [1, m]$ and for all $k \in [-N_j, -1]$. So this fiber product is representable by $V(\{a_{i,j,k}\}) \subset \mathrm{Spec}(A)$ which is of finite presentation because the set $\{a_{i,j,k}\}$ is finite. □

We would like to know more about the ind-scheme $\mathcal{P}$. For example, we would like to know its topology. In the following subsections, we will develop tools that will be useful to study $\mathcal{P}$, and we will see some examples.

2 Inner algebraic structure

2.1 Schematic closure

This section is devoted to schematic closure of a closed subgroup of G_K inside a prolongation. The classical schematic closure, in the sense of the Zariski closure is not flat in general. Then because we will need a closure that is an object of $\mathcal{P}$, we introduce in this section a *flat closure*. We recall that we fixed a finite K-group scheme G_K, and if it is not otherwise stated we write $\mathcal{P}$ instead of $\mathcal{P}_{G_K}$.

2.1.1. Definition. Let A be a k-algebra and $G \in \mathcal{P}(A)$. Let H_K be a closed subgroup of G_K. We call *flat closure of H_K in G*, if it exists, a closed flat subgroup scheme $\overline{H_K}^G$ of G such that $\overline{H_K}^G \otimes A(\!(t)\!) \simeq H_K \otimes A(\!(t)\!)$. When there will be no ambiguity on the group scheme G, we will only write $\overline{H_K}$. We define in a dual way the corresponding notion for a Hopf algebra $\mathcal{H} \in \mathcal{P}_{\mathcal{H}_K}(A)$.

2.1.2. Lemma. *The flat closure of H_K in G exists if and only if the classical Zariski closure of H_K in G is flat, and in this case these objects coincide. In particular, when the flat closure exists, it is unique.*

Proof. Let us suppose that the flat closure of H_K exists. Let us write H_2 for the schematic closure of $i : H_K \hookrightarrow G \otimes A((t))$. By the universal property of the Zariski closure, we have an inclusion $H_2 \subset \overline{H_K}$, then we have a surjection $\mathcal{O}_{\overline{H_K}} \twoheadrightarrow \mathcal{O}_{H_2}$. We have a commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_{\overline{H_K}} & \twoheadrightarrow & \mathcal{O}_{H_2} \\ \downarrow & & \downarrow \\ \mathcal{O}_{\overline{H_K}} \otimes A((t)) & \longrightarrow & \mathcal{O}_{H_2} \otimes A((t)) \end{array}$$

but because the schematic closure commutes with flat base change, we know that

$$H_2 \otimes A((t)) \simeq H_K \otimes A((t)) \simeq \overline{H_K} \otimes A((t)).$$

Then the arrow at the bottom of the diagram is an isomorphism, so $\mathcal{O}_{\overline{H_K}} \twoheadrightarrow \mathcal{O}_{H_2}$ is injective, and the lemma is proved. □

2.1.3. Definition. We call *functor of flat closure of H_K in G* the functor which maps an A-algebra A' to the set of flat closures of H_K in $G \otimes A'[[t]]$, and we write it $F_{\overline{H_K}^G} : \mathrm{Aff}_A \to \mathrm{Set}$.

2.1.4. Theorem. *The functor of flat closure is representable by an ind-scheme over* $\mathrm{Spec}(A)$*, which is an open in the ind-projective scheme of prolongations.*

Proof. Let us call $f_K : H_K \otimes A((t)) \longleftrightarrow G_K \otimes A((t))$ the natural closed immersion. Let $\mathcal{P}^{\mathrm{CI}}_{f_K}$ be the sub-functor of $\mathcal{P}_{f_K}$ defined as follows:

$$\begin{aligned} \mathcal{P}^{\mathrm{CI}}_{f_K} : \{\mathrm{Aff}_A\} &\to \mathrm{Set} \\ A' &\mapsto \{f \in \mathcal{P}_{f_K}(A') \text{ such that } f \text{ is a closed immersion}\}. \end{aligned}$$

Then by definition, $\mathcal{P}^{\mathrm{CI}}_{f_K}(A') = \{\text{flat closure of } H_K \text{ in } G \otimes A'[[t]]\} = F_{\overline{H_K}}(A')$. Let us show that the inclusion $\mathcal{P}^{\mathrm{CI}}_{f_K} \hookrightarrow \mathcal{P}_{f_K}$ is representable by an open immersion. Let $\mathrm{Spec}(A') \to \mathcal{P}_{f_K}$ be a point of $\mathcal{P}_{f_K}$ given by $f : H \to G$ where $H \in \mathcal{P}_{H_K}(A')$ and $G \in \mathcal{P}_{G_K}(A')$. Let $\tilde{A}$ be an A'-algebra and let us write $X := \mathcal{P}^{\mathrm{CI}}_{f_K} \times_{\mathcal{P}_{f_K}} \mathrm{Spec}(A')$. Then

$$X(\tilde{A}) = \begin{cases} \{\emptyset\} & \text{if } f \otimes \tilde{A}((t)) \text{ is a closed immersion} \\ \emptyset & \text{otherwise} \end{cases}.$$

Because being representable is local over $\mathrm{Spec}(A')$, thanks to Lemma 1.2.12, we can suppose $\mathrm{Spec}(A')$ small enough so that $\mathcal{O}_H$ and $\mathcal{O}_G$ are free $A'[[t]]$-modules, and let us write $m = \mathrm{rk}(\mathcal{O}_G)$. In this case, f is a closed immersion if and only if the corresponding morphism $f^{\#}$ on the rings of functions is surjective. Let us write M for the corresponding matrix and let us write $d_1, \cdots, d_k$ for its minors of size m. Then f is surjective if and only if there exists a minor d_i which is invertible if and only if there exists a minor d_i such that its constant coefficient d_i^0 is invertible in A'. Then X is representable by $D(d_1^0) \cup D(d_2^0) \cup \cdots \cup D(d_k^0) \subset \mathrm{Spec}(A')$. □

We present here some useful properties about the functor of flat schematic closure.

2.1.5. Lemma. *The functor $F_{\overline{H_K}^G}$ commutes with base change.*

Proof. Let $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be an A-scheme. We need to show that

$$F_{\overline{H_K}^G} \times \mathrm{Spec}(B) \simeq F_{\overline{H_K}^{G \otimes B}}.$$

This is by definition of the fiber product of a functor. □

2.1.6. Lemma. *Let us fix a closed subgroup H_K of G_K. Let $k \subset \kappa$ be a field extension, and let $G \in \mathcal{P}(\kappa)$. Then in this case*

$$F_{\overline{H_K}} \simeq \mathrm{Spec}(\kappa).$$

Proof. Let A be a κ-algebra. Let us see that $F_{\overline{H_K}}(A) = \mathrm{Hom}_\kappa(\mathrm{Spec}(A), \mathrm{Spec}(\kappa)) = \{\emptyset\}$. This is true because here the flat closure always exists and is given by the Zariski closure: indeed this one is flat because it is without torsion on $\kappa[\![t]\!]$. See [Ra74], section 2 for more details. □

2.1.7. Corollary. *Let H_K be a closed subgroup of G_K. Let A be a k-algebra and let $G \in \mathcal{P}(A)$. Then the morphism of ind-schemes $\Phi : F_{\overline{H_K}^G} \to \mathrm{Spec}(A)$ is a bijective monomorphism.*

Proof. Let $x \in \mathrm{Spec}(A)$ and let $\mathrm{Spec}(\kappa) \to \mathrm{Spec}(A)$ be the corresponding morphism. The formation of flat closure commutes with base change thanks to Lemma 2.1.5, so we have that

$$\Phi^{-1}(x) := F_{\overline{H_K}^G} \times_A \mathrm{Spec}(\kappa) = F_{\overline{H_K}^{G \otimes \kappa}} \simeq \mathrm{Spec}(\kappa)$$

thanks to Lemma 2.1.6. □

2.1.8. Lemma. *The flat closure is functorial in the following sense: let $G_1, G_2 \in \mathcal{P}(A)$ and let H_1, H_2 be two closed subgroups of G_K. Let $f : G_1 \to G_2$ be a prolongation map such that $f_{|H_1}$ factorises through H_2. Let us suppose that $\overline{H_1}^{G_1}$ and $\overline{H_2}^{G_2}$ exist. Then in this case:*

$$f\left(\overline{H_1}^{G_1}\right) \subset \overline{H_2}^{G_2}.$$

Proof. For $j = 1, 2$ let us write $H_j = V(I_j) \subset G_j \otimes A(\!(t)\!)$. Then the morphism

$$H_j \subset G_j$$

is given on the rings by the morphism

$$\phi_j : \mathcal{O}_{G_i} \to \mathcal{O}_{G_i} \otimes A[\![t]\!]/I_j = \mathcal{O}_{H_j}.$$

For $j = 1, 2$, let us write $K_j := \ker(\phi_j)$. Then $\overline{H_j}^{G_j} = V(K_j) \subset G_j$. Because $f_{|H_1}$ factorises through H_2, we have this commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_{G_1} & \xleftarrow[f^\#]{} & \mathcal{O}_{G_2} \\ \Big\downarrow{\scriptstyle \phi_1} & & \Big\downarrow{\scriptstyle \phi_2} \\ \mathcal{O}_{H_1} & \xleftarrow[f^\# \otimes \mathrm{id}]{} & \mathcal{O}_{H_2}. \end{array}$$

In order to see that $f\left(\overline{H_1}^{G_1}\right) \subset \overline{H_2}^{G_2}$, it suffices to show $f^\#(K_2) \subset K_1$. Let us take $x \in K_2$. Then $\phi_2(x) = 0$ so $\phi_1(f^\#(x)) = 0$, hence we obtain $f^\#(x) \in K_1$. □

2.2 Order relation, supremum and infimum

In this subsection, we will study the existence of morphisms between prolongations. Because we will use the Cartier duality, we start by defining the order relation on the space of prolongations of a Hopf algebra.

2.2.1. Definition. Let $\mathcal{H}_K$ be a K-Hopf algebra, and let A be a k-algebra. Let $\mathcal{H}$ and $\mathcal{H}' \in \mathcal{P}_{\mathcal{H}_K}(A)$. We write $\mathcal{H}' \geq \mathcal{H}$ if there exists a prolongation morphism $\mathcal{H}' \leftarrow \mathcal{H}$.

Likewise, for two group schemes G and G' prolongations of a finite group scheme G_K above A, we write $G' \geq G$ if there exists a prolongation morphism $G' \to G$.

2.2.2. Lemma. *The relation defined above is an order relation on $\mathcal{P}_{\mathcal{H}}$. Moreover the Cartier duality inverts the order.*

Proof. This is left to the reader. □

We will use the theory of reflexive module in order to prove that if A is a regular ring of dimension at most 1, the set $\mathcal{P}(A)$ is connected for the order relation. Let us recall a flatness criterion for the modules over a regular ring of dimension 2.

2.2.3. Definition. We say that a finite module M over a ring A is *reflexive* if the morphism $M \to M^{\vee\vee}$ is an isomorphism.

2.2.4. Proposition. *Let S be a regular scheme. Then a reflexive coherent sheaf $\mathcal{F}$ on S is locally free except along a closed subset S' of codimension ≥ 3. In particular, a reflexive sheaf on a regular scheme of dimension 2 is locally free.*

Proof. See Corollary 1.4 in [H80]. □

In particular, for any finite module M over a regular ring of dimension 2, the module $M^{\vee\vee}$ is flat (because it is reflexive).

2.2.5. Proposition. *Let A be a regular ring of dimension lower than 1. Let $\mathcal{H}, \mathcal{H}' \in \mathcal{P}_{\mathcal{H}_K}(A)$. Then, there exist a supremum and an infimum for $\mathcal{H}$ and $\mathcal{H}'$ for the order relation on $\mathcal{P}(A)$.*

Proof. We will adapt the proof of Proposition 2.2.2 in [Ra74], because here we do not suppose that the Hopf algebra $\mathcal{H}_K$ is co-commutative.

- Let us first show that there exists a supremum for $\mathcal{H}$ and $\mathcal{H}'$. Let us consider the tensor product $\mathcal{H} \otimes \mathcal{H}'$, and let us write

$$i : \mathcal{H} \otimes \mathcal{H}' \hookrightarrow \mathcal{H}_K \otimes \mathcal{H}_K \otimes A(\!(t)\!).$$

Let us consider the diagonal morphism:

$$\Delta : \mathcal{H}_K \otimes \mathcal{H}_K \otimes A(\!(t)\!) \twoheadrightarrow \mathcal{H}_K \otimes A(\!(t)\!),\ x \otimes y \otimes 1 \mapsto xy \otimes 1.$$

We denote by $I := \ker(\Delta \circ i)$ and let us write

$$\mathcal{H}_1 := (\mathcal{H} \otimes \mathcal{H}'/I)^{\vee\vee}.$$

Then we have that

$$(\mathcal{H} \otimes \mathcal{H}'/I)^{\vee\vee} \otimes A(\!(t)\!) \simeq (\mathcal{H} \otimes \mathcal{H}'/I) \otimes A(\!(t)\!) \simeq \mathcal{H}_K \otimes A(\!(t)\!).$$

Moreover, using Proposition 2.2.4, we know that $\mathcal{H}_1$ is flat over $A[\![t]\!]$.
Let us consider the diagram

$$\begin{array}{ccccc} \mathcal{H} \longrightarrow \mathcal{H}^{\vee\vee} & & & & \\ & \searrow & & & \\ & & (\mathcal{H} \otimes \mathcal{H}')^{\vee\vee} & \xrightarrow{j} & \mathcal{H}_1. \\ & \nearrow & & & \\ \mathcal{H}' \longrightarrow \mathcal{H}'^{\vee\vee} & & & & \end{array}$$

So $\mathcal{H}_1$ is greater than or equal to $\mathcal{H}$ and $\mathcal{H}'$. Now let us show that $\mathcal{H}_1$ is the supremum of $\mathcal{H}$ and $\mathcal{H}'$. Let $\mathcal{M} \in \mathcal{P}(A)$ such that there exist two morphisms of prolongations $u : \mathcal{H} \to \mathcal{M}$ and $v : \mathcal{H}' \to \mathcal{M}$. Let us write

$$f := u \times v : \mathcal{H} \otimes \mathcal{H}' \xrightarrow{u \otimes v} \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}.$$

We have this commutative diagram, where we embed $\mathcal{M}$ in its generic fiber that we identify with $\mathcal{H}_K \otimes A(\!(t)\!)$:

$$\begin{array}{ccc} \mathcal{H}_K \otimes \mathcal{H}_K \otimes A(\!(t)\!) & \xrightarrow{\Delta} & \mathcal{H}_K \otimes A(\!(t)\!) \\ {\scriptstyle i}\uparrow & & \uparrow \\ \mathcal{H} \otimes \mathcal{H}' & \xrightarrow{\quad f \quad} & \mathcal{M}. \end{array}$$

Then $f(I) = \{0\}$ where $I = \ker(\Delta \circ i)$, and so the morphism f factorises :

$$f : \mathcal{H} \otimes \mathcal{H}'/I \to \mathcal{M}$$

and by functoriality of the double dual we obtain a morphism of prolongations :

$$f : \mathcal{H}_1 = (\mathcal{H} \otimes \mathcal{H}'/I)^{\vee\vee} \to \mathcal{M}^{\vee\vee} = \mathcal{M}$$

because $\mathcal{M}$ is finite locally free. Then $\mathcal{M} \geq \mathcal{H}_1$ and $\mathcal{H}_1$ is a supremum for $\mathcal{H}$ and $\mathcal{H}'$.

