Maker-Maker Domination Game

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1 INTRODUCTION

1.1 Positional games

Positional games were introduced in 1973 by Erdős and Selfridge in their article [ES73]. These games are two-player sequential games with perfect information played on hypergraphs. A hypergraph can be defined by a set of vertices and a set of hyperedges which are subsets of the vertex set. The hyperedges are called the *winning sets*. Alternately, both players choose a vertex that has not been chosen yet. The objective of both players is given by the convention of the game. For example, in the Maker-Maker convention, each player tries to claim all the vertices of a winning set and in the Maker-Breaker convention, one player (called Maker) tries to claim an entire winning set while the other (called Breaker) tries to prevent him from doing that.

By opposition to the usual meaning of a "game", the point is not to try and win by being stronger than the adversary, but to be certain of the victory of a player, whatever they opponent does. The assumption here is that both players play perfectly to win or at least not to lose. As the game is finite, we can describe all the sequences of play that could happen. A tree can be made of this, where each vertex contains the hypergraph of the game, the player whose turn it is ($\mathcal{P}_1$ or $\mathcal{P}_\in$, and the previously claimed vertices of both players. The children of a vertex v are the vertices that contains the same hypergraph, the player $\mathcal{P}_1$ if the one of v is $\mathcal{P}_2$ and vice-versa, and the same claimed sets except there is one more vertex in the set of the player of v. The starting situation where no player has played corresponds to the root, which has empty claimed sets. When the situation is won by one of the players, or when there is a draw (all the vertices are claimed and nobody wins), there is no child to this vertex and we can tag the situation with its outcome (Player $\mathcal{P}_1$ won or player $\mathcal{P}_2$ won or draw). Once all the children of a vertex v are tagged with an outcome, we can then tag the vertex itself: the player in v, call it $\mathcal{P}$, will choose its favorite outcome ($\mathcal{P}$ prefers "$\mathcal{P}$ wins" to "draw" to "$\mathcal{P}$ loses") so the vertex will get the best tag according to $\mathcal{P}$. By tagging all the vertices of the tree, we finally tagged the root (the situation where no player has played) and the tag of the root is called the outcome of the game.

So the outcome of a game with a starting player can be either the victory of one of the two players or a draw. In the case of the same objective, changing the starting player will not change the outcome, so the outcome of such a game is one of the following:

- First player wins
- Second player wins
- Draw

Yet, in the other cases there is no reason to keep the same outcome, so when we add this up, a positional game with a convention where the two players have different objectives ($\mathcal{A}$ and $\mathcal{B}$), has an outcome among:

- First player wins
- Second player wins
- Draw
- Player with the objective $\mathcal{A}$ wins
- Player with the objective $\mathcal{B}$ wins

This method to prove the existence of the outcome also describes what is called a winning strategy (or a drawing strategy): a succession of moves played by one of the players to reach a given outcome, no matter what the other player plays. Describing a strategy is a solution to compute the outcome of a game, but it is not necessary. There are then two separated questions:

"What is the outcome of this game ?" and "How does a player enforce this outcome ?". Describing a strategy is generally in the complexity class called "PSPACE" (PSPACE-complete) since it can be done in polynomial space but not polynomial time : it is necessary to define the next move for every situation and every move from the opponent.

Several well-known games are positional games. For instance, as depicted in Fig. 1., Tic-Tac-Toe is a positional game: they are 9 vertices and 8 hyperedges (3 columns, 3 rows and 2 diagonals). One player is the circle and the other the cross. Tic-Tac-Toe is mostly known in its Maker-Maker convention, which is known to be a draw. Of course, there are more interesting games than Tic-Tac-Toe: Hex or multidimensional Tic-Tac-Toe for instance !

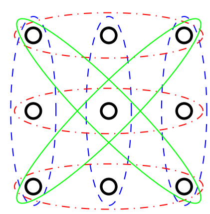

Fig. 1. Tic Tac Toe as an hypergraph.

Games have applications outside of entertainment or mediation. The clique game (a positional game played on the edges of a complete graph of size n, where a player tries to achieve a k-clique while the other tries to prevent it) relates to Ramsey's numbers and Ramsey's Theorem. One of the results is : A graph of size 6 contains at least a 3-clique or 3 independent vertices.

While having the same objective (i.e. the Maker-Maker convention) seems more natural for players, the Maker-Breaker convention is usually easier to study because each player only has one single objective to focus on. The links between the two conventions may be counterintuitive. You probably expect from Tic-Tac-Toe in Maker-Breaker convention to be won by Breaker, as there is a draw in the Maker-Maker convention. But the Breaker player does not have a threat-to-win power as she would in Maker-Maker: see a (partial) tree of the game in Fig 2.

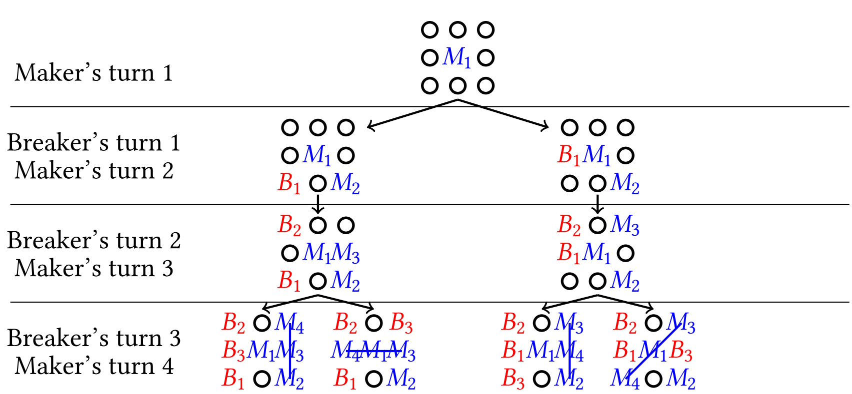

Fig. 2. Tic Tac Toe in Maker-Breaker convention.

1.2 Domination game

The domination game is a positional game that is played on a graph. Let $G = (V, E)$ be a non-directed graph. We define the domination game on G with the hypergraph $H = (V, \mathcal{E})$ where $\mathcal{E}$ are the dominating sets of G, i.e. the sets F such that every element of V is either in F or a neighbor of an element of F.
In the following, an instance of the domination game, is defined by a graph instead of a hypergraph, except if it is expressly precised.
As this game is played on a graph, it is easier to consider more complex graphs while the others games of the literature are often played on complete graphs or on simple grids. This new way to make the problem more complex requires more combinatorial and algorithmic reasoning and arguments. The Maker-Breaker domination game has been introduced recently by Eric Duchêne, Valentin Gledel, Aline Parreau and Gabriel Renault in [Duc+20]. The first and the third authors are co-supervising the internship that is the object of this report. This article is the one explained in Section 3. In particular, it contains and computes the outcome of the Maker-Breaker domination game on cographs and trees, brings useful results for general graphs, and proves that determining the outcome of a Maker-Breaker Domination Game is a *PSPACE-complete problem. That means that the problem is in* PSPACE i.e. that it can be solved using a polynomial amount of memory in the input size, here the graph and that there a reduction from any problem in PSPACE to Maker-Breaker Domination Game.
An example of a finished game is given in Fig. 3.: Maker (in red) played a_1 at his first turn and dominate the vertices in the ellipse around a_1, then Breaker played b_1, and Maker won by playing a_2, dominating the two missing vertices: a_2 and b_1.

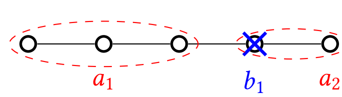

Fig. 3. Maker-Breaker domination game example, won by Maker.

1.3 Objectives

The objective of the internship is to study the domination game in its Maker-Maker convention. As said before, we expect to find the outcome of the Maker-Maker domination game, and to exhibit winning strategies for one of the players, or strategies leading to a draw for both of them.
Considering the difficulty of the domination game, it is natural to deal with specific classes of graphs. The studied classes here are paths, union of paths, subdivided stars and cycles.