- Let us show now that there exists $\mathcal{H}_2 \in \mathcal{P}(A)$ such that $\mathcal{H}_2 = \inf(\mathcal{H}, \mathcal{H}')$. Thanks to the previous point, we know that there exists $\mathcal{H}_3 \in \mathcal{P}(A)$ such that

$$\mathcal{H}_3 = \sup(\mathcal{H}^*, \mathcal{H}'^*).$$

Then the Hopf algebra $\mathcal{H}_2 := \mathcal{H}_3^*$ still belongs to the set $\mathcal{P}(A)$ and is the infimum of $\mathcal{H}$ and $\mathcal{H}'$.

□

2.2.6. Corollary. *Let A be a regular ring of dimension lower than* 1*. Let $G, G' \in \mathcal{P}_{G_K}(A)$ be two prolongations of a group scheme. Then there exist an infimum and a supremum in $\mathcal{P}_{G_K}(A)$ for the order relation.*

Proof. Thanks to Proposition 2.2.5, we know that there exist two $A[\![t]\!]$-Hopf algebras $\mathcal{H}_1$ and $\mathcal{H}_2$, finite locally free, prolongation of $\mathcal{O}_{G_K}$ (hence commutative thanks to Proposition 1.1.5) such that $\mathcal{H}_1 = \sup(\mathcal{O}_G, \mathcal{O}_{G'})$ and $\mathcal{H}_2 = \inf(\mathcal{O}_G, \mathcal{O}_{G'})$. Then we obtain

$$G_1 := \mathrm{Spec}(\mathcal{H}_1) = \inf(G, G')$$

and

$$G_2 := \mathrm{Spec}(\mathcal{H}_2) = \sup(G, G').$$

□

2.2.7. Definition. We denote by $\mathcal{R}$ the subfunctor of $\mathcal{P} \times \mathcal{P}$ of prolongations which are in relation, i.e. we define:

$$\begin{aligned} \mathcal{R} : \mathrm{Aff}_k &\to \mathrm{Set} \\ A &\mapsto \{(G, G'), \text{ such that } G \leq G'\}. \end{aligned}$$

We would like to know more about the structure of this order relation in the moduli space of prolongations.

2.2.8. Theorem. *The morphism*

$$\begin{aligned} u : \mathcal{P} &\overset{u}{\hookrightarrow} \mathcal{R} \\ G &\longmapsto (G,G) \end{aligned}$$

is a quasi-compact open and closed immersion. Moreover, the inclusion morphism

$$\mathcal{R} \hookrightarrow \mathcal{P} \times \mathcal{P}$$

is a closed immersion of finite presentation.

Proof.

- Let us show that the inclusion $\mathcal{R} \hookrightarrow \mathcal{P} \times \mathcal{P}$ is representable by a closed immersion of finite presentation. This is a special case of Proposition 1.2.16 with $H_K = G_K$ and $f_K = \mathrm{id}$.
- Now let us prove that the morphism u is a quasi-compact open immersion. Let A be a k-algebra. Let $v : G_1 \to G_2$ be a point of $\mathcal{R}(A)$. Like above, we can write:

$$\begin{aligned} \mathcal{P} \times_{\mathcal{R}} \mathrm{Spec}(A) : \{A\text{-algebras}\} &\to \mathrm{Set} \\ A' &\mapsto \begin{cases} \{\emptyset\} & \text{if } v \otimes A'[\![t]\!] \text{ is an isomorphism} \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

  Because being representable is local on A and thanks to Lemma 1.2.12, we can suppose $\mathrm{Spec}(A)$ small enough so that G_1 and G_2 are free on $A[\![t]\!]$. Let us write $\mathcal{O}_{G_1} = \bigoplus_{i=1}^{n} A[\![t]\!] e_i$ and $\mathcal{O}_{G_2} = \bigoplus_{i=1}^{n} A[\![t]\!] f_i$. For all i, let us write $b_{i,j} \in A[\![t]\!]$ such that $v^{\#}(f_i) = \sum_j b_{i,j} e_j$. Then v is an isomorphism if and only if the matrix of the $b_{i,j}$ is invertible, i.e. if and only if

$$\delta := \det((b_{i,j})_{i,j}) \in A[\![t]\!]^*,$$

  i.e. if and only if $\delta(0) \in A$ is invertible. Then, the morphism u is representable by $D(\delta(0))$ on $\mathrm{Spec}(A)$ so it is a quasi-compact open immersion.
- For the end, let us prove that the morphism

$$\mathcal{P} \hookrightarrow \mathcal{R}$$

  is also representable by a closed immersion. Because $\mathcal{P}$ is a colimit of separated schemes, where all the transition maps are closed immersion, it is separated in the sense that the diagonal

$$\mathcal{P} \to \mathcal{P} \times \mathcal{P}$$

is representable by a closed immersion. Hence the diagonal is representable by a proper morphism. Moreover the morphism

$$\mathcal{R} \hookrightarrow \mathcal{P} \times \mathcal{P}$$

is separated because it is a monomorphism. Then let us take a morphism $\mathrm{Spec}(A) \to \mathcal{R}$. This defines a morphism $\mathrm{Spec}(A) \to \mathcal{P} \times \mathcal{P}$ (thanks to the inclusion $\mathcal{R} \hookrightarrow \mathcal{P} \times \mathcal{P}$). Let us write $X := \mathcal{P} \times_{\mathcal{R}} \mathrm{Spec}(A)$, $X_1 := \mathcal{P} \times_{\mathcal{P}\times\mathcal{P}} \mathrm{Spec}(A)$ and $X_2 := \mathcal{R} \times_{\mathcal{P}\times\mathcal{P}} \mathrm{Spec}(A)$ the corresponding fiber products. Then $X_1 \to \mathrm{Spec}(A)$ is proper and $X_2 \to \mathrm{Spec}(A)$ is separated. Thanks to [SP23, Tag 01W6], we know that the morphism $X_1 \to X_2$ is proper. But because $\mathrm{Spec}(A) \to \mathcal{P} \times \mathcal{P}$ is a point of $\mathcal{R}$ by hypothesis, we obtain that $X_2 = \mathrm{Spec}(A)$. Then, because $X_1 \simeq X \times_{\mathrm{Spec}(A)} X_2$, we obtain that $X_1 \simeq X$ and so we proved that $X \to \mathrm{Spec}(A)$ is proper. Then because it is also a monomorphism, we obtain that this morphism is also a closed immersion (see [SP23, Tag 04XV]).

□

2.2.9. Corollary. *For any $G_K \to \mathrm{Spec}(K)$, we have an isomorphism:*

$$\mathcal{R} \simeq \mathcal{P} \amalg \mathcal{R}'$$

where here we identify $\mathcal{P}$ with its image in $\mathcal{P} \times \mathcal{P}$, and where $\mathcal{R}'$ is the functor of pairs of prolongations G and G' such that there exists a non-isomorphic prolongation morphism $G \to G'$.

□

2.3 Dilatations

We start our study of dilatations with a review of the classical case. Dilatations were initially defined over a discrete valuation ring (or a Dedekind scheme); we review here some landmarks from the book [BLR90] and the article [WW80]. Classical dilatations are the solution to a universal problem among all flat schemes, and they are defined as follows. Let us recall the notations $R := k[\![t]\!]$ and $K := k(\!(t)\!)$.

2.3.1. Definition. Let $G \to \mathrm{Spec}(R)$ be a flat group scheme. Let $H \hookrightarrow G_k$ be a closed subgroup of G_k. We call *classical dilatation of H in G_k* a group morphism $D_G(H) \to G$ sending the special fiber into H, such that $D_G(H)$ is flat over R and such that it is final for these properties.

2.3.2. Remark. Actually the term *dilatation* is used is the case of general schemes. When we are working with group schemes, it is usually the term *Néron blowup* which is used in the literature.

Because being flat on a DVR is equivalent to being torsion-free, the classical dilatation of any finite type subgroup exists and it can be constructed explicitly.

2.3.3. Proposition. *Let $G \to \mathrm{Spec}(R)$ be a flat group scheme. Let $H \hookrightarrow G_k$ be any closed subgroup, of finite type over R. Let $I = (t, g_1, \cdots, g_n)$ be the ideal of H in $\mathcal{O}_G$. Then the dilatation of H in G exists and is given by $D_G(H) = \mathrm{Spec}(A')$ where*

$$A' = \mathcal{O}_G[y_1, \cdots, y_n]/\left(ty_1 - g_1, \cdots, ty_n - g_n\right)^{\mathrm{sat}}$$

where

- *For any ideal $J \subset \mathcal{O}_G[y_1, \cdots, y_n]$, we write $J^{\mathrm{sat}} := J/J^{t\text{-}\mathrm{tor}}$.*
- *The morphism of dilatation is given by the inclusion $\mathcal{O}_G \hookrightarrow A'$.*

Proof. See [BLR90], Chapter 3, §3.2/1. □

In the following, we would like to use the dilatation as a tool to understand our moduli space $\mathcal{P}$. Using the description from Proposition 2.3.3, we see that the classical dilatation of a group is an isomorphism on the generic fiber: that is, if we dilate a subgroup of an element of $\mathcal{P}(k)$, and if the obtained dilatation is finite, it is also an element of $\mathcal{P}(k)$. Moreover, any prolongation morphism can be generated by dilatation morphisms in the sense that we can associate to any prolongation morphism a canonical sequence of dilatation, called *standard blow-up sequence*:

2.3.4. Theorem. *Any prolongation morphism $G' \to G$ in $\mathcal{P}(k)$ is canonically isomorphic to a finite composition of classical dilatations, that can be constructed explicitly.*

Proof. The construction is the following: if the closed image H of $G'_{|t=0}$ is all $G_{|t=0}$, one can see that the morphism $G' \to G$ is an isomorphism. If not, the morphism $G' \to G$ factors through $G_1 := D_G(H)$. Again if $G'_{|t=0}$ does not map onto $(G_1)_{|t=0}$, the morphism $G' \to G_1$ factorises through $D_{G_1}(\mathrm{im}(G'_{|t=0}))$, and so on. One can see that this process stops because G' is of finite type. See [WW80], Theorem 1.4. for more details. □

Over more general bases, the situation is more complicated. This is considered in the article [MRR20] where the existence of dilatations of flat group schemes with center in a regularly embedded flat subgroup scheme is established. However, in the context of finite locally free group schemes, major difficulties occur: starting from the explicit construction of the dilatation as in Proposition 2.3.3, both finiteness and flatness are hard to obtain. What is more, these two difficulties are interrelated.

Let us start now with the finiteness issue. It turns out that the classical dilatation may not be finite:

2.3.5. Example. Let us suppose that $\mathrm{char}(k) = p > 2$ and let us define

$$G = \mathrm{Spec}(R[X]/X^p - tX).$$

Then the classical dilatation of $V(t, X)$ is given by

$$\mathrm{Spec}(R[Y]/(t^{p-2}Y^p - Y))$$

which is not a finite as a R-module.

We will see that restricting to prolongations of an infinitesimal group, the situation is better thanks to Lemma 2.3.14.

We now come to the flatness issue that will appear whenever we will not be working on a DVR. The first observation here is that strict subgroups of finite locally free group schemes are never regularly embedded, and the flatness result of [MRR20] usually fails. Here is an example.

2.3.6. Example. Let A be a k-algebra and $a \neq 0$ not invertible. Let us write

$$G = \mathrm{Spec}(A[\![t]\!][X]/(X^p - aX))$$

and let us consider $H = V(t, X)$. If we wanted to use the expression of Proposition 2.3.3, this would give the following scheme

$$\mathrm{Spec}(A[\![t]\!][Y]/(t^{p-1}Y^p - aY)),$$

which is not flat over $A[\![t]\!]$, even if the group G was a finite flat group scheme over $A[\![t]\!]$.

Below we shall see how we deal with them, but it is time to give the definition of dilatations that is prompted by the setting of the moduli space $\mathcal{P}$.

2.3.7. Definition. Let A be any k-algebra. Let us fix a prolongation $G \in \mathcal{P}(A)$. We write $u_G : \mathrm{Spec}(A) \to \mathcal{P}$ for the corresponding morphism.

- We define the moduli space $\mathcal{P}_{\geq G}$ of prolongations which dominate G by this fiber product:

$$\begin{array}{ccc} \mathcal{P}_{\geq G} & \hookrightarrow & \mathcal{P} \times \mathrm{Spec}(A) \\ \downarrow & & \downarrow \scriptstyle{\mathrm{id} \times u_G} \\ \mathcal{R} & \hookrightarrow & \mathcal{P} \times \mathcal{P}. \end{array}$$

- For H a subgroup scheme of $G_{t=0}$, we write $\mathcal{P}^H_{\geq G}$ for the subspace of $\mathcal{P}_{\geq G}$ of prolongations whose morphism to G sends the special fiber into H.
- Likewise, we define the moduli space $\mathcal{P}_{\leq G}$ of prolongations which are dominated by G by this fiber product:

$$\begin{array}{ccc} \mathcal{P}_{\leq G} & \hookrightarrow & \mathrm{Spec}(A) \times \mathcal{P} \\ \downarrow & & \downarrow \scriptstyle{u_G \times \mathrm{id}} \\ \mathcal{R} & \hookrightarrow & \mathcal{P} \times \mathcal{P}. \end{array}$$

2.3.8. Remark.

1. The formations of $\mathcal{P}^H_{\geq G}$ commute with base change: indeed let $T \to S$ be a S-scheme. Then by definition, $\mathcal{P}^H_{\geq G} \times_S T$ is the following functor:

$$\begin{aligned}\mathcal{P}^H_{\geq G} \times_S T(T') &:= \{G' \in \mathcal{P}(T') \text{ s.t. } \exists f : G' \to G_{T'} = G_{T_{|T'}} \text{ s.t. } f(G'_{|t=0}) \subset H_{T'}\} \\ &= \mathcal{P}^{H_T}_{\geq G_T}(T').\end{aligned}$$

2. There exists a morphism of functors $\mathcal{P}^H_{\geq G} \hookrightarrow \mathcal{P}_{\geq G}$, where we just forget the condition on the given morphism in the definition.

The functor $\mathcal{P}^H_{\geq G}$ is representable for any prolongation G, as a closed immersion of $\mathcal{P}_{\geq G}$ defined above.

2.3.9. Proposition. *Let A be a k-algebra and $G \in \mathcal{P}(A)$. Let H a finite locally free closed subgroup scheme of $G_{|t=0}$. The morphism $\mathcal{P}^H_{\geq G} \hookrightarrow \mathcal{P}_{\geq G}$ is representable by a closed immersion, of finite presentation.*

Proof. Let A' be an A-algebra, and let us take a morphism $\operatorname{Spec}(A') \to \mathcal{P}_{\geq G}$, i.e. let us take a prolongation $G' \in \mathcal{P}(A')$ such that there exists a prolongation morphism $f : G' \to G_{A'}$. We want to show that

$$\begin{aligned}X := \mathcal{P}^H_{\geq G} \times_{\mathcal{P}_{\geq G}} \operatorname{Spec}(A') : \operatorname{Aff}_{A'} &\to \operatorname{Set} \\ T &\mapsto \begin{cases} \{\emptyset\} & \text{if } f_T(G'_{t=0}) \subset H_T \\ \emptyset & \text{otherwise}\end{cases}\end{aligned}$$

is representable by a closed subscheme of $\operatorname{Spec}(A')$. First of all, both H and G are of finite presentation, hence the morphism $H_{A'} \hookrightarrow G_{A'|t=0}$ is of finite presentation. Then let us denote $I = (t, f_1, \cdots f_m)$ the finite type ideal of $H_{A'}$. Because everything is local on A and thanks to Lemma 1.2.12 we can suppose that $\mathcal{O}_G$ is a free $A[\![t]\!]$-module, and we can write

$$\mathcal{O}_G \hat{\otimes} A'[\![t]\!] = \bigoplus_{i=1}^{n} A'[\![t]\!] e_i,$$

with all $e_i \in \mathcal{O}_G \otimes A'$. Similarly, we can write

$$\mathcal{O}_{G'} = \bigoplus_{i=1}^{m} A'[\![t]\!] e'_i.$$

Then the morphism $f : G' \to G_{A'}$ satisfies $f(G'_{|t=0}) \subset H_{A'}$ if and only if $f^{\#}(I) \subset t \cdot \mathcal{O}_{G'}$ if and only if for all $i \in [1, n]$ we have $f^{\#}(f_i) \in (t)$. But let $i \in [1, n]$. Then we can write

$$f^{\#}(f_i) = \sum_{j=1}^{m} a'_{i,j}(t) e'_j \text{ with } a'_{i,j}(t) \in A'[\![t]\!]$$

and for all i, j we can write

$$a'_{i,j} = a'_{i,j,0} + a'_{i,j,1}t + a'_{i,j,2}t^2 + \cdots$$

where all the coefficients $a'_{i,j,k}$ belong to A'. Then we see that $f^{\#}(f_i) \in (t)$ if and only if for all $j \in [1, m]$, $a'_{i,j,0} = 0$. If we write

$$J := \langle \{a'_{i,j,0}, (i, j) \in [1, n] \times [1, m]\} \rangle,$$

then X is representable by $\mathrm{Spec}(A'/J)$, and $X \to \mathrm{Spec}(A')$ is a closed immersion of finite presentation.

□

Summary: To sum up, we have showed:

$$\mathcal{P}^H_{\geq G} \hookrightarrow \mathcal{P}_{\geq G} \hookrightarrow \mathcal{P} \xrightarrow[\text{ind-projective}]{\text{ind-scheme}} S.$$

In particular, $\mathcal{P}^H_{\geq G}$ and $\mathcal{P}_{\geq G}$ are two ind-schemes, both ind-projective over S.

2.3.10. Definition. Let A be a k-algebra, and let $G \in \mathcal{P}(A)$. Let $H \subset G_{|t=0}$ be a closed subgroup of $G_{t=0}$, finite locally free over $\mathrm{Spec}(A)$. We we call *dilatation of H in G*, and we write it $D_G(H)$, if it exists:

$$D_G(H) := \min \mathcal{P}^H_{\geq G}(A).$$

2.3.11. Notation. Let $G \to \mathrm{Spec}(A[\![t]\!])$ be an $A[\![t]\!]$-group scheme, and let $D_G(H_1)$ be the dilatation of G by a closed subgroup H_1 of the special fiber. Let H_2 be a closed subgroup of the special fiber of $D_G(H_1)$. Then we write $D_G(H_1, H_2)$ for the dilatation of H_2 in $D_G(H_1)$.

We can remark that there is a functoriality between dilatation morphisms:

2.3.12. Lemma. *(Functoriality of the dilatation) Let G_1 and G_2 be two $A[\![t]\!]$-group schemes with A a k-algebra. Let H_1 and H_2 be two subgroup schemes, finite locally free over Spec(A) of respectively $G_{1,|t=0}$ and $G_{2,|t=0}$. Let us suppose $D_{G_1}(H_1)$ and $D_{G_2}(H_2)$ exist. Let $u : G_1 \to G_2$ be such that $H_1 \subset u_k^{-1}(H_2)$. Then there exists a morphism* $\mathrm{dil}(u) : D_{G_1}(H_1) \to D_{G_2}(H_2)$ *such that the following diagram commutes:*

$$\begin{array}{ccc} D_{G_1}(H_1) & \xrightarrow{\mathrm{dil}(u)} & D(G_2)(H_2) \\ \downarrow{\scriptstyle \mathrm{dil}_1} & & \downarrow{\scriptstyle \mathrm{dil}_2} \\ G_1 & \xrightarrow[u]{} & G_2. \end{array}$$

Proof. Let us write $\phi := u \circ \mathrm{dil}_1$. Then, $\phi_k(D_{G_1}(H_1)_{t=0}) \subset u_A(H_1) \subset H_2$ Then by definition of the dilatation, the lemma is proved. □

In the following, we present two different results on existence of dilatations in $\mathcal{P}(A)$, where A is a regular ring of dimension at most 1. We will use again the fact that flatness is well-behaved in this context. Let B be a $k[\![t]\!]$-algebra, let $g_1, \cdots, g_n \in B$ and let us write $I = (t, g_1, \cdots, g_n)$. Then we write

$$B[t^{-1}I] := B[X_1, \cdots X_n]/(g_1 - tX_1, \cdots, g_n - tX_n)^{\mathrm{sat}}$$

where we recall that for any ideal J in B, $J^{\mathrm{sat}} = J/J^{t\text{-}\mathrm{tor}}$. Then in this context, if the dilatation of $H = V(I) \subset G_{|t=0}$ exists, it is given by $\mathrm{Spec}((\mathcal{O}_G[t^{-1}I])^{\vee\vee})$. In other words:

2.3.13. Proposition. *Let A be a regular k-algebra of dimension at most 1. Let $G \in \mathcal{P}(A)$ and let H be a subgroup scheme of $G_{|t=0}$. The following are equivalent:*

1. *$D_G(H)$ exists in $\mathcal{P}(A)$*
2. *The set $\mathcal{P}^H_{\geq G}$ is not empty*
3. *The $A[\![t]\!]$-module $\mathcal{O}_G[t^{-1}I]$ is finite.*

Proof. - 1) $\implies$ 2) by definition.

- Let us show that 2) $\implies$ 3). Let $G' \to G$ be an element of $\mathcal{P}^H_{\geq G}$. Then the inclusion $\mathcal{O}_G \subset \mathcal{O}_{G'}$ factorises through $\mathcal{O}_G[t^{-1}I]$:

$$\begin{array}{ccc} \mathcal{O}_G & \hookrightarrow & \mathcal{O}_{G'} \\ \downarrow & \nearrow & \\ \mathcal{O}_G[t^{-1}I] & & \end{array}.$$

  But by hypothesis $\mathcal{O}_{G'}$ is a finite $A[\![t]\!]$-module, and because $A[\![t]\!]$ is Noetherian, then $\mathcal{O}_G[t^{-1}I]$ is also finite.
- Let us show that 3) $\implies$ 1). Because $\mathcal{O}_G[t^{-1}I]$ is a finite $A[\![t]\!]$-module, we know that $\mathcal{O}_G[t^{-1}I]^{\vee\vee}$ is reflexive in the sense of Definition 2.2.3. Then because $A[\![t]\!]$ is a regular ring of dimension at most 2, this module is locally free thanks to Proposition 2.2.4. Now let us show that this group respects the universal property of the dilatation. Let $g : G' \to G$ be a morphism in $\mathcal{P}(A)$, such that $g(G'_{|t=0}) \subset H$. Then in this case we have

$$I \cdot \mathcal{O}_{G'} = (t) \cdot \mathcal{O}_{G'}.$$

  Let $i \in [1, n]$. Then $g^{\#}(g_i) \in I \cdot \mathcal{O}_{G'}$ then let us write

$$g^{\#}(g_i) = ta_i$$

  with $a_i \in \mathcal{O}_{G'}$. Because $t \nmid 0$ in $\mathcal{O}_{G'}$, the element a_i is uniquely determined. Then the morphism $g^{\#}$ factorises through $\mathcal{O}_G[t^{-1}I]$, sending the indeterminate X_i on the a_i corresponding. Let us write this morphism ϕ. By dualizing this morphism twice, we obtain a morphism $\phi^{\vee\vee} : \mathcal{O}_G[t^{-1}I]^{\vee\vee} \to \mathcal{O}_{G'}^{\vee\vee}$. But G' is flat then we have an isomorphism $i : \mathcal{O}_{G'} \to \mathcal{O}_{G'}^{\vee\vee}$, so if we write $j : \mathcal{O}_G[t^{-1}T] \to \mathcal{O}_G[t^{-1}T]^{\vee\vee}$, the morphism $g^{\#}$ factorises through $\mathcal{O}_G[t^{-1}I]^{\vee\vee}$ via $i^{-1} \circ \phi^{\vee\vee}$. This can be summarized by this diagram:

$$\begin{array}{ccccc} \mathcal{O}_G & \xrightarrow{g^{\#}} & \mathcal{O}_{G'} & \xleftarrow{i^{-1}} & \mathcal{O}_{G'}^{\vee\vee} \\ & \searrow & \uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle \phi^{\vee\vee}} \\ & & \mathcal{O}_G[t^{-1}I] & \xrightarrow{j} & \mathcal{O}_G[t^{-1}I]^{\vee\vee}. \end{array}$$

Hence the dilatation of $H = V(I)$ is given by $\mathrm{Spec}(\mathcal{O}_G[t^{-1}I]^{\vee\vee})$.

□

This proposition allows us to say when there does not exist a dilatation. For instance, as we saw in Example 2.3.5, we know that for any k-algebra A, regular of dimension at most 1, the dilatation of $V(t, X)$ in $\mathrm{Spec}(A[\![t]\!][X]/(X^p - tX))$ does not exist.

For the finiteness condition, we need to work with infinitesimal groups, where a group scheme $G \to S$ is called *infinitesimal* when its structure morphism $G \to S$ is a universal homemorphism.

Let us introduce a lemma first:

2.3.14. Lemma. *Let S be any scheme and $f : X \to S$ be a surjective, quasi-compact, separated S-scheme. Let $U \subset S$ be an open subscheme whose preimage $X_U = f^{-1}(U)$ is schematically dense in X. Assume that $f_U : X_U \to U$ is a homeomorphism and that either*

(i) f is flat, or

(ii) f has a section.

Then f is a universal homeomorphism. If moreover f is of finite type, then it is finite.

Proof. By [SP23, Tag 0CEX], the property that f is a universal homeomorphism is local on S for the fpqc topology. It follows that in case (i), it suffices to show that the base change of f with itself is a universal homeomorphism; in this way we are reduced to case (ii) where f has a section $e : S \to X$. Because f is separated, the section e is closed; let us write $I \subset \mathcal{O}_X$ for its ideal sheaf. Since the open immersion $j : X_U \to X$ is schematically dense, the morphism of sheaves $j^{\#} : \mathcal{O}_X \to j_*\mathcal{O}_{X_U}$ is injective. Besides, by the assumption on f_U the morphism $e_U : U \to X_U$ is a nil-immersion (as a section of a nil-immersion), from which follows that the images of the local sections of I in $j_*\mathcal{O}_{X_U}$ are nilpotent. Because $j^{\#}$ is injective, the local section of I themselves are nilpotent, so e is a universal homeomorphism. Because f is a retraction of e, it is also a universal homeomorphism. Finally if moreover f is of finite type, it is finite (see [EGA4], (18.12.11)). □

Then we obtain:

2.3.15. Proposition. *Let us suppose G_K is infinitesimal. Let A be a k-algebra. Let $G \in \mathcal{P}(A)$. Let $H = V(I)$ be a finite locally free subgroup scheme of $G_{|t=0}$. Then if $\mathcal{O}_G[t^{-1}I]^{\vee\vee}$ is flat over $A[\![t]\!]$ (for example if A is regular of dimension at most 1), the dilatation of H in G exists in*

$\mathcal{P}(A)$ and it is given by $\mathrm{Spec}(\mathcal{O}_G[t^{-1}I]^{\vee\vee})$. Moreover, in this case the formation of the dilatation commutes with flat base change.

Proof. Let us write $S := \mathrm{Spec}(A[\![t]\!])$, its generic fiber $U = \mathrm{Spec}(A(\!(t)\!))$ and $D := \mathrm{Spec}(\mathcal{O}_G[t^{-1}I])$. We consider this morphism

$$f : \mathrm{Spec}(\mathcal{O}_G[t^{-1}I]) \to S.$$

Let us show that f is finite. Because f is affine, it is quasi-compact and separated. By construction,

$$D_U \simeq G \otimes A(\!(t)\!)$$

which is schematically dense in G by flatness of G. Moreover, $G \otimes A(\!(t)\!) \simeq G_K \otimes A(\!(t)\!)$ by hypothesis so f_U is a homeomorphism because G_K is infinitesimal. Then because f has a section, we can apply Lemma 2.3.14 to get that f is finite, so $\mathcal{O}_G[t^{-1}I]$ is a finite $A[\![t]\!]$-module, and thanks to Proposition 2.3.13, we know that the dilatation of H in G exists and it is given by $\mathrm{Spec}(\mathcal{O}_G[t^{-1}I])^{\vee\vee})$.