Section 2 introduces classic results on general positional games.
Section 3 presents a recent article on the Maker-Breaker domination game: [Duc+20].
Section 4 presents the work made during this internship on the Maker-Maker positional game.

2 BACKGROUND

This part is dedicated to the presentation of classical results on general positional games from the book [Hef+14]. It consists in two general principles and a study of complexity about determining the outcome of positional games.

2.1 Strategy stealing principle

Among the possible outcomes, some of them do not seem realizable: in a game where both players have the same role (as in Maker-Maker for instance), it is easy to figure that the first one has an advantage over the second.
This intuition is confirmed by the following result: the Strategy stealing principle.

Theorem 1. *The outcome of a positional game in the Maker-Maker convention cannot be SECOND PLAYER WINS.*

Proof. The idea is to use the so-called strategy stealing principle.
Assume that the second player has a winning strategy $\mathcal{S}$. First player plays on an arbitrary vertex, call it a. He then plays following the winning strategy $\mathcal{S}$, assuming he is the second player. Whenever the strategy tells him to play on the vertex a, he plays on an arbitrary vertex a' instead and acts from now as if he played a' on the first turn.
The vertex claimed on the first turn (or its substitute) can only benefit the first player, and as $\mathcal{S}$ is a winning strategy, so the first player has, at some point, claimed an entire winning set of the hypergraph (even excluding the vertex he claimed arbitrarily) before the second player. The first player has a winning strategy now, so the second player cannot have one. □

This theorem leads to define the two possible outcomes in the Maker-Maker convention :

(1) $\mathcal{FP}$ (for First Player), is the outcome previously called FIRST PLAYER WINS: No matter what Second Player does, First Player can win.
(2) DRAW, is the outcome previously called the same way: No matter what First Player does, Second Player can enforce a draw.

Remark. This theorem expresses that the best outcome the second player can get in a Maker-Maker domination game is a DRAW, so she will try to find a strategy to enforce a DRAW, and not to win. These strategies can be identical to the one of a Breaker strategy, but not necessarily.

Remark. This result led to the introduction of the Maker-Breaker convention: the second player cannot win if she plays with the same objective as the first player, so what if her objective is to make Maker lose?

Now that the possible outcomes for Maker-Maker positional games have been reduced, we will see a similar result for Maker-Breaker positional games.

Theorem 2. *The outcome of a positional game in its Maker-Breaker convention cannot be SECOND PLAYER WINS.*

Proof. If Maker has a winning strategy as second player, he can ignore (exactly as in the previous proof) his first move and plays his winning strategy. He then wins even by playing first, so the game is MAKER WINS.
The same stands for Breaker, if she has a winning strategy as second player, the outcome of the game is BREAKER WINS.
So in any way, the outcome of a Maker-Breaker positional game is never SECOND PLAYER WINS. □

Remark. In addition to this, the objectives of Maker and Breaker are complementary: if Breaker does not win, then Maker does and if Maker does not win then Breaker wins. That also excludes the result DRAW from this convention.

This remark and the second theorem leads to define the three possible outcomes in the Maker-Breaker convention :

(1) $\mathcal{M}$ (for Maker), is the outcome previously called MAKER WINS: No matter Maker plays first or second, and no matter what Breaker does, Maker can win.
(2) $\mathcal{N}$ (for Next Player), is the outcome previously called FIRST PLAYER WINS: If Maker plays first, no matter what Breaker does, Maker can win.
Otherwise, Breaker plays first and no matter what Maker does, Breaker can win.
(3) $\mathcal{B}$ (for Breaker), is the outcome previously called BREAKER WINS: No matter Breaker plays first or second, and no matter what Maker does, Breaker can win.

And the second, the two possible outcomes in the Maker-Maker convention :

(1) $\mathcal{FP}$ (for First Player), is the outcome previously called FIRST PLAYER WINS: No matter what Second Player does, First Player can win.
(2) DRAW, is the outcome previously called the same way: No matter what First Player does, Second Player can enforce a draw.

Now that the possible outcomes of both conventions are clearly defined, we can look at the links between the Maker-Breaker and the Maker-Maker outcomes.

Proposition 3. *If a positional game is $\mathcal{B}$ in the Maker-Breaker convention, then it is a DRAW in the Maker-Maker convention.*

Proof. Assume a positional game is $\mathcal{B}$ in the Maker-Breaker convention. So Breaker has a strategy to win playing second (i.e. to prevent Maker from fulfilling a winning set), call it $\mathcal{T}$.

On the same positional game, but in the Maker-Maker convention, the second player can play following the strategy $\mathcal{T}$. That way, she will prevent the first player from filling a winning set, i.e. she will prevent the first player from winning. As second player cannot win either, this positional game is a DRAW in its Maker-Maker convention. □

Corollary 4. *If a positional game is $\mathcal{FP}$ in the Maker-Maker convention, then it is either $\mathcal{N}$ or $\mathcal{M}$ in the Maker-Breaker version.*

Proof. Assume that a Maker-Maker positional game is $\mathcal{FP}$ and the corresponding Maker-Breaker game is $\mathcal{B}$. By the previous proposition, the Maker-Maker game is a DRAW. But it's also $\mathcal{FP}$, so that's absurd. So if a positional game is $\mathcal{FP}$ in its Maker-Maker version, then it is not $\mathcal{B}$ in the Maker-Breaker version, so it is either $\mathcal{N}$ or $\mathcal{M}$. □

2.2 Pairing strategies

Another useful result that reduces the cases to study would be general strategies, potentially working in several conventions, on several graph classes.

The pairing strategy is such a strategy. It consists in defining, before anyone played, a partition of the vertices made of pairs. When the adversary plays on a vertex a belonging to the pair $\{a, a'\}$, the player following the pairing strategy claims a'. That way, the player using this strategy assure to have one vertex in each pair.

One of the most simple utilities is for Breaker in Maker-Breaker positional games. Indeed, in order to win, Breaker only has to claim a vertex in every winning set. So if she achieves to make a partition of the hypergraph (made of pairs) with a pair included in each winning set, Breaker has a winning strategy.

The previously given definition of a pairing strategy can be enlarged. Sometimes, the player may need to see the first moves of his opponent before defining the pairs. Also, the pairs may not need to cover the whole vertex set but only a subset. Advanced pairing strategies are much more flexible and useful. An example of such a strategy is presented for the Maker-Breaker 9-in-a-row game.

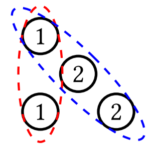

Fig. 4. Pairing partition of an hypergraph whose hyperedges are represented by ellipses.

Definition 5. n-in-a-row positional game's board is $\mathbb{N}^2$ considered as squares on a plane, and the winning sets are the sets made of n squares aligned on a line, a column, or a diagonal.

Remark. Here the board is infinite and there is an infinite number of winning sets, which breaks the proof given for the existence of the outcome. Yet, we can assume that in order to win, Maker has to do so in finite time. As the following strategy consists in preventing him from winning. Maker will not be able to win and as there is no Draw in Maker-Breaker, it is a winning strategy for Breaker.

Proposition 6. *9-in-a-row in its Maker-Breaker version is $\mathcal{B}$.*

Proof. H-shaped modules of seven vertices can tile the plane as shown in Fig 5.a). For each winning set of the 9-in-a-row game, there is a module that contains at least three vertices of the winning set. These three vertices are either the middle line, one of the vertical columns or one of the diagonals. If the three vertices are the left column of the module, then there is another module in which the right column also belongs to the winning set, therefore Breaker can ignore the left column. So if Breaker achieved to claim a vertex in the middle line, the right column and the two diagonals of each module (as depicted in Fig 5.b), she can prevent Maker from winning. When Maker plays on an unplayed module, if he plays in the middle, Breaker answers in the upper right corner and has the pairing strategy (on the module) made of the two remaining vertices of the middle line and of the up-left to down-right diagonal as shown in Fig 5.c). If Maker plays elsewhere, Breaker plays in the middle, and has the pairing strategy (on the module) made of (the) 2 unclaimed vertices of the right column as shown in Fig 5.d).

By playing this strategy, Breaker prevents Maker from claiming one of the essential sets in a module to achieve a 9-in-a-row win. So 9-in-a-row is $\mathcal{B}$ □

Remark. 8-in-a-row has been proven $\mathcal{B}$ by a similar argument, but determining the outcome of 6-in-a-row and 7-in-a-row are open problems.

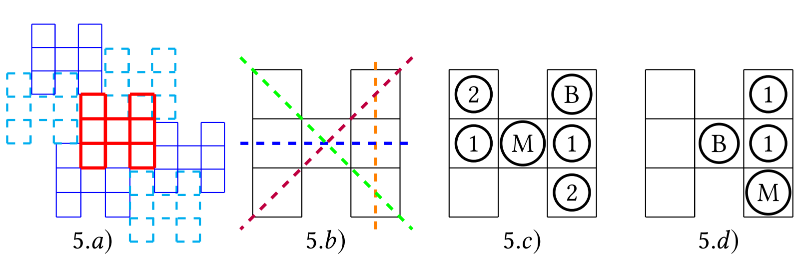

Fig. 5. 9-in-a-row in Maker-Breaker is $\mathcal{B}$

2.3 Complexity

The objective of this part is to determine the complexity of finding the outcome of a positional game.

Definition 7 (PSPACE). A decision problem D is in PSPACE if a Turing machine can solve it using $O(P(n))$ space, where P is a polynomial and n the size of the entries.

Definition 8 (PSPACE-Completeness). A decision problem D is PSPACE-Complete if it is in PSAPCE and if there is a polynomial reduction from any decision problem in PSPACE to D.

Definition 9 (MAKER-BREAKER POSITIONAL GAME). MAKER-BREAKER POSITIONAL GAME is the following decision problem :
Inputs: A hypergraph G, a first player $\mathcal{P}$ (Maker or Breaker).
Output: $\mathcal{M}$ if the Maker-Breaker positional game played on G with $\mathcal{P}$ as first player has $\mathcal{M}$ for outcome. $\mathcal{B}$ else.

Definition 10 (MAKER-MAKER POSITIONAL GAME). MAKER-MAKER POSITIONAL GAME is the following decision problem :
Input: A hypergraph G.
Output: $\mathcal{FP}$ if the Maker-Maker positional game played on G has $\mathcal{FP}$ for outcome. DRAW else.

MAKER-BREAKER POSITIONAL GAME and MAKER-MAKER POSITIONAL GAME are both PSPACE-complete problems. The original result is from [Sch78], but [Bys04] proved it more simply, the results and proof ideas are from that second article.

Theorem 11. *MAKER-BREAKER reduces to MAKER-MAKER.*

Proof. Let (V, E) be an instance of MAKER-BREAKER and (d_1, d_2) two vertices not in V.
Let $V' = V \cup \{d_1, d_2\}$ and $E' = \{(d_1, d_2)\} \cup (E \times \{d_1\})$ and $G' = (V', E')$. Consider G' as a MAKER-MAKER instance. In every game, the first player has to play on d_1, else the second can claim d_1, and there is no winning set without a vertex claimed by the second player, so the first one cannot win. The second player now has to play on d_2 else the first player can win by playing on d_2. There are only the vertices of G left unclaimed and the second player cannot win anymore. If Breaker had a winning strategy on G, then by applying such a strategy from now, the second player has a drawing strategy on G'. If Maker had a winning strategy, the first player has then a winning strategy on G'. □

Definition 12 (QUANTIFIED BINARY FORMULA). QUANTIFIED BINARY FORMULA is the following decision problem : Input: A quantified binary formula P (i.e. a first order closed formula in which all the quantifiers are before the variables)
Output: TRUE if P is satisfiable. FALSE if P is not satisfiable.

An example of non-satisfiable QUANTIFIED BINARY FORMULA instance is:

$$\forall a, \exists b, \exists c, \forall d, a \wedge (b \vee c \vee d) \wedge (a \vee c)$$

.
And an example of a satisfiable one is:

$$\exists a, \exists b, \forall c, \forall d, a \wedge (b \vee c \vee d) \wedge (a \vee c)$$

.

Theorem 13. *MAKER-BREAKER is PSPACE-complete.*

Remark. This result is the main one of this part, but the proof is quite extensive, so it is not fully presented here. To see the proper reduction, please refer to [Bys04].

PROOF. MAKER-BREAKER is PSPACE: a certificate would be a winning strategy for one of the players, and describing such a strategy is possible in polynomial space but with an unlimited time.

For the hardness, the idea is to reduce from QUANTIFIED BINARY FORMULA which is the reference PSPACE-complete problem.

□

Corollary 14. *MAKER-MAKER is PSPACE-complete.*

PROOF. The PSPACEness comes from the same argument used for MAKER-BREAKER. For the hardness, according to Theorems 11 and 13, MAKER-MAKER is PSPACE-complete. □

3 PUBLICATION

This part explains some results from [Duc+20]. This article is about the Maker-Breaker domination game and deals with trees, cographs, union of games in this convention and the complexity of the Maker-Breaker domination game.

3.1 Union of graphs

Definition 15 (Game on a union of graphs)**.** A game played on a union of two graphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ is the same game played on $(V_1 \cup V_2, E_1 \cup E_2)$.

Proposition 16. *The outcome of the union of two graphs G and H in the Maker-Breaker domination game is the one given in the following array :*

$G \setminus H$	$\mathcal{M}$	$\mathcal{N}$	$\mathcal{B}$
$\mathcal{M}$	$\mathcal{M}$	$\mathcal{N}$	$\mathcal{B}$
$\mathcal{N}$	$\mathcal{N}$	$\mathcal{B}$	$\mathcal{B}$
$\mathcal{B}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{B}$

This proposition can be summarized into: $\mathcal{M}$ graphs are neutral, $\mathcal{B}$ ones are absorbent, and the union of two $\mathcal{N}$ is a $\mathcal{B}$ graph. The outcome of any non-connected graph is easy to determine as long as you know the outcome of each component.

This result shows once again that the Maker-Breaker version is easier to study than the Maker-Maker version: In the Maker-Maker version, knowing the outcome of each component is not enough, even for simple graphs as paths.

3.2 Trees

The results of this part are applications of the useful pairing strategy again. If there is a leaf u whose neighbor v is of degree 2, and it is Breaker's turn, she can play on v and has Maker plays on u. If it is Maker's turn, he knows that he cannot lose because of these vertices, because when Breaker will play on one of them, he will be able to play on the other one and dominate both. Both players do not have to worry about these vertices and can ensure the same outcome as on $G \setminus \{u; v\}$. If Maker is winning on $G \setminus \{u; v\}$, he then wins on G also. And if Breaker is winning on $G \setminus \{u; v\}$, she also wins on G. So the outcome of G is the same as the outcome of $G \setminus \{u; v\}$.

Such a leaf and neighbor are called pendant P_2. Once all pendant P_2 are removed, the tree is called P_2-irreducible. Authors now get the following Lemma :

Lemma 17. *Every P_2-irreducible tree has one of the following form :*

- K_1
- P_2
- $K_{1,n}$ *with* $n \geq 3$
- *Trees where there are at least two vertices with more than two leaves as neighbors.*

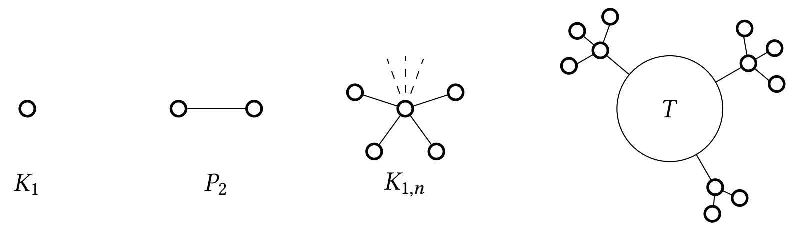

Fig. 6. Different possible reductions for trees.