In this case the formation of the dilatation commutes with flat base change because the formation of the double dual commutes with flat base change. □

Now we can prove the main result of this subsection, which gives a rational morphism between the space of subgroups of the special fiber of a prolongation and $\mathcal{P}$. Let us write $p = \mathrm{char}(k)$ and let $G \to \mathrm{Spec}(k[\![t]\!])$ be a finite locally free group scheme of order p^n. For $0 \leq d \leq n-1$, let us write $\mathcal{S}_d$ for the moduli space of the finite locally free subgroups of order p^d of $G_{|t=0}$. Let us recall that it is representable by a closed subscheme of a Grassmannian: see [DG70], Chapitre III, §1, n°2, 2.6 for more details. Thanks to the existence criteria (Proposition 2.3.15), we can deduce this result:

2.3.16. Corollary. *Let us suppose that G_K is infinitesimal. Let $G \in \mathcal{P}(k)$ of order p^n. For any $0 \leq d \leq n-1$, there exists a dense open subscheme U of $\widetilde{\mathcal{S}_d^{\mathrm{red}}}$ in the normalisation of $\mathcal{S}_d^{\mathrm{red}}$, such that U contains the points of codimension 1, and such that the dilatation gives a morphism $\phi : U \to \mathcal{P}$:*

$$\begin{array}{ccc} U & \xrightarrow{\ \phi\ } & \mathcal{P} \\ \big\downarrow & \nearrow & \\ \widetilde{\mathcal{S}_d^{\mathrm{red}}} & & \end{array}.$$

Proof. Covering $\widetilde{\mathcal{S}_d^{\mathrm{red}}}$ with affine schemes, and working on one connected component of it, we can suppose that $\widetilde{\mathcal{S}_d^{\mathrm{red}}} = \mathrm{Spec}(A)$ is affine and irreducible over $\mathrm{Spec}(k)$. Let $H = V(I) \in \widetilde{\mathcal{S}_d^{\mathrm{red}}}(k)$. By "Théorème de platitude générique" (see [EGA4], seconde partie, Théorème 6.9.1), we know that there exists a non-empty open subset (hence dense) of $\mathrm{Spec}(A[\![t]\!])$ such that $(\mathcal{O}_G \hat{\otimes} A[\![t]\!])[t^{-1}I]^{\vee\vee}$ is flat over it. Let us denote by W for the largest one with this property. Then thanks to Proposition 2.2.4, we know that this open subset contains the points of codimension at most 1, and thanks to Proposition 2.3.15, we know that on this open subset, the dilatation exists and it is given by $\mathrm{Spec}(\mathcal{O}_G[t^{-1}I]^{\vee\vee})$. Then this defines a morphism from the nonempty open subscheme $U := (W \cap \mathrm{Spec}(A))$ of $\mathrm{Spec}(A)$ to $\mathcal{P}$. □

Thanks to this corollary, some natural questions appear. We would like to know more about this morphism. For example, is this morphism injective ? In general, it is not, i.e. doing dilatations give us "too much" prolongations, in the sense that two dilatations of different subgroup schemes (even of same order) can give the same prolongation:

2.3.17. Example. Let us write $\mathrm{char}(k) = p > 0$ and let us consider the prolongation G of its generic fiber, defined over R by

$$G = \ker \begin{pmatrix} \mathbb{G}_a^4 & \longrightarrow & \mathbb{G}_a^4 \\ (x,y,z,w) & \longmapsto & (ty - x^p, y^p, tw - z^p, w^p) \end{pmatrix}$$

Then $G_{|t=0} = (\alpha_p)^4$. For $u = [\alpha : \beta] \in \mathbb{P}^1(k)$, let us write

$$H_u := V(x, z, \alpha y + \beta w).$$

Using Proposition 2.3.3, we obtain that the dilatations of H_u does not depend of the point $u \in \mathbb{P}^1(k)$, and it is given by:

$$\begin{aligned} D_G(H_u) = \alpha_{p^2} \times \alpha_{p^2} &\longrightarrow G \\ (x,z) &\longmapsto (tx, t^{p-1}x^p, tz, t^{p-1}z^p) \end{aligned}$$

2.4 Action of the automorphism group and covering tree of prolongations

Studying examples of moduli spaces of prolongations (like Example 3.1.4), we see that in general $\mathcal{P}$ is "big", in the sense that it is of infinite dimension. But one can see easily that if we have a model (G,i) of G_K, then any automorphism ϕ of G_K gives another model of G_K, by composing i and ϕ. So the automorphism group of G_K acts on $\mathcal{P}$, and it is natural to study its orbits, and to see if the quotient is of finite dimension. Let us write:

$$\Gamma = \mathrm{Res}_{k((t))/k}(\mathrm{Aut}_{k((t))}(G_K))$$

where $\mathrm{Res}_{k((t))/k}$ is the functor of Weil restriction associated to the inclusion $k \hookrightarrow k((t))$.

2.4.1. Definition. The group Γ acts on $\mathcal{P}$ as follows: for any k-algebra A,

$$\begin{aligned} \Gamma(A) \times \mathcal{P}(A) &\to \mathcal{P}(A) \\ (\Phi, (G, i)) &\mapsto G^{\Phi} := (G, i \circ \Phi^{-1}). \end{aligned}$$

2.4.2. Lemma. *Let A be a k-algebra. Let $(G, i) \in \mathcal{P}(A)$. The stabiliser of G for the action of $\Gamma \times \operatorname{Spec}(A)$ is given by*

$$\Sigma_G := \operatorname{Res}_{A[\![t]\!]/A}(\operatorname{Aut}(G)).$$

Proof. Let B be an A-algebra, and let $\gamma \in \Gamma \times \operatorname{Spec}(A)(B)$ such that $\gamma \cdot G = G$ as a prolongation. This means that there exists an $B[\![t]\!]$-isomorphism $u : G \otimes B[\![t]\!] \to G \otimes B[\![t]\!]$ such that:

$$\begin{array}{ccc} G \otimes B(\!(t)\!) & \xrightarrow{u_{B(\!(t)\!)}} & G \otimes B(\!(t)\!) \\ {\scriptstyle i}\uparrow & & {\scriptstyle i}\uparrow \\ G_K \otimes B(\!(t)\!) & \xrightarrow{\gamma^{-1}} & G_K \otimes B(\!(t)\!) \end{array}.$$

This means that γ is in the image of this injective morphism:

$$\begin{aligned} \operatorname{Res}_{A[\![t]\!]/A}(\operatorname{Aut}(G))(B) &\hookrightarrow \operatorname{Res}_{A(\!(t)\!)/A}(\operatorname{Aut}(G_K))(B) \\ u &\mapsto i \circ u_{B(\!(t)\!)} \circ i^{-1} \end{aligned}$$

so we get the result. □

2.4.3. Corollary. *For any $G \in \mathcal{P}(A)$, the orbit of G under the action of $\Gamma \times \operatorname{Spec}(A)$ is the twisted affine Grasmannian of G (see Pappas and Rapoport [PR08]), that is to say, the fpqc sheaf Γ/Σ_G associated to the presheaf*

$$\begin{aligned} \Omega(G) = \{A\text{-}\operatorname{Alg}\} &\to \operatorname{Set} \\ B &\mapsto \Gamma(B)/\Sigma_G(B) = \operatorname{Aut}_{k(\!(t)\!)}(G)(B(\!(t)\!))/\operatorname{Aut}_{k[\![t]\!]}(G)(B[\![t]\!]). \end{aligned}$$

□

In the following, we would like to understand the set $\mathcal{P}(k)$ of all prolongations of G_K, defined over $k[\![t]\!]$. In order to do this, a strategy consists in finding an interesting family of prolongations, that *generates* all the prolongations under iterated dilatations.

2.4.4. Definition. Let G_K be a K-group scheme. We say that a prolongation G of G_K is *minimal* if for any prolongation $\tilde{G}$, there exists a prolongation map $\tilde{G} \to G$.

We say that a family of prolongations $\{G_i\}_{i \in I}$ is a *minimal family of prolongations of G_K* if for any prolongation $\tilde{G}$, there exists $i \in I$ and a morphism of prolongations $\tilde{G} \to G_i$.

2.4.5. Example. In general, there does not exist a minimal prolongation. But we can see that $\mathbb{G}_{m,R}$ is a minimal prolongation of $\mathbb{G}_{m,K}$ (see [WW80], Corollary 2.4). Moreover, there always exists a minimal family of prolongations (we can just take all the prolongations), but of course we are looking for *interesting* families, in the sense that we are looking for "small" families that allow us to calculate all the prolongations of a group. We will see in Section 3.2 that the group $\alpha_{p^n,K}$ has an interesting minimal family of prolongations that can be easily calculated. It is not difficult to see that this is also true for μ_{p^n}, and their products.

Then after finding (if it exists) an interesting minimal family F of prolongations for G_K, the strategy to obtain all the prolongations is the following: for all group schemes G in the family

- dilate all the strict subgroups of the special fiber of G
- dilate again the new groups we find
- (optionally) stop if we find a group G' which is in the orbit of G.

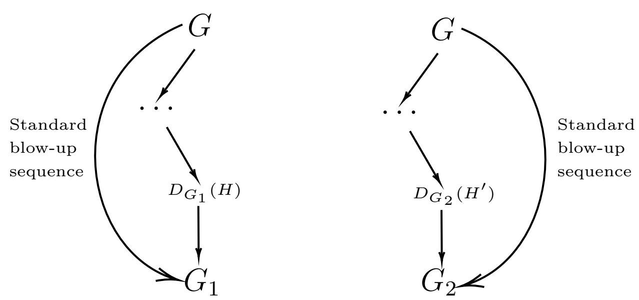
Indeed, we can stop the dilatation process whenever we find a group scheme that is in the orbit of G thanks to the following observation:

2.4.6. Proposition. *Let A be a k-algebra, and let $(G, i_1), (G, i_2) \in \mathcal{P}(A)$. Let us suppose that these two prolongations are in the same orbit for the action of the automorphism group, i.e. let $\gamma : G_K \otimes A(\!(t)\!) \to G_K \otimes A(\!(t)\!)$ be an isomorphism such that $i_1 = i_2 \circ \gamma^{-1}$. Let H be a subgroup scheme of $G_{|t=0}$ be such that its dilatation exists in $\mathcal{P}(A)$. Let us write $\phi : D_G(H) \to G$. Then $(D_G(H), \phi^{-1}_{A(\!(t)\!)}(\!(t)\!) \circ i_1)$ and $(D_G(H), \phi^{-1}_{A(\!(t)\!)} \circ i_2)$ are in the same orbit for the automorphism group of G_K.*

□

Then the process described above constructs a tree that we can call *covering tree of prolongations associated to the family F*, denoted by T_F.

2.4.7. Remark. The morphism $T_F \to \mathcal{P}(k)$ is surjective thanks to the standard blow-up sequence. But this morphism is not injective in general. Indeed let G_1 and G_2 be two elements of T_F. Let us write $G := \sup(G_1, G_2)$. Then G possesses a prolongation morphism to G_1 and another one to G_2. So G appears in two different spaces of the tree:



3 Illustrations and examples

3.1 Case of height 1 group schemes

As we did in Chapter II, we would like to use the equivalence of categories between height 1 group schemes and their Lie p-algebras. Then in all this subsection, we suppose that the characteristic of k is $p > 0$ and that G_K is a height 1 finite group scheme over $\mathrm{Spec}(K)$. Let us denote by $L_K := \mathrm{Lie}(G_K)$, its Lie algebra equipped with its canonical p-mapping. As we did for Hopf algebras and morphisms of finite group schemes, we denote by $\mathcal{P}_{L_K}$ the moduli space of prolongations of L_K, as Lie p-algebra vector bundle, i.e. for any k-algebra A, let us write

$$\mathcal{P}_{L_K}(A) := \left\{ \begin{array}{c} L \to \mathrm{Spec}(A[\![t]\!]) \text{ Lie } p\text{-algebra, finite locally free with an} \\ \text{isomorphism } i : L_K \otimes A(\!(t)\!) \xrightarrow{\sim} L \otimes A[\![t]\!] \text{ of Lie } p\text{-algebras} \end{array} \right\}.$$

We would like to show that the equivalence mentioned above gives an equivalence between $\mathcal{P}_{G_K}$ and $\mathcal{P}_{L_K}$. We need this lemma first:

3.1.1. Proposition. *Let G_K be a finite group scheme over $\mathrm{Spec}(K)$ of height n with $n \in \mathbb{N}^*$. Let A be a k-algebra, and let G be an element of $\mathcal{P}_{G_K}(A)$. Then, G is of height $\leq n$. In particular, all prolongations of an infinitesimal group scheme are infinitesimal.*

3.1.2. Remark. We can also find the result "all prolongations of an infinitesimal group scheme are infinitesimal" thanks to Lemma 2.3.14.

Proof. Again we know that $G \otimes A(\!(t)\!)$ is schematically dense in G, and because the trivial morphism and the n-th Frobenius morphism coincide on $G \otimes A(\!(t)\!)$, we know that the n-th Frobenius morphism is trivial on G. □

3.1.3. Corollary. *Now let us suppose that G_K is of height 1. Let us denote by $L_K := \mathrm{Lie}(G_K)$, its Lie algebra equipped with its canonical p-mapping. Then the functor* Lie *induces an equivalence of functors between $\mathcal{P}_{G_K}$ and $\mathcal{P}_{L_K}$.*

Proof. Thanks to Proposition 3.1.1, we know that for any k-algebra A, any group $G \in \mathcal{P}(A)$ is of height 1. Let $G \in \mathcal{P}(A)$. Then we have an isomorphism:

$$G_K \otimes A(\!(t)\!) \xrightarrow{\sim} G \otimes A(\!(t)\!)$$

so we obtain an isomorphism on their Lie algebras. Because the formation of the Lie algebra commutes with base change, we obtain an isomorphism

$$\mathrm{Lie}(G_K) \otimes A(\!(t)\!) \xrightarrow{\sim} \mathrm{Lie}(G) \otimes A(\!(t)\!).$$

Moreover, because G is of height 1, its Lie algebra is finite locally free. Then the functor Lie gives a morphism $\mathcal{P}_{G_K} \to \mathcal{P}_{\mathrm{Lie}(G_K)}$. Conversely, we can use the functor $G_p := \mathrm{Spec}^*(U_p(-))$, the inverse functor of Lie. Doing the same reasoning, we obtain an inverse map $\mathcal{P}_{\mathrm{Lie}(G_K)} \to \mathcal{P}_{G_K}$, so the corollary follows. □

In some cases, it would be easier to calculate the moduli space of prolongations of Lie p-algebras instead of the one of a group scheme. Let us show an example:

3.1.4. Example. Let $n \in \mathbb{N}^*$ and let us define $G_K := (\alpha_{p,K})^n$. Because G_K is commutative, its Lie algebra L_K is abelian. Moreover, the p-mapping given on L_K is the null morphism (See [DG70], II, §7 n°2, exemple 2.2.). In this case, the following morphism is an isomorphism

$$\begin{aligned} \mathcal{P}_{G_K} &\xrightarrow{\sim} \mathcal{L}att_n \\ G &\longmapsto \mathcal{O}_{\mathrm{Lie}(G)} \end{aligned}$$

where $\mathcal{L}att_n$ is the moduli space of *lattices*, as defined in Definition 1.2.10: indeed, in this case the functor Lie gives an equivalence $\mathcal{P}_{G_K} \xrightarrow{\sim} \mathcal{P}_{L_K}$ thanks to Corollary 3.1.3, and $L_K \simeq K^n$ as a K-vector space. Then for any k-algebra A, any element $L \in \mathcal{P}_{L_K}(A)$ is an $A[\![t]\!]$-module, locally free of rank n, such that $L \otimes A(\!(t)\!) \simeq L_K \otimes A(\!(t)\!) \simeq A(\!(t)\!)^n$. Then L is an element of $\mathcal{L}att_n$. Conversely, any lattice M can be viewed as a element of $\mathcal{P}_{L_K}$, equipped with the null bracket and the null p-mapping.