Theorem 18. *The trees that reduce to the P_2 are those whose outcome is $\mathcal{M}$.*
The trees that reduce to $K_{1,n}$ or K_1 are the trees with outcome $\mathcal{N}$.
The other trees are the trees whose outcome is $\mathcal{B}$.

Corollary 19. *A tree has the outcome $\mathcal{M}$ if and only if it admits a pairing strategy.*

3.3 Complexity

The fact that the existence of a pairing strategy is a necessary and sufficient condition on the trees simplify the complexity of finding the outcome of the Maker-Breaker game on them:

Definition 20 (MAKER-BREAKER DOMINATION GAME)**.** MAKER-BREAKER DOMINATION GAME is the following decision problem :
Inputs: A graph G, the first player $\mathcal{P}$ (Maker or Breaker).
Output: $\mathcal{M}$ if the Maker-Breaker domination game played on G with $\mathcal{P}$ as first player has $\mathcal{M}$ for outcome. $\mathcal{B}$ else.

Theorem 21. *MAKER-BREAKER DOMINATION GAME is P on trees.*

Remark. While on the DOMINATION GAME problem, the input is a hypergraph (of size $O(2^n)$ where n is the number of vertices), here the input is a graph (of size $O(n^2)$ which is much smaller. So the following result is clearly stronger than the one seen in Section 2.

Theorem 22. *MAKER-BREAKER DOMINATION GAME is PSPACE-complete*

4 RESULTS

We will study here the Maker-Maker domination game played on specific graphs. We name the first player $\mathcal{A}$, and the second one $\mathcal{B}$.
We also name a_k and b_k the vertices respectively claimed by $\mathcal{A}$ and $\mathcal{B}$ at the turn number k and A and B the sets of all the vertices claimed by $\mathcal{A}$ and by $\mathcal{B}$.

In the Maker-Maker convention, First Player cannot use pairing strategy as Maker did in Maker-Breaker because, for instance, on the path, the second can dominate a pair without playing in it. See for instance Fig. 7, where two vertices with the same number are in the same pair. The ellipses are the dominating zone of each vertex.

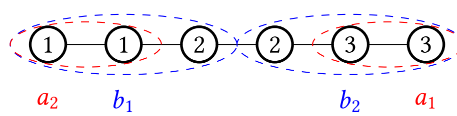

Fig. 7. Failed pairing strategy by First-Player.

The large majority of the following results do not have proof in this section, you can find them in the Appendix section.

4.1 Paths

We study here the Maker-Maker domination game played on the path of length n. We name this path P_n. We name the vertices $(1, ..., n)$ from a leaf to the other.
In this section, we will define a strategy for $\mathcal{A}$, and with this strategy, prove that P_n is $\mathcal{FP}$.

Definition 23 (Trap)**.** A vertex e is a trap if e is unclaimed and if every neighbor of e is claimed by $\mathcal{A}$.

Definition 24 (Potential trap)**.** A vertex e is a potential trap if e is unclaimed and if every neighbor of e is claimed by $\mathcal{A}$ except one that is unclaimed.

Remark. This definition is equivalent to say that e is a potential trap if $\mathcal{A}$ can make a trap on e in a single turn.

Let this strategy be the strategy of $\mathcal{A}$ for this section :

$$\begin{aligned} a_1 &= 2 \\ \forall k \in \mathbb{N}^*, a_{k+1} &= b_k + 1 \text{ if } b_k < n \text{ and } b_k + 1 \notin A \cup B \\ & \quad b_k - 1 \text{ if } b_k > 1 \text{ and } b_k - 1 \notin A \cup B \\ & \quad i \pm 1 \text{ (the one which is unclaimed) where } i \text{ is a potential trap else} \end{aligned}$$

Proposition 25. *For all $n \geq 4$, after $\mathcal{A}$ played his k-th turn, if $\mathcal{A}$ does not dominate, there is at least one trap and one potential trap.*

Definition 26 (Threat)**.** A set of vertices E from a graph G is a threat if E is a connected component of $G \backslash B$ and if $A \cap E = \emptyset$.
On paths, a threat can be *on the left* (or *on the right*) of $i \in [\![1; n]\!]$ when i is the neighbor greater than (or lower than) every vertex of E.

Proposition 27. *After $\mathcal{B}$ played his turn k, there is no threat on the left of b_k.*

The two previous proposition ensure that with the given strategy, the next result is true:

Theorem 28. *For all n, P_n is $\mathcal{FP}$.*

4.2 Union of paths

We study here the Maker-Maker domination game played on specific graphs: the union of $m > 1$ paths of respective length n_i for $i \in [\![1; m]\!]$. We name this union $\sum P_{n_i}$. We name the j-th vertices of the path i, (i, j) where $i \in [\![1; m]\!]$ and $j \in [\![1; n_i]\!]$.

Definition 29 (Neutral)**.** Let G be a graph. G is said neutral for the union if, for all graphs H, the outcome of $H + G$ is the same as H.

Proposition 30. *P_2 and P_4 are neutrals.*

Proposition 31. *$\sum P_{n_i} + P_1$ and $\sum P_{n_i} + P_3$ have the same outcome as $\sum P_{n_i}$ in the Maker-Breaker convention with Breaker playing first : If all n_i are even, $\sum P_{n_i}$ is $\mathcal{M}$, else $\sum P_{n_i}$ is $\mathcal{B}$.*
So, $\sum P_{n_i} + P_1$ and $\sum P_{n_i} + P_3$ are $\mathcal{FP}$ if all n_i are even and DRAW else.

Lemma 32. *In the Maker-Maker convention, on a graph $G = (V, E)$, if $V \backslash B$ contains a component that is a path where there is no vertex claimed by $\mathcal{A}$ before $\mathcal{B}$ plays, then if the length of this path is odd, $\mathcal{B}$ can ensure a draw and if it's even, $\mathcal{B}$ can dominate the component before $\mathcal{A}$ and forces $\mathcal{A}$ playing on this component.*

This lemma is useful to get this interesting result :

Proposition 33. *If $\sum P_{n_i}$ contains only paths of length $n_i > 4$, and at least two of them, $\sum P_{n_i}$ is DRAW.*

Theorem 34. *If $m > 1$:*
If one of the n_i is 1 or 3 and all the others are even, then $\sum_{i=1} P_{n_i}$ is $\mathcal{FP}$.
If there is at most one path of length that is not 2 or 4, then $\sum_{i=1} P_{n_i}$ is $\mathcal{FP}$.
Else, $\sum_{i=1} P_{n_i}$ is DRAW.

Remark. In the Maker-Breaker strategy, the union was simple and only needed to know the outcome of each component, while here it is more fine, even for paths: for instance the union of two graphs $\mathcal{FP}$ can be $\mathcal{FP}$ or DRAW.

4.3 Cycles

We study here the Maker-Maker domination game played on the cycles of n vertices. We name this cycle C_n. We name the vertices of C_n from 1 to n, where for all $i \in [\![1; n]\!]$, i has for neighbors $(i-1)[n]$ and $(i+1)[n]$
In this section, we give a winning strategy on C_n for $\mathcal{B}$ when $n \geq 10$ and $n \equiv 1[3]$ and for $\mathcal{A}$ otherwise. This implies that the outcome of C_n is DRAW if these conditions are met, and $\mathcal{FP}$ else.

Proposition 35. *If $n \not\equiv 1[3]$, $\mathcal{A}$ is able to keep these properties true after he played:*
-All the vertices claimed by $\mathcal{B}$ are dominated by $\mathcal{A}$
-There is no unclaimed path limited by two elements of B
-If $\mathcal{A}$ has not won, there is at least one unclaimed path limited by elements of A of length k, where $k \not\equiv 0[3]$

Proof. For all $k \in \mathbb{N}^*$, let H_k be "The properties are all verified after $\mathcal{A}$ played his k-th turn."

H_1: $\mathcal{A}$ plays on 1. B is empty, so the first and second proprieties are true. The path that contains all the vertices from 2 to n is limited by 1 on both ends and 1 is in A so the third propriety is true. So is H_1.