3.2 Prolongations of α_{p^2}

In this subsection, let k be any field of characteristic $p > 0$, and let us write again $R := k[\![t]\!]$ and $K := k(\!(t)\!)$. We would like to study the k-points of the moduli space of prolongations of $\alpha_{p^2,K}$. First of all, we can give a minimal family of prolongations of $\alpha_{p^n,K}$ above A for any k-algebra A and for any $n \in \mathbb{N}$. For $m \in \mathbb{Z}$, we write $\alpha_{p^n}^{(m)}$ for the prolongation of $\alpha_{p^n,K}$ defined over $k[\![t]\!]$, where $\alpha_{p^n}^{(m)}$ is equal to $\alpha_{p^n,R}$ as a group scheme, and the isomorphism on the generic fiber with $\alpha_{p^2,K}$ is given by:

$$\begin{aligned} i_m : \alpha_{p^n,K} &\longrightarrow \alpha_{p^n}^{(m)} \hat{\otimes} K \\ x &\longmapsto t^m x. \end{aligned}$$

Note that for any $m \in \mathbb{Z}$, there exists a prolongation morphism $\alpha_{p^n}^{(m)} \to \alpha_{p^n}^{(m+1)}$ given by $x \mapsto tx$.

3.2.1. Proposition. *The family* $\left(\alpha_{p^n}^{(m)}, i_m\right)_{m\in\mathbb{Z}}$ *is a minimal family of prolongations for* $\alpha_{p^n,K}$ *on* R.

Proof. Let G be a prolongation of $\alpha_{p^n,K}$ defined over R. There exists an isomorphism $\alpha_{p^n,K} \simeq G \otimes K$ given on the rings by an isomorphism

$$A(\!(t)\!)[X]/X^{p^n} \simeq \mathcal{O}_G \hat{\otimes} K,$$

which is uniquely determined by the image $a \in \mathcal{O}_G\hat{\otimes}K$ of the indeterminate X. Let us write m for the t-valuation of a. Then $t^m a \in \mathcal{O}_G$. Therefore the morphism $G \to \alpha_{p^n}^{(m)}$, given on the rings by $X \mapsto t^m a$ is a morphism of prolongations. □

3.2.2. Remark. For a prolongation $G \in \mathcal{P}_{\alpha_{p^n},K}(k)$, we can set:

$$m_G := \min\{m \in \mathbb{Z}, \exists\ G \to \alpha_{p^n}^{(m)}\},$$

in order to obtain a "canonical" prolongation morphism from G to one of the prolongations that appear in the minimal family.

We would like to study the set $\mathcal{P}_{\alpha_{p^2},K}(k)$. Thanks to Proposition 3.2.1, we already know that the family $(\alpha_{p^2}^{(m)})_{m\in\mathbb{Z}}$ is a minimal family of prolongations of $\alpha_{p^2,K}$. Then because every prolongation map in $\mathcal{P}(k)$ is a composition of dilatations (standard blow-up sequence, Theorem 2.3.4), we are led to studying the different compositions of dilatations coming from $\alpha_{p^n}^{(m)}$ for all m. Because the source of any dilatation can be turned canonically into a prolongation, we do not even need to specify the prolongation morphisms but only the dilatation morphisms. Then as said in Proposition 2.4.6, we are led to studying only the dilatations of $\alpha_{p^2}^{(0)}$.
Then in the following, we will use the strategy explained in the last subsection: we will study all the dilatations and the composite of dilatations of the group scheme $\alpha_{p^2,R}$. These dilatations can be calculated explicitely, thanks to Proposition 2.3.3.

3.2.3. Proposition. *Let us write* $\alpha_{p^2} = \operatorname{Spec}\left(R[X]/X^{p^2}\right)$. *Then* $\alpha_{p^2|t=0}$ *has two strict subgroup schemes, and these subgroups give two different dilatations:*

- $D_{\alpha_{p^2}}(\{e\})$ *is isomorphic to* α_{p^2} *as a group scheme, and the dilatation morphism is given by*

$$\begin{aligned} \mathrm{dil} : D_{\alpha_{p^2}}(\{e\}) = \alpha_{p^2} &\longrightarrow \alpha_{p^2} \\ x &\longmapsto tx. \end{aligned}$$

- $D_{\alpha_{p^2}}(\alpha_p)$ *is isomorphic to* $\beta_1 := \operatorname{Spec}\left(R[X,Y]/(tY - X^p, Y^p)\right)$ *as a subgroup scheme of* $\mathbb{G}_a^2$. *The dilatation morphism is given by*

$$\begin{aligned} \mathrm{dil} : D_{\alpha_{p^2}}(\alpha_p) = \beta_1 &\longrightarrow \alpha_{p^2} \\ (x,y) &\longmapsto x. \end{aligned}$$

Proof. The two strict subgroup schemes of $\alpha_{p^2|t=0}$ are given by the trivial one and α_p. Let us calculate the corresponding dilatations:

- With the description of α_{p^2} given above, we can write that $\{e\} = V(X)$, so using the construction given in Proposition 2.3.3 we can calculate:

$$\begin{aligned} D_{\alpha_{p^2}}(\{e\}) &= \operatorname{Spec}\left(R[X,Y]/(X^{p^2}, tY - X)^{\mathrm{sat}}\right) \\ &\simeq \operatorname{Spec}\left(R[Y]/Y^{p^2}\right) \end{aligned}$$

given on the ring by $X \mapsto tY$. For the group law, the comultiplication of Y should verify:

$$\Delta(tY) = 1 \otimes tY + tY \otimes 1, \text{ i.e. } \Delta(Y) = 1 \otimes Y + Y \otimes 1, \text{ because } t \nmid 0 \text{ in } R[Y]/Y^{p^2}.$$

- Likewise, we can write that $\alpha_p \hookrightarrow \alpha_{p^2}$ is given by $V(X^p)$, then

$$\begin{aligned} D_{\alpha_{p^2}}(\alpha_p) &= \operatorname{Spec}\left(R[X,Y]/(X^{p^2}, tY - X^p)^{\mathrm{sat}}\right) \\ &= \operatorname{Spec}\left(R[X,Y]/(tY - X^p, Y^p)\right) := \beta_1. \end{aligned}$$

For the group law, the comultiplication of Y should verify:

$$\begin{aligned} \Delta(tY) = \Delta(X^p) &= 1 \otimes X^p + X^p \otimes 1 \\ &= 1 \otimes tY + tY \otimes 1 \end{aligned}$$

i.e. again $\Delta(Y) = 1 \otimes Y + Y \otimes 1$ because $t \nmid 0$ in $R[X,Y]/(tY - X^p, Y^p)$.

□

So now, we go to the next step, i.e. we can focus on the calculations of the dilatations of the group scheme we denoted by β_1, because the group scheme α_{p^2} already appeared in the covering tree of prolongations. We need first to know the subgroups of its special fiber

$$\beta_{1|t=0} \simeq (\alpha_p \times \alpha_p)_{|t=0}\,.$$

3.2.4. Lemma. *The strict subgroups schemes of* $(\alpha_p \times \alpha_p)_{|t=0} = \operatorname{Spec}(k[X,Y]/(X^p, Y^p))$ *over* $\operatorname{Spec}(k)$ *are given by* $H_u := V(aX + bY)$ *with* $u = [a:b] \in \mathbb{P}^1_k(k)$*, and by the trivial subgroup* $\{e\} = V(X,Y)$*.*

Proof. Like in Example 3.1.4, we have that the Lie algebra of $\alpha_p \times \alpha_p$ is the commutative Lie algebra of rank 2 (because the group is commutative), and the canonical p-mapping on it is the zero map. Hence in this case, the sub-Lie p-algebras are given by the submodules, which are the trivial one, and the lines. Finally because $\alpha_p \times \alpha_p$ is of height 1, there is an equivalence between its subgroups schemes and the sub-Lie p-algebras of its Lie algebra. □

Now we can calculate all the dilatations of β_1, and we sum up all these calculations in the following proposition.

3.2.5. Proposition. *Let us write* $\beta_1 = \mathrm{Spec}\,(R[X,Y]/(tY - X^p, Y^p))$*. Let* $u = [a:b] \in \mathbb{P}^1(k)$ *and let us write* $H_u := V(aX + bY)$ *for the corresponding subgroup of its special fiber. Then the dilatation of* H_u *in* β_1 *is given by:*

- *If* $u = [1:b]$*,* $D_{\beta_1}(H_u)$ *is isomorphic to* α_{p^2} *as group scheme, and the dilatation morphism is given by:*

$$\begin{aligned} \alpha_{p^2} &\longrightarrow \beta_1 \\ x &\longmapsto (tx + bt^{p-1}x^p, t^{p-1}x^p). \end{aligned}$$

- *If* $u = 0 \in \mathbb{P}^1(k)$ *then* $D_{\beta_1}(H_0) = \beta_2$ *where we define*

$$\beta_2 := \mathrm{Spec}\left(R[X,Y]/(t^2Y - X^p, Y^p)\right),$$

  and the dilatation morphism is given by:

$$\begin{aligned} \beta_2 &\longrightarrow \beta_1 \\ (x,y) &\longmapsto (x, ty). \end{aligned}$$

Moreover, the dilatation of the trivial subgroup is given by

$$\begin{aligned} \alpha_{p^2} &\longrightarrow \beta_1 \\ x &\longmapsto (tx, t^{p-1}x^p). \end{aligned}$$

Proof. Let $u \in \mathbb{P}^1_k(k)$.

- If $u = [1:b]$, then the function ring of $D_{\beta_1}(H_{[1:b]})$ is given by:

$$\begin{aligned} R[X,Y,Z]/(tY - X^p, Y^p, tZ - X - bY)^{\mathrm{sat}}) &\simeq R[Y,Z]/(tY - t^pZ^p + b^pY^p, Y^p)^{\mathrm{sat}}) \\ &\simeq R[Y,Z]/(tY - t^pZ^p, Y^p)^{\mathrm{sat}}) \\ &\simeq R[Y,Z]/(Y - t^{p-1}Z^p, Y^p)^{\mathrm{sat}} \\ &\simeq R[Z]/Z^{p^2}. \end{aligned}$$

  The dilatation morphism is given by the composite of the above isomorphisms, and we see that the group law of $D_{\beta_1}(H_u)$ is given by

$$\begin{aligned} \Delta(tZ) = \Delta(X + bY) &= 1 \otimes X + X \otimes 1 + 1 \otimes bY + bY \otimes 1 \\ &= 1 \otimes tZ + tZ \otimes 1 = t(1 \otimes Z + Z \otimes 1) \end{aligned}$$

  i.e. $\Delta(Z) = 1 \otimes Z + Z \otimes 1$.

- If $u = 0 \in \mathbb{P}^1(k)$, then the function ring of $D_{\beta_1}(H_0)$ is given by:

$$\begin{aligned} R[X,Y,Z]/(tY - X^p, Y^p, tZ - Y)^{\text{sat}}) &\simeq R[X,Z]/(t^2Z - X^p, t^pZ^p)^{\text{sat}} \\ &\simeq R[X,Z]/(t^2Z - X^p, Z^p). \end{aligned}$$

Then let us use the notation $\beta_2 := \operatorname{Spec}\Big(R[X,Z]/(t^2Z - X^p, Z^p)\Big)$, and again, we find the dilatation morphism by keeping in mind the isomorphisms above. We obtain again in β_2 that the group law is given by

$$\Delta(X) = 1 \otimes X + X \otimes 1 \text{ and } \Delta(Z) = 1 \otimes Z + Z \otimes 1.$$

Finally, the dilatation morphism given by the trivial subgroup $\{e\}$ is given on the ring by:

$$\begin{aligned} R[X,Y]/(tY - X^p, Y^p) \hookrightarrow\ & R[X,Y,Z_1,Z_2]/(tY - X^p, Y^p, tZ_1 - X, tZ_2 - Y)^{\text{sat}} \\ \simeq\ & R[Z_1,Z_2]/(t^2Z_2 - t^pZ_1, t^pZ_2^p)^{\text{sat}} \\ \simeq\ & R[Z_1,Z_2]/(Z_2 - t^{p-2}Z_1, Z_2^p)^{\text{sat}} \\ \simeq\ & R[Z_1]/(t^{p(p-2)}Z_1^p)^{\text{sat}} \\ \simeq\ & R[Z_1]/(Z_1^{p^2}). \end{aligned}$$

□

Thanks to what we have done above, we can deduce this result:

3.2.6. Proposition. *For all $n \in \mathbb{N}$, let us denote*

$$\beta_n := \ker \begin{pmatrix} \mathbb{G}_a^2 & \longrightarrow & \mathbb{G}_a^2 \\ (x,y) & \longmapsto & (t^n y - x^p, y^p) \end{pmatrix}.$$

Let $H_u := V(aX + bY)$ be a subgroup scheme of its special fiber, with $u = [a:b] \in \mathbb{P}^1(k)$. Then the dilatation of H_u in β_n is given by:

- *If $u = [1:b]$, then the dilatation $D_{\beta_n}(H_u)$ if either isomorphic to α_{p^2} as group scheme or to β_{n-p} and the dilatation morphisms are given by:*
  — *if $n \leq p$:*

$$\begin{aligned} D_{\beta_n}(H_{[1:b]}) = \alpha_{p^2} &\longrightarrow \beta_n \\ x &\longmapsto (tx + bt^{p-n}x^p, t^{p-n}x^p) \end{aligned}$$

  — *if $n > p$:*

$$\begin{aligned} D_{\beta_n}(H_{[1:b]}) = \beta_{n-p} &\longrightarrow \beta_n \\ (x,y) &\longmapsto (tx + by, y) \end{aligned}$$

- *If* $u = [0:1]$ *then* $D_{\beta_n}(H_0) = \beta_{n+1}$, *and the dilatation morphism is given by:*

$$\begin{aligned} \beta_{n+1} &\longrightarrow \beta_n \\ (x,y) &\longmapsto (x, ty) \end{aligned}$$

The last subgroup that we can dilate is the trivial one. The obtained dilatation depends on n*:*

- *If* $n < p$*:*

$$\begin{aligned} D_{\beta_n}(\{e\}) = \alpha_{p^2} &\longrightarrow \beta_n \\ x &\longmapsto (tx, t^{p-n}x^p) \end{aligned}$$