Let $k \in \mathbb{N}^*$, assume H_k. H_{k+1} :

First case, $\mathcal{B}$ played b_k in an unclaimed path delimited by elements of A, of length n, where $n \not\equiv 0[3]$. There is now two paths delimited by an element of A and one of B, of length i and j, such as $i + 1 + j \not\equiv 0[3]$, i.e. $i + j \not\equiv 2[3]$. So either $i \not\equiv 1[3]$ or $j \not\equiv 1[3]$. Assume, without loss of generality, that $i \not\equiv 1[3]$, and if $j \not\equiv 1[3]$, $i \geq j$. If $i > 1$, $\mathcal{A}$ answers on the path of length i, next to b_k. The new path created is not empty, of length $i - 1 \not\equiv 0[3]$, b_k is dominated by a_{k+1} and there is still no path delimited only by elements of B.

If $i \leq 1$, then $i = 0$ so b_k is already dominated by $\mathcal{A}$, the first and third proprieties are still true. If $\mathcal{A}$ did not win at the turn k, there is at least a vertex not dominated by $\mathcal{A}$. Let l be such a vertex. One of its nearest claimed vertices is claimed by A because of the second propriety. Let it be l'. l' is at least at a distance 2 of l, so by playing at a distance 2 of l', going toward l, $\mathcal{A}$ create a new path delimited by elements of A, of length $1 \not\equiv 0[3]$. So the second propriety is also true.

Second case, $\mathcal{B}$ played b_k in an unclaimed path not delimited by two elements of A. By the second propriety, there is at most one path delimited by elements of B. So $\mathcal{A}$ plays next to b_k, in the path delimited by elements of B (if it is not empty, else he plays on the other neighbor of b_k if it is possible, else he plays anywhere). b_k is now dominated by A, there is no path delimited only by elements of B anymore, and the third propriety is still true.

Last case, $\mathcal{B}$ played b_k in an unclaimed path delimited by elements of A but of length n, where $n \equiv 0[3]$. The second and third proprieties are still true. There is at least one vertex unclaimed next to b_k. So $\mathcal{A}$ plays in this vertex and the first propriety is also true. □

Proposition 36. *C_k where $k \equiv 1[3]$ and $k \geq 10$ is DRAW.*

Proof. For $i \in \mathbb{N}^*$, let H_i be "$\mathcal{B}$ can enforce a draw on cycle of length $k = 1 + 3 * i$ where she already dominates a_1"

H_1: $\mathcal{A}$ plays on a vertex a_1. $\mathcal{B}$ answers on $(a_1 + 2)[4]$. $\mathcal{B}$ dominates before $\mathcal{A}$ so she ensures a draw.

H_2: $\mathcal{A}$ plays on a vertex a_1. $\mathcal{B}$ answers on $(a_1 + 2)[7]$. If $\mathcal{A}$ does not answer on $(b_1 - 1)[7]$, $(b_1 + 1)[7]$ or $(b_1 + 2)[7]$, $\mathcal{B}$ plays on $(b_1 + 1)[7]$ and at the next turn $\mathcal{A}$ can only play on one of the neighbors of $\{b_1; b_2\}$ so $\mathcal{B}$ can play on the other and enforce a draw. If $\mathcal{A}$ does answer on one of these vertices, $\mathcal{B}$ can play on $(b_1 + 3)[7]$ and dominate C_7 before $\mathcal{A}$, so there is a draw.

Let $i \in \mathbb{N}^*$. Assume H_i. H_{i+2} :
$\mathcal{B}$ answers to a_1 by playing on $(a_1 + 5)[k]$.

If $\mathcal{A}$ does not answer next to b_1, $\mathcal{B}$ can play next to b_1 in the opposite direction to a_2. $\mathcal{A}$ can only play on one of the neighbors of $\{b_1, b_2\}$, so $\mathcal{B}$ can play on the other and forces a draw.

If $\mathcal{A}$ answers on $(b_1 - 1)[k]$:
If there are an even number of vertices between b_1 and a_1 (rotating in the direction that avoid a_2), $\mathcal{B}$ plays on $(b_1 + 2 * j)[k]$, for $j \geq 0$ as long as $(b_1 + 2 * j)$ and $(b_1 + 2 * j - 1)$ are not claimed by $\mathcal{A}$. Each of her plays leave only one possible move $((b_1 + 2 * j - 1)[k])$ for $\mathcal{A}$, otherwise $\mathcal{B}$ can make a draw instantly. After she made all these moves, $\mathcal{B}$ dominates a_1, and can play on $(a_1 + 2)[k]$ so she dominates everything and assure a draw.
If there are an odd number of vertices, $\mathcal{B}$ do the same strategy but stop before: she stops when 3 vertices are still unclaimed before getting to a_1, then plays on $(a_1 + 2)[k]$. If $\mathcal{A}$ answers on $(a_1 - 1)[k]$, $\mathcal{B}$ plays on $(a_1 - 2)[k]$ and forces $\mathcal{A}$ to play on $(a_1 - 3)[k]$. So she can play on $(a_1 + 1)[k]$, dominates a_1 and every other vertex. If $\mathcal{A}$ does not, $\mathcal{B}$ claims $(a_1 - 1)[k]$ and dominates all the vertices. So $\mathcal{B}$ can assure a draw.

If $\mathcal{A}$ answers on $(b_1+1)[k]$, $\mathcal{B}$ can play on $(b_1-2)[k]$, so $\mathcal{A}$ has to play on $(b_1-1)[k]$, then $(b_1-4)[k]$ so $\mathcal{A}$ has to respond on $(b_1-3)[k]$. $\mathcal{B}$ already dominates a_1, a_2 and every vertex between them (going the shortest path), so she only has to dominate a path of length $1+3*(i+2)-4$ without having to dominate the extremities. That is exactly dominating a cycle of length $k' = 1+3*(i+2)-6 = 1+3*i$, where she does not have to dominate a_1. So according to H_i, $\mathcal{B}$ can enforce a draw.

With the previously described strategy, the domination of a_1 is granted when $k \geq 10$, so $\mathcal{B}$ can assure a draw on all the C_k where $k \equiv 1[3]$ and $k \geq 10$. □

Theorem 37. *C_k is $\mathcal{FP}$ if and only if $k \not\equiv 1[3]$ or $k \leq 10$*

Proof. According to **Proposition 36**, $\mathcal{B}$ can assure a draw if $k \equiv 1[3]$ and $k \geq 10$.

According to **Proposition 35**, $\mathcal{A}$ is able to keep some proprieties true after he played a turn on C_k where $k \not\equiv 1[3]$. Assume that $\mathcal{B}$ wins at the turn number n with $\mathcal{A}$ keeping the proprieties true. So $\mathcal{B}$ played in an unclaimed path limited by elements of A of length i, where $i \not\equiv 0[3]$. $\mathcal{B}$ won by playing only once on this path, so the length of the path is 1 or 2, so the path is already dominated by $\mathcal{A}$. As $\mathcal{B}$ won, $\mathcal{A}$ would not be able to do a new path of that type, else $\mathcal{B}$ would not dominate the vertices included in this path, but as shown in the proof of **Proposition 35**, such a path is buildable using one vertex not dominated by $\mathcal{A}$. So as $\mathcal{B}$ did not claim a vertex not dominated by $\mathcal{A}$, there is no vertex not dominated by $\mathcal{A}$ at the end of his turn number n, so he wins before $\mathcal{B}$ does. So C_k with these conditions is $\mathcal{FP}$.

C_4 is won by $\mathcal{A}$ by playing on 1, then any other vertex not taken by $\mathcal{B}$.

C_7 is won by $\mathcal{A}$ by playing on 1, then on 4 if $\mathcal{B}$ played on 2, 3 or 5 and on 5 if $\mathcal{B}$ played on 7, 6 or 4. And finally be filling the gap in the domination of $\mathcal{A}$ ($\mathcal{B}$ did not dominate anything at this time because she can dominate at most 6 vertices and $\mathcal{B}$ did not prevent $\mathcal{A}$ from dominating because she needs 3 consecutive vertices to do so on a cycle).