- *If* $n \geq p$*:*

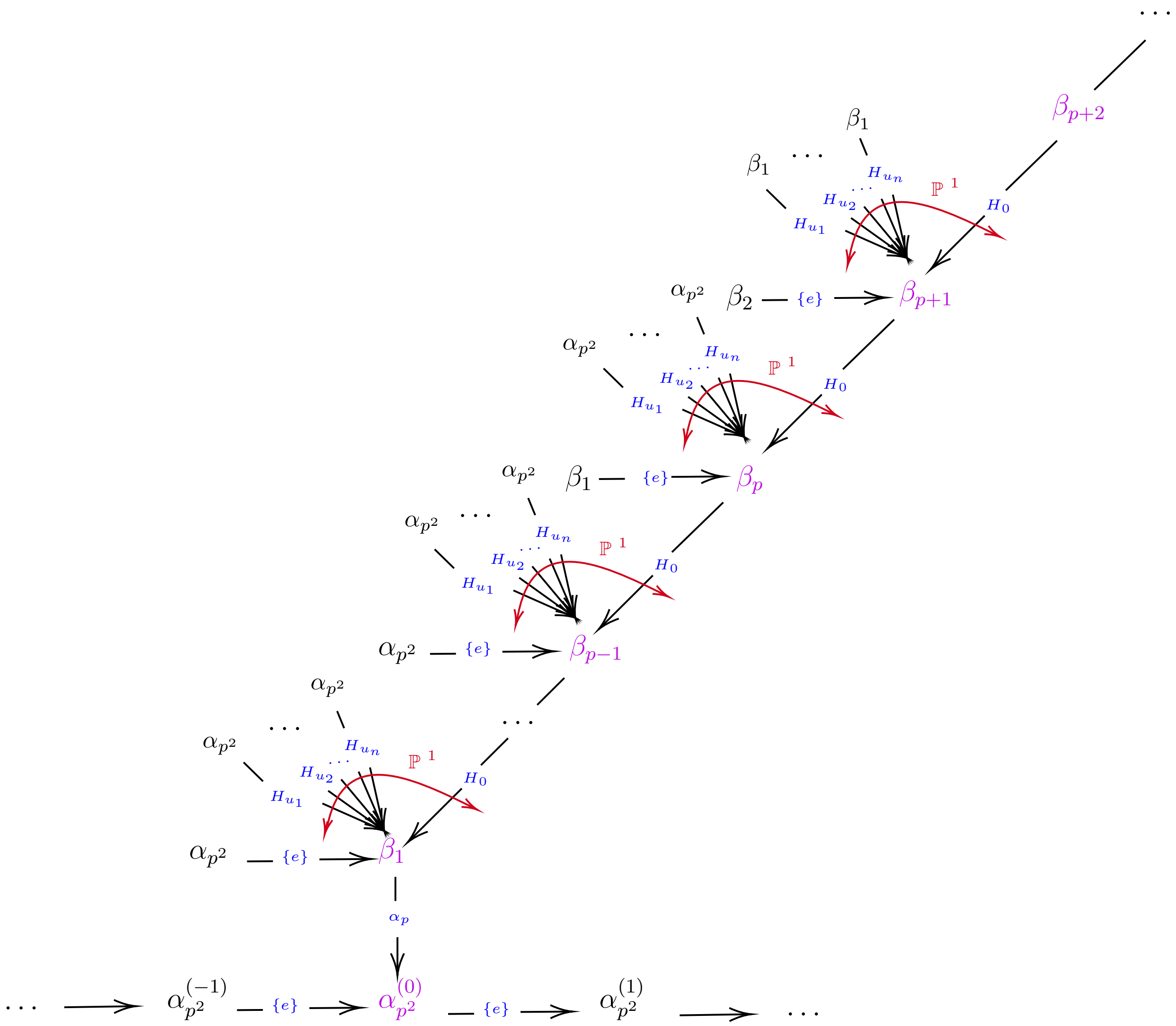
$$\begin{aligned} D_{\beta_n}(\{e\}) = \beta_{n+1-p} &\longrightarrow \beta_n \\ (x,y) &\longmapsto (tx, ty) \end{aligned}$$

Proof. The proof is the same as the one of 3.2.5. □

Here all the group schemes that appear in the sequence of dilatations are group schemes that have already appeared before. So we can stop our calculations here: we now know all the prolongations of $\alpha_{p^2,K}$ defined on $k[\![t]\!]$, i.e. with our notations, we know all the points of $\mathcal{P}_{\alpha_{p^2}}(k)$.

Summary: a picture of the k-points of $\mathcal{P}_{\alpha_{p^2}}$

Here is a summary of the different prolongations of $\alpha_{p^2,K}$ over $k[\![t]\!]$ in a tree, called *covering tree* above. The arrows represent dilatation morphism, and we color in purple the prolongations that do not appear before in the tree. We write on the arrows which subgroup is dilated, and let us recall that for $u = [a : b] \in \mathbb{P}^1(k)$, we write H_u for the subgroup of $\alpha_p \times \alpha_p$ given by $V(ax + by)$. We stop the process of dilatation whenever we see a group that has already appear before thanks to Proposition 2.4.6.



To conclude, we prove that $\mathcal{P}_{\alpha_{p^2,K}}$ contains a Γ-orbit of infinite dimension, so that

$$\dim(\mathcal{P}_{\alpha_{p^2,K}}) = +\infty.$$

3.2.7. Lemma. *The automorphism group of α_{p^2} is $\mathbb{G}_a \rtimes \mathbb{G}_m$, where the action of $\mathbb{G}_m$ over $\mathbb{G}_a$ is given by $v \cdot u = v^{p-1}u$.*

Proof. Let A be a k-algebra. Let $f \in \mathrm{Aut}_A(\alpha_{p^2})(A)$. Let us see that the corresponding morphism of Hopf algebras

$$\phi : A[X]/X^{p^2} \to A[X]/X^{p^2}$$

is of the form

$$X \mapsto a_1 X + a_p X^p$$

with $a_1, a_p \in A$. Indeed, in $A[X]/X^{p^2}$, the comultiplication of X is given by

$$\Delta(X) = X \otimes 1 + 1 \otimes X.$$

Let us write $\phi(X) = \sum_{k=0}^{n} a_k X^k$ with $n < p^2$. Then

$$\begin{aligned}\Delta(\phi(X)) &= \sum_{k=0}^{n} a_k (X \otimes 1 + 1 \otimes X)^k \\ &= \sum_{k=0}^{n} a_k \sum_{i=0}^{k} \binom{k}{i} \left(X^i \otimes X^{k-i}\right).\end{aligned}$$

But $\Delta(\phi(X)) = (\phi \otimes \phi)(\Delta(X))$ hence we obtain

$$\begin{aligned}\sum_{k=0}^{n} a_k \sum_{i=0}^{k} \binom{k}{i} \left(X^i \otimes X^{k-i}\right) &= \left(\sum_{k=0}^{n} a_k X^k\right) \otimes 1 + 1 \otimes \left(\sum_{k=0}^{n} a_k X^k\right) \\ &= \sum_{k=0}^{n} a_k (X^k \otimes 1 + 1 \otimes X^k).\end{aligned}$$

But the family $\{X^i \otimes X^j\}_{i,j \in [0,n]}$ form a basis of the A-module $A[X]/X^{p^2} \otimes A[X]/X^{p^2}$, then $a_k = 0$ for all k but $k = 1$ and $k = p$.

Moreover, ϕ is bijective if and only if there exists $\psi \in \mathrm{Aut}(A[X]/X^{p^2})$ such that $\phi \circ \psi = \mathrm{id}$. This means that $a_1 \in A^*$. Hence we have an isomorphism of group schemes:

$$\begin{aligned}\mathbb{G}_a \rtimes \mathbb{G}_m &\to \mathrm{Aut}(\alpha_{p^2}) \\ (u, v) &\mapsto (x \mapsto v(x + ux^p)).\end{aligned}$$

□

3.2.8. Corollary. *The orbit of the standard prolongation $\alpha_{p^2,k[\![t]\!]}$ contains the infinite-dimensional affine ind-space $\mathbb{A}_k^\infty$. In particular it is an infinite-dimensional ind-scheme over* $\mathrm{Spec}(k)$.

Proof. Let us write $G_K := \alpha_{p^2,K}$. Let $G := \alpha_{p^2,k[\![t]\!]}$ be the standard prolongation of $\alpha_{p^2,K}$ given by the identity on the generic fiber. Let us write $\Gamma := \mathrm{Aut}_{\alpha_{p^2}}$. Then thanks to Corollary 2.4.3, we know that the orbit of G inside $\mathcal{P}$ is the affine Grassmannian Gr_Γ. Because Γ contains $\mathbb{G}_a$, its Grassmannian contain $\mathrm{Gr}_{\mathbb{G}_a}$ which is infinite-dimensional because it is the sheaf associated to the presheaf

$$A \mapsto A(\!(t)\!)/A[\![t]\!] = \bigcup_{N\in\mathbb{N}^*} \bigoplus_{i=1}^{N} At^{-i}$$

which is the sheaf $\mathbb{A}_k^\infty$. □

3.3 Appendix. Classification of free Lie p-algebras of rank 3 over $k(\!(t)\!)$

As stated in Corollary 3.1.3, we have an equivalence between the prolongation space of a height 1 group scheme and the one of its Lie p-algebra. Then in this section, as we did in Chapter II, we propose to classify the Lie p-algebras of rank 3 over the field $k(\!(t)\!)$, up to isomorphisms. Actually in the following, we classify the dimension 3 Lie algebras over $k(\!(t)\!)$, with k algebraically closed and $\mathrm{char}(k) \neq 2$, and we give some results about the p-mappings on them. We explain how the calculations work, but we do not explicit all of them, because at some point it starts to be cumbersome and difficult.

Then let us first classify the Lie algebras over $k(\!(t)\!)$. Thanks to the work we have done in Subsection 3.1, we already know the classification of the Lie algebras over $\overline{k(\!(t)\!)}$. Hence we only have to compute the forms of these Lie algebras. In order to do this, we will use the characterisation of the forms by the cohomology groups. For this reason, let us recall the following result:

3.3.1. Proposition. *Let K be any field, and let $G \to \mathrm{Spec}(K)$ be a group scheme over K. Then, the forms over $\overline{K}$ of G are classified by the first fppf cohomology group of its automorphism group scheme.*

Proof. In [SP23, Tag 03AJ], an isomorphism in given between the following groups

$$\mathrm{H}^1(K, \mathrm{Aut}(G)) \simeq \{\mathrm{Aut}(G)\text{-torsors}\}/\sim .$$

Moreover, we can construct an isomorphism between $\{\mathrm{Aut}(G)\text{-torsors}\}/\sim$ and the forms of G as follows : we associate to a class of $\mathrm{Aut}(G)$-torsor $\overline{E}$ the the contracted product $G \overset{\mathrm{Aut}(H)}{\times} E$, and to any form G' of G, we associate the class of $\mathrm{Aut}(G)$-torsor $\overline{\mathrm{Isom}_K(G, G')}$. The proof of this isomorphism is given in [DG70], Chapitre III, §5, n°1. □

So in the following, if it is not precised before, all the exact sequences will be understood as sequences of fppf sheaves, and all the schemes will be seen as fppf sheaves. Likewise, all the cohomology groups will be cohomology groups for the fppf topology. Because it will be used in the following, let us recall this theorem:

3.3.2. Proposition. *Let K be any field. Then*

$$\mathrm{H}^1(K, \mathbb{G}_a) = \mathrm{H}^1(K, \mathbb{G}_m) = 1.$$

Moreover, for all $n \in \mathbb{N}^*$, we have $\mathrm{H}^1(K, \mathrm{GL}_n) = 1$.

Proof. As a sheaf, we have $\mathbb{G}_a = \mathcal{O}_{\mathrm{Spec}(k)}$ which is quasi-coherent, so its cohomology is 1. The second equality is known as Hilbert's theorem 90, and the last one from Speiser, see [CTS21], 1.3, Theorem 1.3.2 and Theorem 1.3.1. □

Thanks to theses theorems, we can study the different forms of Lie p-algebras of dimension 3. We will first generalise the classification of Lie algebras we have made in Theorem 3.1.4, and then focus on the ones which are restrictable.

3.3.3. Notation. Let us denote by $\mathfrak{f}_t$ the three-dimensional Lie algebra over K, with bracket given on a basis by $[x, y] = z$, $[x, z] = ty$ and $[y, z] = 0$.

3.3.4. Theorem. *Let $K = k((t))$, with $\mathrm{char}(k) \neq 2$. Then any Lie algebras of dimension 3 over K is isomorphic to exactly one in the following table.*

Name		*Structure*	*Orbit dimension*	*Center dimension*	*Restrictable*
$\mathfrak{ab}_3$		*abelian*	*0*	*3*	*yes*
$\mathfrak{h}_3$		*nilpotent*	*3*	*1*	*yes*
$\mathfrak{r}$		*solvable*	*5*	*0*	*no*
$\mathfrak{s}$		*simple*	*6*	*0*	*yes*
$\mathfrak{f}_t$		*solvable*	*5*	*0*	*yes*
$\mathfrak{l}_\alpha$	$\overline{\alpha} \notin \mathbb{F}_p/\sim$	*solvable*	*5*	*0*	*no*
	$\overline{\alpha} \in \mathbb{F}_p/\sim \smallsetminus \{\overline{0}, \overline{1}\}$	*solvable*	*5*	*0*	*yes*
	$\overline{\alpha} = \overline{0}$	*solvable*	*5*	*1*	*yes*
	$\overline{\alpha} = \overline{1}$	*solvable*	*3*	*0*	*yes*

That is, there is only one non-trivial form of Lie algebra of dimension 3 over $k((t))$.

In the following, we will detail the proof of this result, by calculating the automorphism groups of the different Lie algebras over $\overline{k((t))}$, and their cohomology.

The forms of $\mathfrak{r}$

Let $\mathfrak{r}$ denote the three-dimensional Lie algebra vector bundle over $\mathrm{Spec}(K)$, with bracket given on a basis by $[x,y]=y$, $[x,z]=y+z$ and $[y,z]=0$.

3.3.5. Proposition. *With these notations, we have*

$$\mathrm{Aut}(\mathfrak{r})=\mathbb{G}_a^2\rtimes(\mathbb{G}_m\rtimes\mathbb{G}_a),\ \textit{then}\ \mathrm{H}^1(K,\mathrm{Aut}(\mathfrak{r}))=1.$$

Proof. Let us calculate the sheaf $\mathrm{Aut}(\mathfrak{r})$. For this, let A be a K-algebra, and let $\phi\in\mathrm{Aut}(\mathfrak{r})(R)$. Then ϕ can be represented by a matrix

$$\phi=\begin{pmatrix}a&d&g\\b&e&h\\c&f&i\end{pmatrix}\ \text{in a basis}\ \{x,y,z\}.$$

But the derived Lie algebra $\mathfrak{r}'$ is given by $\mathfrak{r}'=\mathrm{Vect}(y,z)$ and we always have $\phi(\mathfrak{r}')\subset\mathfrak{r}'$, so $d=g=0$.

Now, using the fact that $\phi([v_1,v_2])=[\phi(v_1),\phi(v_2)]$ for all $v_i\in\mathfrak{r}$, we obtain this system:

$$\left\{\begin{array}{rc}a(e+f)=&e\\af=&f\\a(h+i)=&e+h\\ai=&f+i\\a(ei-fh)\in&A^*\end{array}\right.\Leftrightarrow\left\{\begin{array}{rc}(a-1)e=&-af\\(a-1)f=&0\\(a-1)h=&e-ai\\(a-1)i=&f\\a(ei-fh)\in&A^*.\end{array}\right.$$

so we can calculate $(a-1)^2(ei-fh)=(a-1)^2ei=-(a-1)afi=0$. But because $ei-fh$ is invertible, we obtain $(a-1)^2=0$. Now we are able to give a description of the functor $\mathrm{Aut}(\mathfrak{r})$:

$$\mathrm{Aut}(\mathfrak{r}):\{K\text{-algebras}\}\to\mathrm{Set}$$

$$A\mapsto\left\{\begin{pmatrix}a&0&0\\b&(a-1)h+ai&h\\c&-(a-1)(h(a-1)+ai)&i\end{pmatrix}\in\mathrm{GL}_3(A),(a-1)^2=0,i\in A^*\right\}.$$

Moreover,

$$\begin{aligned}e-i&=(a-1)h+ai-i\\&=(a-1)(h+i)\end{aligned}$$

so $(e-i)(a-1)=0$. But

$$\begin{aligned}(e-i)(a-1) &= e(a-1)-i(a-1)\\ &= -af-f\\ &= -f-f\\ &= -2f.\end{aligned}$$

Hence $2f=0$ so because $\operatorname{char}(k)\neq 2$ we obtain $f=0$. Then in this case, aei is invertible, so e is invertible, so we obtain $a=1$ and so $e=i$. To conclude, we obtain for any K-algebra A,

$$\operatorname{Aut}(\mathfrak{r})(A)\subset \operatorname{GL}_3(A)=\left\{\begin{pmatrix}1&0&0\\ b&i&h\\ c&0&i\end{pmatrix}\in \operatorname{GL}_3(A)\right\}.$$

For more convenience, let us write the subfunctor

$$B=\left(A\mapsto\left\{\begin{pmatrix}1&0&0\\ 0&i&h\\ 0&0&i\end{pmatrix}\in \operatorname{GL}_3(A)\right\}\right)\simeq(\mathbb{G}_m\rtimes\mathbb{G}_a)(A).$$

Then we have $\operatorname{Aut}(\mathfrak{r})=\mathbb{G}_a^2\rtimes B$. So in cohomology we obtain that this sequence

$$\mathrm{H}^1(K,\mathbb{G}_a)^{\oplus 2}\to \mathrm{H}^1(K,\operatorname{Aut}(\mathfrak{r}))\to \mathrm{H}^1(K,B)$$

is exact, but thanks to Proposition 3.3.2 we know that both $\mathrm{H}^1(K,\mathbb{G}_a)$ and $\mathrm{H}^1(K,B)$ are 1. Hence $\mathrm{H}^1(K,\operatorname{Aut}(\mathfrak{r}))=1$.

□

The forms of $\mathfrak{l}_\alpha$

Let $\alpha\in K$ and let $\{x,y,z\}$ be a basis of $\mathfrak{l}_\alpha$ where the bracket is given by $[x,y]=y$, $[x,z]=\alpha z$ and $[y,z]=0$. In order to calculate the forms of l_α, we need a lemma before:

3.3.6. Lemma. *Let k be an algebraically closed field of characteristic $\neq 2$. Then*

$$k(\!(t)\!)^*/k(\!(t)\!)^{*2}\simeq \mathbb{Z}/2\mathbb{Z}.$$

Proof. Let $f(t)\in k(\!(t)\!)$. Let $n\in\mathbb{Z}$ be the valuation of f. So we have

$$f(t)=t^n(a_0+a_1t+a_2t^2+\cdots)$$

with $a_i \in k$, and $a_0 \neq 0$. Then

$$\begin{aligned} a_0 + a_1 t + a_2 t^2 + \cdots =& a_0(1 + t\tilde{a_1} + t^2\tilde{a_2} + \cdots) \\ =& b_0^2(1 + tg(t)) \end{aligned}$$

with $g(t) \in k(\!(t)\!)$ and a certain $b_0 \in k$ because k is algebraically closed. But in $k(\!(t)\!)$, the series $1 + tg(t)$ is a square, for any power series g.

So we prove in this case that $a_0 + a_1 t + a_2 t^2 + \cdots$ is a square in $k(\!(t)\!)$. Then we have now to distinguish two cases.

- First, if $n \equiv 2[2]$ then t^n is also a square in $k(\!(t)\!)$ and so $f(t)$ is a square.
- Now if $n \equiv 1[2]$ then t^n is not a square. Then if f was a square, because we have

  $$t^n = f(t)(a_0 + a_1 t + \cdots)^{-1},$$

  then t^n would be a square too. This is false so in this case we obtain that f is not a square.

Hence, for any algebraically closed field k,

$$k(\!(t)\!)^*/k(\!(t)\!)^{*2} \simeq \mathbb{Z}/2\mathbb{Z}.$$

□

3.3.7. Proposition. *With these notations, we have*

$$\mathrm{Aut}(\mathfrak{l}_\alpha) = \begin{cases} \mathbb{G}_a^2 \rtimes \mathbb{G}_m^2 & \text{if } \alpha \neq \pm 1 \\ \mathbb{G}_a^2 \rtimes \mathrm{GL}_2 & \text{if } \alpha = 1 \\ \mathbb{G}_a^2 \rtimes \mathbb{G}_m^2 \rtimes \mu_2 & \text{if } \alpha = -1. \end{cases}$$

And then,

$$\mathrm{H}^1(K, \mathrm{Aut}(\mathfrak{l}_\alpha)) = \begin{cases} 1 & \text{if } \alpha \neq -1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \alpha = -1. \end{cases}$$

Proof. Let A be a K-algebra. Let $\phi \in \mathrm{Aut}(\mathfrak{l}_\alpha)(A)$ be represented by this matrix

$$\phi = \begin{pmatrix} a & 0 & 0 \\ b & e & h \\ c & f & i \end{pmatrix}, \text{ in the basis } \{x, y, z\}.$$

Doing similar calculations as we did for $\mathrm{Aut}(\mathfrak{r})$, we obtain that the coefficients of ϕ should verify:

$$\begin{cases} (a-1)e & = 0 \\ (\alpha a-1)f & = 0 \\ (a-\alpha)h & = 0 \\ (a-1)i\alpha & = 0 \\ a(ei-fh) & \in A^*. \end{cases}$$

Then this implies

$$\begin{cases} (a-1)(a\alpha-1)(ei-fh) & = 0 \\ (a-1)(a-\alpha)(ei-fh) & = 0. \end{cases}$$

Therefore, $(a-1)(a\alpha-\alpha^2)(ei-fh)=0$ so $(a-1)(\alpha^2-1)(ei-fh)=0$ but because $ei-fh$ is invertible, we obtain

$$(a-1)(\alpha^2-1)=0.$$

- That is, if $\alpha^2\neq 1$, i.e. $\alpha\neq\pm 1$, we obtain $a=1$, so $f=0$ and $h=0$. Therefore, for all $\alpha\in K$ such that $\alpha\neq\pm1$, we have

$$\mathrm{Aut}(\mathfrak{l}_\alpha)(A)=\left\{\begin{pmatrix}1&0&0\\ b&e&0\\ c&0&i\end{pmatrix}\in \mathrm{GL}_3(A)\right\}\simeq(\mathbb{G}_a^2\rtimes\mathbb{G}_m^2)(A).$$

  Hence, $\mathrm{H}^1(K,\mathrm{Aut}(\mathfrak{l}_\alpha))=1$ in this case thanks to Proposition 3.3.2.
- If $\alpha=1$, we obtain $(a-1)(ei-fh)=0$ hence $a=1$, and

$$\mathrm{Aut}(\mathfrak{l}_1)(R)=\left\{\begin{pmatrix}1&0&0\\ b&e&h\\ c&f&i\end{pmatrix}\in \mathrm{GL}_3(A)\right\}\simeq(\mathbb{G}_a^2\rtimes\mathrm{GL}_2)(A).$$

  Again here, $\mathrm{H}^1(K,\mathrm{Aut}(\mathfrak{l}_\alpha))=1$ thanks to Proposition 3.3.2.
- Now let us suppose $\mathrm{char}(K)\neq 2$, and $\alpha=-1$. In this case, the conditions can be written as

$$\begin{cases} (a-1)e = & 0 \\ (a+1)f = & 0 \\ (a+1)h = & 0 \\ (a-1)i = & 0 \\ a(ei-fh) & \in A^*. \end{cases}$$

  Therefore, $(a+1)(a-1)(ei-fh)=0$ i.e. $(a-1)(a+1)=0$ i.e. $a^2=1$.

Hence we have this exact sequence:

$$1 \to \left\{ \begin{pmatrix} 1 & 0 & 0 \\ b & e & 0 \\ c & 0 & i \end{pmatrix} \in \mathrm{GL}_3 \right\} \hookrightarrow \mathrm{Aut}(\mathfrak{l}_{-1}) \xrightarrow{\psi} \mu_2 \to 1$$

where the morphism ψ is the one which maps a matrix on its first coefficient. So

$$\mathrm{H}^1(K, \mathrm{Aut}(\mathfrak{l}_{-1})) \simeq \mathrm{H}^1(K, \mu_2).$$

Let us calculate $\mathrm{H}_1(K, \mu_2)$. We have the following Kummer fppf exact sequence:

$$1 \to \mu_2 \to \mathbb{G}_m \xrightarrow{\wedge 2} \mathbb{G}_m \to 1$$

which gives a long exact sequence in cohomology:

$$1 \to K^{*2} \to K^* \to \mathrm{H}^1(K, \mu_2) \to \mathrm{H}^1(K, \mathbb{G}_m) = 1$$

where $K^{*2} = \{f \in K^*, \exists g \in K^*, g^2 = f\}$. So we obtain

$$\mathrm{H}^1(K, \mu_2) \simeq K^*/K^{*\wedge 2}.$$

Here we have $K = k((t))$ with k algebraically closed, so we obtain

$$\mathrm{H}^1(k((t)), \mathrm{Aut}(\mathfrak{l}_{-1})) = k((t))^*/k((t))^{*2} \simeq \mathbb{Z}/2\mathbb{Z}$$

where the last isomorphism comes from Lemma 3.3.6.

□

Let us find the only non trivial form of $\mathfrak{l}_{-1}$. Let $T \in \overline{k((t))}$ such that $T^2 = t$. Let us denote by $\mathfrak{f}_t$ the three-dimensional Lie algebra over $k((t))$, whose bracket is given on a basis $\{x', y', z'\}$ by $[x', y'] = z'$, $[x', z'] = ty'$ and $[y', z'] = 0$. Then, this Lie algebra is a form of the Lie algebra $\mathfrak{l}_{-1}$. Indeed, the isomorphism on $\overline{k((t))}$ is given by

$$\begin{aligned} \phi : \mathfrak{f}_t \otimes \overline{K} &\to \mathfrak{l}_{-1} \otimes \overline{K} \\ x' &\mapsto Tx \\ y' &\mapsto y + z \\ z' &\mapsto T(y - z). \end{aligned}$$

The details are left to the reader. Let us show that this form is indeed non trivial: let us suppose there exists a isomorphism ϕ defined on $k((t))$ from $\mathfrak{l}_{-1}$ to this Lie algebra $\mathfrak{f}_t$. Let us write ϕ as a matrix: $\phi = \begin{pmatrix} a & 0 & 0 \\ b & e & h \\ c & f & i \end{pmatrix}$. We obtain this system:

$$\begin{cases} taf = & e \\ ae = & f \\ ah = & -i \\ tai = & -h \\ a(ei - fh) \neq & 0. \end{cases}$$

Hence we obtain $e(ta^2 - 1) = 0$. If $e = 0$ then a and f are invertible, but this is impossible because $taf = e$. Then $e \neq 0$ and $ta^2 - 1 = 0$, i.e. $a^2 = t^{-1}$ but this is impossible because t^{-1} is not a square in $k(\!(t)\!)$. Then there is no isomorphism defined on $k(\!(t)\!)$ so $\mathfrak{f}_t$ is the unique non-trivial form of $\mathfrak{l}_{-1}$.

The forms of $\mathfrak{h}_3$

Let us consider the three-dimensional Lie algebra $\mathfrak{h}_3$, whose bracket is given on a basis by $[x, y] = [x, z] = 0$ and $[y, z] = x$.

3.3.8. Proposition. *With these notations, we have* $\mathrm{Aut}(\mathfrak{h}_3) = \mathbb{G}_a \rtimes \mathrm{GL}_2$*, and so*

$$\mathrm{H}^1(K, \mathrm{Aut}(\mathfrak{h}_3)) = 1.$$

Proof. Doing the same kind of calculations as we did for the proof of Proposition 3.3.5 and Proposition 3.3.7, the result follows. □

The forms of $\mathfrak{s}$

For the end, let us study the forms of the Lie algebra $\mathfrak{s}$. We recall that the Lie algebra $\mathfrak{s}$ is the three-dimensional one, whose bracket is defined on a basis by $[x, y] = z$, $[x, z] = -x$ and $[y, z] = y$. In order to study its forms, we need to recall that a discrete valuation ring is strictly henselian if it is henselian and its residue field is separably algebraically closed. In our case, $R = k[\![t]\!]$ with k algebraically closed so R is strictly henselian.

3.3.9. Proposition. *Let R be a strictly henselian discrete valuation ring with fraction field K and algebraically closed residue field. Then*

$$\mathrm{H}^i(K, \mathbb{G}_m) = 1, \textit{ for all } i \geq 1.$$

Proof. See [CTS21], 1.4, Proposition 1.4.5. □

3.3.10. Proposition. *Because* $\mathrm{char}(k) \neq 2$*, we have* $\mathfrak{s} \simeq \mathfrak{sl}_2$*, then*

$$\mathrm{Aut}(\mathfrak{s}) = \mathrm{PSL}_2 \textit{ and } \mathrm{H}^1(K, \mathrm{Aut}(\mathfrak{s})) = 1.$$

Proof. Because $\mathfrak{s} \simeq \mathfrak{sl}_2$, we obtain $\mathrm{Aut}(\mathfrak{s}) = \mathrm{Aut}(\mathfrak{sl}_2) = \mathrm{Aut}(\mathrm{SL}_2) = \mathrm{PSL}_2$. Indeed the last equality comes from [SGA3], Exposé XXIV, 3.6, where it is proved that for any simply-connected reductive group scheme, we have an isomorphism $\mathrm{Out}(G) \xrightarrow{\sim} \mathrm{Aut}(\underline{\mathrm{Dyn}}(G))$. But here, the Dynkin diagram of SL_2 is A_1, which has no automorphism. Hence we have (See [Mi17] §23. e. for more details) $\mathrm{PSL}_2 \simeq \mathrm{Inn}(\mathrm{SL}_2) \simeq \mathrm{Aut}(\mathrm{SL}_2)$.

Then let us calculate $\mathrm{H}^1(K, \mathrm{PSL}_2)$. We have this exact sequence:

$$1 \to \mathbb{G}_m \to \mathrm{GL}_2 \to \mathrm{PGL}_2 = \mathrm{PSL}_2 \to 1$$

and because $\mathrm{H}^2(K, \mathbb{G}_m) = 0$ thanks to Proposition 3.3.9 and $\mathrm{H}^1(K, GL_2) = 1$, we obtain $\mathrm{H}^1(K, \mathrm{PSL}_2) = 1$. □

Then now we can prove the theorem of the classification of Lie algebras over $k(\!(t)\!)$.