□

4.4 Subdivided stars

We study here the Maker-Maker domination game played on particular trees: the subdivided star. There are graphs made of a central vertex and at least 3 paths linked to it. We name the branches from 1 to the number of branches, and the vertices of the branch i from 1 (the leaf) to the length of the branch i (the neighbor of the central vertex). A vertex is then (i,k) where i is the number of its branch, and k the position of the vertex on this branch.
In this section, we give some useful results, especially on the position of the first move of $\mathcal{A}$ and a result about the outcome of subdivided stars with at least a branch of size 1 (the corresponding strategy can be found in Appendix.

Proposition 38. *A subdivided star with at least one branch of length 1 is $\mathcal{FP}$.*

As we have this, the following results will consider only star without any branch of length 1. Finding a drawing strategy requires looking at every first move, and there are many. So we prove that only some are not good moves for $\mathcal{A}$, on specific stars.

Lemma 39. *If there is no branch of length 1 and if there is at least one branch of even length or at least a branch of length at least 5, $\mathcal{A}$ cannot claim the central vertex as his first vertex.*

Lemma 40. *On a subdivided-star without branch of length* 1*, if* $\mathcal{A}$ *claims his first vertex on a branch* i *of size greater than* 5*, it has to be the vertex* $(i, 2)$

Remark. These two results considerably reduce the possibilities of the $\mathcal{A}$ first move. Applications of these are still to be studied, but the problem remains complex and the several parameters of subdivided stars do not help to get a proper classification.

4.5 PSPACE-Completeness

Definition 41 (MAKER-MAKER DOMINATION GAME)**.** MAKER-MAKER DOMINATION GAME is the following decision problem :
Inputs: A graph G.
Output: $\mathcal{FP}$ if the domination game played on G in its Maker-Maker convention is $\mathcal{FP}$. DRAW else.

Theorem 42. *MAKER-MAKER DOMINATION GAME is PSPACE-Complete.*

Proof. MAKER-MAKER is PSPACE, so it is PSPACE in particular on domination games. For the PSPACE-hardness, there is a reduction from MAKER-BREAKER on domination games (this problem is known to be PSPACE-complete thanks to [Duc+20]). Let $G = (V, E)$ be an instance of MAKER-BREAKER and d a vertex that is not in G. Let $G' = (V \cup \{d\}, E)$ be an instance of MAKER-MAKER domination game. First player has to play on d, else second player plays on d and enforces a draw.
Now, if G is $\mathcal{M}$, as the second player cannot win anymore, the first player applies the same strategy as Maker on G and wins. If G is $\mathcal{FP}$ or $\mathcal{B}$, the second player can use the strategy for Breaker when Breaker starts on G and enforce a draw. □

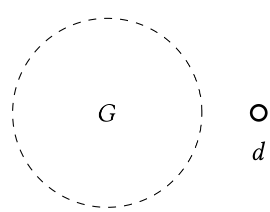

Fig. 8. MAKER-BREAKER to MAKER-MAKER reduction.

5 CONCLUSION

5.1 Results

This internship led to finding the outcome and a strategy leading to this outcome for four types of graphs: Paths, union of paths, cycles and subdivided stars.

- All paths are $\mathcal{FP}$.
- A union of paths is $\mathcal{FP}$ if one of the path is of length 1 or 3 and all the others are even, or if all the paths are of length 2 or 4 except at most one. In every other case, the union is DRAW.
- Cycles of n vertices are $\mathcal{FP}$ when $n \not\equiv 1[3]$ or $n < 10$ and are DRAW when $n \equiv 1[3]$ and $n \geq 10$
- Subdivided stars are $\mathcal{FP}$ if at least one of branches is of length 1, and not determined yet in other cases.

Furthermore, the MAKER-MAKER DOMINATION GAME decision problem is PSPACE-COMPLETE.

5.2 Next steps

The natural continuity of this work is to study the domination game on trees: Paths and subdivided stars are particular trees. As the subdivided stars are trees, and as they were quite difficult to classify, one would expect trees to be divided between $\mathcal{FP}$ and DRAW with precise criteria, and such criteria may be hard to determine. Another extension can be to study: cographs, but as the union of simple graphs, as paths is not obvious, cographs may also be a challenging class of graphs.

Studying other conventions on the domination game can also be very interesting. Avoider-Enforcer is a convention where one of the players (Avoider) try to not fill a winning set, and the other (Enforcer) try to enforce him to fill one. Else, adding to a Maker-Breaker or even to a Maker-Maker domination game, the convention named Picker-Chooser is certainly stimulating towards the pairing strategies previously explained. This convention is the following: at each turn, one of the players (Picker) picks two unclaimed vertices, and the other (Chooser) chooses one of the two vertices, claims it for herself and the other is claimed by Picker.

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APPENDIX

Paths

This section about paths is in French, I do not have time to translate it, I am genuinely sorry about that.

Definition 43 (Piège)**.** Un sommet e est un piège s'il est non revendiqué et si tous les voisins de e sont revendiqués par $\mathcal{A}$.

Definition 44 (Piège potentiel)**.** Un sommet e est un piège potentiel s'il est non revendiqué et si tous les voisins de e sont revendiqués par $\mathcal{A}$ sauf un qui n'est pas revendiqué.

Remark. Cette définition revient à dire qu'il y a un piège potentiel en e si $\mathcal{A}$ peut créé un piège en e en un seul coup.

On définit la stratégie suivante pour $\mathcal{A}$:

$$\begin{aligned} a_1 =& 2 \\ \forall k \in \mathbb{N}^*, a_{k+1} =& b_k + 1 \text{ si } b_k < n \text{ et } b_k + 1 \notin A \cup B \\ & b_k - 1 \text{ si } b_k > 1 \text{ et } b_k - 1 \notin A \cup B \\ & i \pm 1 \text{ (celui qui est libre) où } i \text{ est un piège potentiel sinon} \end{aligned}$$

On suppose dans toute la suite que $\mathcal{A}$ suit cette stratégie.

Proposition 45. *Pour $n \geq 4$, après le coup du joueur $\mathcal{A}$ du tour k, si $\mathcal{A}$ ne domine pas, il y a au moins un piège et un piège potentiel.*

Proof. Soit $n \geq 4$.
On pose H_k l'hypothèse de récurrence : "Après le coup du joueur $\mathcal{A}$ du tour k, si $\mathcal{A}$ ne domine pas, il y a au moins un piège et un piège potentiel."

H_1 : En jouant a_1=2, 1 est un piège et 3 un piège potentiel. Donc H_1 est vérifié.

Soit $k \geq 1$. On suppose H_k.
Supposons que $\mathcal{A}$ ne domine pas après son coup du tour $k + 1$.

Par H_k, il existait au moins un piège et un piège potentiel après le coup (du tour k) de $\mathcal{A}$.
Si $\mathcal{B}$ n'a pas joué dans un des pièges, alors il en existe toujours au moins un.
Si $\mathcal{B}$ a joué dans un piège au tour k, alors $b_k + 1$ et $b_k - 1$ sont revendiqués (ou n'existent pas si $b_k = 1$ ou n), donc $\mathcal{A}$ a joué au tour $k+1$ selon la troisième ligne de sa stratégie : il a transformé un piège potentiel (qui existait bien par H_k) en un piège. Donc il existe bien un piège à la fin de son tour.

Comme $\mathcal{A}$ ne domine pas , il existe $i \in [\![1;n]\!]$ non dominé par $\mathcal{A}$ c'est à dire, tel qu'aucun des voisins de i n'est dans A. On s'intéresse au plus proche sommet revendiqué à gauche de i, que l'on note j.
Si $j \in B$ alors $j + 1 \in A$ d'après la stratégie de $\mathcal{A}$, ce qui contredit le fait que j est le plus proche sommet revendiqué de i. Donc $j \in \mathcal{A}$.
De plus, comme i n'est pas dominé par $\mathcal{A}$, $i \neq j + 1$. Donc $j + 2 \leq i$ n'est pas revendiqué, et $j + 1$ est donc un piège potentiel.