Proof. (of Theorem 3.3.4)

- In order to know all the differents isomorphism classes, we use the theorem of classification of Lie algebras of dimension 3 over an algebraically closed field, Theorem 3.1.4, and the propositions 3.3.5, 3.3.7, 3.3.8 and 3.3.10.
- Let us show that $\mathfrak{f}_t$ is restrictable for any prime $p \neq 2$. Let us write $\{x, y, z\}$ for a basis of $\mathfrak{f}_t$. Then, is this basis, the morphism ad can be written like this:

$$\mathrm{ad}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & t \\ 0 & 1 & 0 \end{pmatrix}, \mathrm{ad}_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & t \\ -1 & 0 & 0 \end{pmatrix} \text{ and } \mathrm{ad}_z = \begin{pmatrix} 0 & 0 & 0 \\ -t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

  Then for all $p \geq 3$ we have $\mathrm{ad}_y^p = \mathrm{ad}_z^p \equiv 0$. Now for any $n \in \mathbb{N}$ such that $n \equiv 1[2]$, we can prove by induction that we have $\mathrm{ad}_x^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & t^{\frac{n+1}{2}} \\ 0 & t^{\frac{n-1}{2}} & 0 \end{pmatrix}$. So for any prime $p \geq 3$ we have $\mathrm{ad}_x^p = \mathrm{ad}_{x'}$ where $x' = t^{\frac{p-1}{2}} x$, and $\mathrm{ad}_y^p = \mathrm{ad}_z^p = \mathrm{ad}_0$. So $\mathfrak{f}_t$ is restrictable thanks to Theorem 1.1.6 (Jacobson's theorem).
- Let us show that its orbit's dimension is 5. In order to do this, let us calculate its stabilizer. Let $A = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \in \mathrm{GL}_3(K)$ be a matrix in the stabilizer of $o(\mathfrak{f}_t)$. Then, we write the relation

$$[Av, Aw] = A[v, w]$$

  for all elements of the basis and we can fin the equations for the stabilizer. We obtain these conditions:

$$\begin{cases} g = d = 0 \\ ae = i \\ aft = h \\ ah = tf \\ ait = te \\ a(ei - fh) \neq 0 \end{cases} \Leftrightarrow \begin{cases} g = d = 0 \\ a^2 i = i \\ a^2 h = h \\ aft = h \\ ah = tf \\ ait = te \\ a(ei - fh) \neq 0 \end{cases}$$

If we suppose $i = 0$ then $e = 0$ so h is invertible, so $a^2 = 1$. If $i \neq 0$, the relation $a^2 i = i$ gives also $a^2 = 1$. Hence in all cases we obtain $a^2 = 1$ and so $a = \pm 1$. Using the relations we found just above, we can write

$$\mathrm{Stab}(o(\mathfrak{f}_t)) = \left\{ A \in \mathrm{GL}_3(K), A = \begin{pmatrix} 1 & 0 & 0 \\ b & e & tf \\ c & e & e \end{pmatrix} \right\} \bigsqcup \left\{ A \in \mathrm{GL}_3(K), A = \begin{pmatrix} -1 & 0 & 0 \\ b & e & -tf \\ c & e & -e \end{pmatrix} \right\}.$$

Then $\dim(\mathrm{Stab}(\mathfrak{f}_t)) = 4$, so $\dim(o(\mathfrak{f}_t)) = 5$.

□

Now in order to classify all the Lie p-algebras over $k((t))$ (and not only the Lie algebras), we also have to classify the different p-mappings that we can have on the Lie algebras over $k((t))$. Let $\mathfrak{l}$ be a restrictable Lie algebra over K. Let $\gamma : \mathfrak{l} \to \mathfrak{l}$ be a p-mapping. We write γ-$\mathrm{Aut}(\mathfrak{l})$ for the functor of the automorphism of Lie p-algebra of $\mathfrak{l}$, with respect to the p-mapping γ. If the p-mapping is unique on $\mathfrak{l}$, we just write p-$\mathrm{Aut}(\mathfrak{l})$.

3.3.11. Lemma. *Let $\mathfrak{l}$ be a restrictable Lie algebra over a field K of characteristic $p > 0$, with $Z(\mathfrak{l}) = \{0\}$. Let us write $\gamma : \mathfrak{l} \to \mathfrak{l}$ the unique p-morphism of $\mathfrak{l}$, and let $\phi : \mathfrak{l} \to \mathfrak{l}$ be an isomorphism of Lie algebra. Then, ϕ is an isomorphism of Lie p-algebras.*

Proof. With simple calculations, one can prove that the morphism $\gamma' :=: f \circ \gamma \circ f^{-1}$ is a p-mapping on $\mathfrak{l}$. But by uniqueness we obtain $\gamma' = \gamma$ i.e. $f \circ \gamma = \gamma \circ f$. □

3.3.12. Corollary. *There is no $\mathfrak{s}$-form as Lie p-algebra over $k((t))$, and no $\mathfrak{l}_\alpha$-form either, with $\alpha \in \mathbb{F}_p^*$.*

□

For the other ones, it is more complicated to find all the forms of Lie p-algebra because there are infinitely many different p-mappings and the calculations of the forms of them start to be difficult. Because this subsection is only here to illustrate and to give some examples, we stop our calculations here.

Bibliography

[Bo14] S. Bosch, *Lectures on formal and rigid geometry.* Lecture Notes in Mathematics, 2105. Springer, Cham, 2014. viii+254 pp.

[BBM82] P. Berthelot, L. Breen, W. Messing, *Théorie de Dieudonné cristalline. II.* Lecture Notes in Mathematics, Springer-Verlag, 1982. x+261 pp.

[Bi98] L. Bianchi, *Sugli spazi a tre dimensioni che ammettono un gruppo continuo di movimenti.* Mem. Soc. Ital. delle Scienze (3) 11, 267-352 (1898).

[BLR90] S. Bosch, W. Lütkebohmert, M Raynaud, *Néron models.* Ergeb. Math. Grenzgeb. 21. Springer-Verlag, 1990.

[Br13] M. Brion, P. Samuel, V.Uma, *Lectures on the structure of algebraic groups and geometric applications.* CMI Lecture Series in Mathematics, 1. Hindustan Book Agency, 2013. viii+120 pp.

[Carl79] R. Carles, *Variétés des algèbres de Lie de dimension inférieure ou égale à 7.* C. R. Acad. Sci., Paris, Sér, 1979. A 289, 263-266.

[Cart62] P. Cartier, *Groupes algébriques et groupes formels.* Colloq. Théorie des Groupes Algébriques (Bruxelles, 1962) pp. 87–111 Librairie Universitaire, Louvain; Gauthier-Villars, Paris.

[CD84] R. Carles, Y. Diakité, *Sur les variétés d'algèbres de Lie de dimension* ≤ 7. J. Algebra 91, 53-63 (1984).

[CTS21] J. Colliot-Thélène, A.N. Skorobogatov, *The Brauer-Grothendieck group.* Springer, Cham, 2021. xv+453 pp

[D93] A.J. de Jong, *Finite locally free group schemes in characteristic p and Dieudonné modules.* Invent. Math. 1993, 89-137.

[DG70] M. Demazure, P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs.* Masson and Cie, Éditeur, Paris; North-Holland Publishing Co, 1970.

[E95] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry.* Graduate Texts in Mathematics, 150. Springer-Verlag, 1995. xvi+785 pp.

[EGA3] A. Grothendieck, *Eléments de géometrie algébrique. I. Le langage des schémas (French)* Inst. Hautes Etudes Sci. Publ. Math. No. 4 (1960) 228 pp.

[EGA4] A. Grothendieck *(with collaboration of* J.Dieudonné), *IV. Eléments de géometrie algébrique. Étude locale des schémas et des morphismes de schémas. (French)* Inst. Hautes Études Sci. Publ. Math. No. 32 (1967), 361 pp. 14.55

[FH91] W. Fulton, J. Harris, *Representation theory. A first course.* Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, 1991. xvi+551 pp.

[FP87] E. M. Friedlander, B. J. Parshall, *Limits of infinitesimal group cohomology.* in Algebraic topology and algebraic K-theory (Princeton, N.J., 1983), 523–538, Ann. of Math. Stud., 113, Princeton Univ. Press, 1987.

[G10] U. Görtz, *Affine Springer fibers and affine Deligne-Lusztig varieties.* (English summary) Affine flag manifolds and principal bundles, 1–50, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2010.

[GW20] U. Görtz, T. Wedhorn, *Algebraic geometry I. Schemes with examples and exercises.* Second edition. Springer Studium Mathematik—Master. Springer Spektrum, Wiesbaden, 2020. vii+625 pp.

[H80] R. Hartshorne, *Stable reflexive sheaves.* Math. Ann. 254, 121-176 (1980).

[Ja03] J. C. Jantzen, *Representations of algebraic groups.* Second edition, Mathematical Surveys and Monographs 107, American Mathematical Society, 2003.

[J62] N. Jacobson, *Lie algebras.* Interscience Tracts in Pure and Applied Mathematics, No. 10 Interscience Publishers (a division of John Wiley and Sons), 1962. ix+331 pp.

[KN84] A. A. Kirillov, Yu. A. Neretin, *The variety A_n of structures of n-dimensional Lie algebras.* Some problems in modern analysis, 42–56, Moskov. Gos. Univ., Mekh.-Mat. Fak., Moscow, 1984.

[L02] Q. Liu, *Algebraic geometry and arithmetic curves,* Oxford Graduate Texts in Mathematics, 6. Oxford Science Publications. Oxford University Press, 2002. xvi+576 pp.

[Ma16] L. Manivel, *On the cariety of four dimensional Lie alegbras.* Journal of Lie Theory, 2016, pp.1-10. hal-01161618.

[Macaulay2] D. R. Grayson, M. E. Stillman, *Macaulay2* a software system for research in algebraic geometry, available at http://www2.macaulay2.com/Macaulay2.

[Mi17] J.S. Milne, *Algebraic groups.* Cambridge Studies in Advanced Mathematics, 170. Cambridge University Press, 2017. xvi+644 pp.

[Mi80] J. S. Milne, *Étale cohomology.* Princeton Mathematical Series 33, Princeton University Press, 1980. xiii+323 pp.

[MRR20] A. Mayeux, T.Richarz, M. Romagny, *Néron blowups and low degree cohomological applications.* Forthcoming in *Algebraic Geometry.*

[MRT13] A. Mézard, M. Romagny, D. Tossici, *Models of group schemes of roots of unity.* Ann. Inst. Fourier(2013), no. 3, 1055–1135.

[PR08] G. Pappas, M. Rapoport, *Twisted loop groups and their affine flag varieties.* Adv. Math. 219 (2008), no. 1, 118–198.

[PS74] C. Peskine, L. Szpiro, *Liaison des variétés algébriques. I.* Invent. Math. 26, 1974. 271–302.

[Ro11] M. Romagny, *Composantes connexes et irréductibles en familles.* Manuscripta Math. 136, 1–32. 2011.

[Ra74] M. Raynaud, *Schémas en groupes de type* $(p, \cdots, p)$. Bull. Soc. Math. France 102 (1974), 241–280.

[RG71] M. Raynaud, L. Gruson, *Critères de platitude et de projectivité. Techniques de "platification" d'un module.* (French) Invent. Math. 13 (1971), 1–89.

[S07] H. Strade, *Lie algebras of small dimension,* Contemp. Math., 442, Amer. Math. Soc., 2007.

[SP23] The Stacks Project Authors, *Stacks Project.* Located at `http://www.math.columbia.edu/algebraic_geometry/stacks-git`.

[SGA3] *Schémas en groupes.* Séminaire de Géométrie Algébrique du Bois Marie 1962–64. Dirigé par A. Grothendieck et M. Demazure avec la collaboration de M. Artin, J.-E. Bertin, P. Gabriel, M. Raynaud et J-P. Serre. Documents Mathématiques 7, Société Mathématique de France, 2011.

[Su78] J.B. Sullivan, *Simply connected groups, the hyperalgebra, and Verma's conjecture.* Amer. J. Math. 100 (1978), no. 5, 1015–1019.

[SF88] H. Strade, R. Farnsteiner, *Modular Lie algebras and their representations.* Monographs and Textbooks in Pure and Applied Mathematics, 116. Marcel Dekker, Inc., 1988. x+301 pp.

[T10] D. Tossici, *Models of* $\mu_{p^2,K}$ *over a discrete valuation ring.* J. Algebra 323 (2010), no. 7, 1908–1957.

[TO70] J. Tate, F. Oort, *Group schemes of prime order.* Ann. Sci. Ecole Norm. Sup. 1970. 1-21.

[V66] M. Vergne, *Réductibilité de la variété des algèbres de Lie nilpotentes.* C. R. Acad. Sci., Paris, Sér. A 263, 4-6, 1966.

[W79] W.C. Waterhouse, *Introduction to affine group schemes.* Graduate Texts in Mathematics, 1979. xi+164 pp.

[WW80] W.C. Waterhouse, B. Weisfeiler, *One-dimensional affine group schemes.* J. Algebra 66 (1980), no. 2, 550–568.

[Z39] H. Zassenhaus, *Über Lie'sche Ringe mit Primzahlcharakteristik.* Abh. math. Sem. Hansische Univ. 13, 1-100 (1939).

COLLEGE DOCTORAL BRETAGNE — MATHS, TELECOMS, INFORMATIQUE, SIGNAL, SYSTEMES, ELECTRONIQUE

Université de Rennes

Titre : Modules de groupes finis plats en caractéristique $p > 0$

Mot clés : Algèbre de Lie restreignable, schéma en groupe fini, espace de modules

Résumé : Les objets principaux de cette thèse sont les schémas en groupes définis sur un schéma de base de caractéristique $p > 0$. Le point de vue adopté ici est l'étude de ces objets en famille. Plus précisément, nous commençons par étudier l'espace de modules des schémas en groupes finis sur un schéma de caractéristique $p > 0$, localement libres de hauteur 1. Ce cadre est plaisant car ces groupes sont caractérisés par leur algèbre de Lie, qui est naturellement munie d'une structure supplémentaire, appelée "p-application". Nous explorons alors en détail l'espace de modules des p-algèbres de Lie localement libres de rang fini. Nous allons voir que les espaces de module qui apparaissent sont des champs non séparés, et nous proposons alors d'étudier leur défaut de séparation en étudiant leurs modèles. Ainsi en deuxième partie nous développons les bases de l'étude des modèles d'un schéma en groupe fini en famille, et nous illustrons les résultats obtenus. Nous faisons notamment le lien avec la première partie en étudiant également les modèles de schémas en groupes de hauteur 1 et ceux de leur p-algèbre de Lie.

Title: Moduli of finite flat group schemes in characteristic $p > 0$

Keywords: Restrictable Lie algebra, finite group scheme, moduli space

Abstract: The main objects of this thesis are the group schemes defined over a based scheme of characteristic $p > 0$. We propose here to study these schemes in families, thanks to tools of moduli spaces. More precisely, we start by studying the moduli space of finite group schemes of height 1. This is a pleasant setting because these groups are determined by their Lie algebra, which is naturally equipped with an additional structure, called a "p-mapping". Then we explore in detail the moduli space of finite locally free Lie p-algebras. We will see that the moduli spaces that appear are non-separated stacks. We propose then to study its lack of separateness by studying models of finite group schemes, in families. We also make the connection with the first part of the thesis studying models of height 1 group schemes and those of their Lie p-algebra.