Donc après que $\mathcal{A}$ ai joué, s'il ne domine pas, il existe au moins un piège et un piège potentiel donc H_{k+1} est vérifié. ◻

Definition 46 (Menace)**.** Un ensemble de sommets E d'un graphe G est une menace si E est une composante connexe de $G \backslash B$ et si $A \cap E = \emptyset$.
Dans le cadre des chemins, on peut préciser qu'une menace est *à gauche* (ou *à droite*) de $i \in [\![1; n]\!]$ lorsque i est le voisin supérieur (ou inférieur) à tous les sommets de la menace.

Proposition 47. *Après le coup du joueur $\mathcal{B}$ du tour k, il n'y a pas de menace à gauche de b_k.*

PROOF. Soit $k \in \mathbb{N}$.
On suppose par l'absurde qu'il y a une menace à gauche de b_k.
Soit donc $k' < k$ tel que $]\!]b_{k'}; b_k[\![\neq \emptyset$ et pour tout $i \in]\!]b_{k'}; b_k[\![$, $i \notin A \cup B$.
$b_{k'} + 1$ n'est pas revendiqué au tour k, donc il ne l'est pas non plus au tour k'.
Donc, d'après le stratégie du joueur $\mathcal{A}$, $a_{k'+1} = b_{k'} + 1$. Et donc $b_{k'} + 1$ n'est pas libre au tour k. C'est absurde. □

Theorem 48. *Pour tout n, $\mathcal{A}$ est gagnant dans P_n.*

PROOF. Soit $n \in \mathbb{N}^*$.
Si $n = 1$, en jouant $a_1 = 1$, $\mathcal{A}$ domine P_1.
Si $n \in \{2, 3\}$, en jouant $a_1 = 2$, $\mathcal{A}$ domine P_n.
Si $n \geq 4$,
$\mathcal{B}$ ne peut pas dominer avant $\mathcal{A}$: Si $\mathcal{B}$ domine, alors il n'y a plus de piège et plus de piège potentiel. Or, d'après la **Proposition 1**, il existe un piège i et un piège potentiel j après que $\mathcal{A}$ ai joué. $\mathcal{B}$ domine i donc il a joué en i, mais il ne peut pas alors pas dominer j au même tour : c'est absurde.
$\mathcal{B}$ ne peut pas empêcher $\mathcal{A}$ de dominer : cela reviendrait à dire qu'il existe $i \in [\![1; n]\!]$ tels que tous les voisins de i sont dans B. Par les deux premières lignes de la stratégie et la **Proposition 2**, ceci n'est possible que si les voisins de i sont revendiqués par $\mathcal{B}$ avant que $\mathcal{B}$ ne joue en i.
Comme $a_1 = 2$, $i > 3$ et donc lors de la revendication de $i - 1$, i est alors encore libre et par la première ligne de la stratégie de $\mathcal{A}$, $\mathcal{A}$ joue alors en i : c'est absurde. □

Union of paths

Definition 49 (Neutral)**.** Let G be a graph. G is said neutral for the union if, for all graphs H, the outcome of $H + G$ is the same as H.

Proposition 50. *P_2 and P_4 are neutrals.*

Proof. Let G be a graph won by $\mathcal{A}$.
On $G + P_2$, $\mathcal{A}$ starts by playing on G as his strategy for G tells him. While $\mathcal{A}$ is not dominating G and $\mathcal{B}$ plays on G, $\mathcal{A}$ answers with his winning strategy on G.
If $\mathcal{B}$ plays on P_2 then, $\mathcal{A}$ answers by taking the vertex $\mathcal{B}$ left on P_2. He now plays with his strategy on G, and dominate G before $\mathcal{B}$, and as he is already dominating P_2, $\mathcal{A}$ is dominating $G + P_2$.
If $\mathcal{B}$ does not play on P_2 until $\mathcal{A}$ dominate G, then at his next turn $\mathcal{B}$ can dominate G or P_2 but not both, so $\mathcal{A}$ can play on P_2 ($\mathcal{B}$ claimed at most one vertex) and dominate $G + P_2$.
For P_4 the main strategy remains, but when $\mathcal{B}$ plays on P_4 on the vertex 1 or 2 then $\mathcal{A}$ answers on the other, and when $\mathcal{B}$ plays on 3 or 4, $\mathcal{A}$ answers on the other.

Let G be a graph leading to a draw.
$\mathcal{B}$ has a strategy to enforce the draw on G. If $\mathcal{A}$ plays on G and $\mathcal{B}$ does not dominate G, then $\mathcal{B}$ answers with her strategy. If $\mathcal{A}$ plays on P_2 (or P_4), then $\mathcal{B}$ plays as described in the first part of the proof (for G won by $\mathcal{A}$). If $\mathcal{B}$ dominate G, it is before $\mathcal{A}$ does. So by playing on P_2 (or P_4) he dominates $G + P_2$ (or $G + P_4$) before $\mathcal{A}$ does. □

Proposition 51. *$\sum P_{n_i} + P_1$ and $\sum P_{n_i} + P_3$ have the same outcome as $\sum P_{n_i}$ in Maker-Breaker with Breaker playing first.*

Proof. Let $\sum P_{n_i}$ be a union of paths.
On $\sum P_{n_i} + P_1$, if $\mathcal{A}$ does not play on P_1 then $\mathcal{B}$ can and assures a draw as $\mathcal{A}$ cannot dominate P_1. So $\mathcal{A}$ has to play his first move on P_1.
$\mathcal{B}$ can not dominate P_1 anymore, so her only hope is to break $\mathcal{A}$'s domination, and she plays first on $\sum P_{n_i}$.

On $\sum P_{n_i} + P_3$, if $\mathcal{A}$ does not play on P_3 then $\mathcal{B}$ can play in the middle of P_3 on her first turn and assures a draw by playing another vertex of P_3 on her second turn. $\mathcal{A}$ can not dominate P_3 and $\mathcal{B}$ wins. So $\mathcal{A}$ has to play his first move on P_3.
If $\sum P_{n_i}$ is won by Maker in Maker-Breaker when Breaker plays first, then $\mathcal{A}$ can follow his strategy to dominate $\sum P_{n_i}$ after his first turn and whenever $\mathcal{B}$ plays on P_3, $\mathcal{A}$ claims the remaining vertex and $\mathcal{B}$ can not dominate anymore.
If $\sum P_{n_i}$ is won by Breaker in Maker-Breaker when Breaker plays first, then $\mathcal{B}$ can ignore P_3 and play her winning strategy on $\sum P_{n_i}$. □

Lemma 52. *On a graph $G = (V, E)$, if $V \backslash B$ contains a component that is a path where there is no vertex claimed by $\mathcal{A}$ before $\mathcal{B}$ plays, then if the length of this path is odd, $\mathcal{B}$ can ensure a draw and if it's even, $\mathcal{B}$ can dominate the component before $\mathcal{A}$ and force $\mathcal{A}$ playing on this component.*

Proof. Let for all n, H_n be "If one of the components of $V \backslash B$ is a path of length n, and it is the turn of $\mathcal{B}$, then if the length of this path is odd, $\mathcal{B}$ can ensure a draw and if it's even, $\mathcal{B}$ can dominate the component before $\mathcal{A}$ and force $\mathcal{A}$ playing on this component until one turn after $\mathcal{B}$ dominated the component."

We name the vertices of such a path from 1 to n, where n is the length of the path.

H_1 : The only vertex of this path has no unclaimed neighbor, so if $\mathcal{B}$ plays on 1, $\mathcal{A}$ cannot dominate 1, so $\mathcal{B}$ enforce the draw. So H_1 is true.

H_2 : $\mathcal{B}$ plays on 2. Then if $\mathcal{A}$ does not play on 1, $\mathcal{B}$ can and 2 in not dominatable by $\mathcal{A}$ anymore. So $\mathcal{A}$ has to play on 2, and $\mathcal{B}$ was dominating the path before $\mathcal{A}$. So H_2 is true.

Let $n \in \mathbb{N}$. Assume H_n.
H_{n+2} : $\mathcal{B}$ plays on vertex 2. If $\mathcal{A}$ does not answer on 1, then $\mathcal{B}$ can play on 1 and this vertex is not dominatable by $\mathcal{A}$ anymore, so $\mathcal{B}$ enforce the draw. So $\mathcal{A}$ has to answer on 1. $V \backslash B$ contains a component that is path of length n, and it is the turn of $\mathcal{B}$. So by, H_n, H_{n+2} is also true. □

Proposition 53. *If $\sum P_{n_i}$ contains only paths of length $n_i > 4$, and at least two of them, $\sum P_{n_i}$ is a draw.*

Proof. If there are i, j such as n_i and n_j are odd. Then $\mathcal{A}$ plays his first turn on at most one of them, we can assume without loss of generality that $\mathcal{A}$ played on the path i. $\mathcal{B}$ only has to play first on the path j, on even vertexes by growing index while $\mathcal{A}$ answer in the traps created this way. $\mathcal{B}$ plays on $(j, n_j - 1)$, there will be a trap on (j, n_j) and a trap on $(j, n_j - 2)$. $\mathcal{A}$ will only be able to answer in one of them, and $\mathcal{B}$ can fill the other one so $\mathcal{A}$ cannot win.

If there is only one path of odd length (the i-th), if $\mathcal{A}$ does not play on it on his first turn, $\mathcal{B}$ can do the same strategy as described above. If $\mathcal{A}$ played on the odd path, then $\mathcal{B}$ dominates all the other paths except one by using the strategy described in **Lemma 1**. Whenever $\mathcal{A}$ does not answer in the trap from $\mathcal{B}$, $\mathcal{B}$ plays in her trap and enforces the draw. When $\mathcal{B}$ dominates all the even paths except one (let k be its number), she begins the same strategy but stops when exactly four vertices of this path are unclaimed. Then she plays on $(k, n_k - 1)$. This move set up a trap for $\mathcal{A}$ so he has to play on (k, n_k) but while $\mathcal{B}$ dominates the path k, $\mathcal{A}$ does not. $\mathcal{B}$ can now play on the i-th path : she begins by following the strategy described in **Lemma 1** on the paths of length at least 2 of $P_i \backslash a_1$. After doing that, a_1 neighbors are either both $\mathcal{B}$ or both unclaimed. In the first case, $\mathcal{B}$ dominates every path but not $\mathcal{A}$ so she wins. In the second case, by playing on a neighbor of a_1 (if one of them is not dominated by $\mathcal{B}$, she plays on this one), $\mathcal{B}$ dominate the whole graph but not $\mathcal{A}$, so she also wins.

If all n_i are even, then $\mathcal{B}$ can execute exactly the same strategy as for one odd path. The only difference is on the final situation once $\mathcal{B}$ did all the traps she could, there is exactly one neighbor of a_1 unclaimed, so if $\mathcal{B}$ does not already dominate, she only has to play on this neighbor.

As if $\mathcal{A}$ does not enforce the draw, then he loses, the issue of $\sum P_{n_i}$ is a draw. □

Theorem 54. *If $m > 1$:*
If one of the n_i is 1 or 3 and all the others are even, then $\sum P_{n_i}$ is won by first player.
If there is at most one path of length that is not 2 or 4, then $\sum P_{n_i}$ is $\mathcal{FP}$.
Else, $\sum P_{n_i}$ is a draw.

Proof. If one of the n_i is 1 or 3, then according to **Proposition 2** and to [Duc+20], $\sum P_{n_i}$ is won by $\mathcal{A}$ if and only if for all $j \neq i$, n_j is even.
If all the n_i are not 1 neither 3, then according to **Proposition 1**, $\sum P_{n_i}$ has the same outcome as $\sum_{i, n_i > 4} P_{n_i}$. So by **Proposition 3**, if there is at least two $n_i > 4$, $\sum P_{n_i}$ is a draw. Else, by the result on the paths, $\sum P_{n_i}$ is won by the first player. □

Subdivided stars

Proposition 55. *A subdivided star with at least one branch of length 1 is won by $\mathcal{A}$.*

Proof. $\mathcal{A}$ plays on the central vertex. When $\mathcal{B}$ plays on a branch that is not of length 1 and not dominated by $\mathcal{A}$, $\mathcal{A}$ then plays according to the strategy on the path made of the branch where $\mathcal{B}$ played, the central vertex and the branch of length 1. When $\mathcal{B}$ plays on a branch of length 1 or in a trap on a branch dominated by $\mathcal{A}$, as $\mathcal{A}$ played as on paths for his previous moves, he either already won (so $\mathcal{B}$ did not play here) or can make a trap on one of the paths. That way, $\mathcal{A}$ can maintain that there is a trap on the graph, and a potential trap, so this star is won by $\mathcal{A}$. □

Lemma 56. *If there is no branch of length* 1 *and if there is at least one branch of even length or at least a branch of length at least* 5*, $\mathcal{A}$ cannot claim his first vertex on the central vertex.*

Proof. Assume that $\mathcal{A}$ claimed the central vertex at its first turn. $\mathcal{B}$ applies the trap strategy on even branches. If there is at least one, she now dominates the central vertex. Then she applies the trap strategy on odd branches. Before her last move, she dominates all the vertices of the branches she is not playing on and the central vertex. She only needs to dominate at most 3 vertices in a row, and the middle one is unclaimed. So she wins if $\mathcal{A}$ answers in her trap, so he will not, so it is a draw. If there are no even branches, but there is at least one branch of length at least 5, let i be its number. $\mathcal{B}$ applies her trap strategy on the branches that are not i. On i, she applies the same strategy but stops when there are only 3 unclaimed vertices. Instead of playing on the middle one, she plays on the neighbor of the central vertex and dominates the whole branch. So as before, she wins if $\mathcal{A}$ answers in her traps so $\mathcal{A}$ will not, so it is a draw. □

Lemma 57. *On a subdivided-star without branch of length* 1*, if $\mathcal{A}$ claims his first vertex on a branch i of size greater than* 5*, it has to be the vertex* $(i, 2)$

Proof. Assume that $\mathcal{A}$ claimed the vertex (i, k) as its first vertex, $k \neq 2$.

If (i, k) is not a neighbor of the central vertex :

Step 1: $\mathcal{B}$ can then use the trap strategy from the leaf of the branch i, $(i, 1)$ to (i, k). She stops when the strategy tells her to play on (i, k) or on $(i, k + 1)$.

Step 2: Then $\mathcal{B}$ uses the same strategy on the other branches, stopping when the strategy tells her to play on or beyond the central vertex. Once she did that on all the branches, she dominates all the branches except i, and potentially, the central vertex.

Step 3: From now on, whenever $\mathcal{B}$ is not dominating only two or three vertices, she plays on $(i, k + 1)$. If one of the branches dominated by $\mathcal{B}$ is of odd length, $\mathcal{B}$ plays on the central vertex and progress using the trap strategy from then central vertex to (i, k). Else, $\mathcal{B}$ plays on the last neighbor of the central vertex and applies her trap strategy from this vertex to (i, k).

When $\mathcal{B}$ sets up a trap or a large trap next to (i, k), $\mathcal{A}$ does not dominate everything, but $\mathcal{B}$ will: she dominates already the vertices from the leaf to $(i, k - 1)$ by Step 1, all the branches that are not n by Step 2, the central vertex, and the vertices from the central vertex to (i, k) by Step 3.

If (i, k) is a neighbor of the central vertex : First, $\mathcal{B}$ begins by the previously called Step 2. Then, if one the dominated branch is of odd length, plays in the central vertex.
Now, $\mathcal{B}$ plays the trap strategy on the branch n from the leaf to (i, k). If the strategy tells her to play on $(i, k - 2)$, she plays on $(i, k - 1)$ instead. She now dominates the whole star. □